A Composite feedback approach to stabilize nonholonomic systems with time varying time delays and nonlinear disturbances

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Abstract: In this work, we propose a robust stabilizer for nonholonomic systems with time varying time delays and nonlinear disturbances. The proposed approach implements a composite nonlinear feedback structure in which a linear controller is designed to yield a fast response and a nonlinear feedback control law is considered to increase the system's damping ratio. This structure results in the simultaneous improvement of the steady-state accuracy and transient performance of time-delay nonholomic systems.

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Asymptotic stability of the proposed feedback control approach is derived using a Lyapunov–Krasovskii functional aimed at reaching a compromise between system's transient performance and asymptotic stability. Simulation and analytical results are considered to highlight the robustness and superior performance of the proposed approach in controlling high-order-time-delay nonholonomic systems with nonlinear disturbances.

Keywords: Composite nonlinear feedback, nonholonomic systems, time-delay, nonlinear disturbances.

1. Introduction

Though system stabilization is widely considered in the literature, reaching a tradeoff between stability, steady-state performance and transient response for highly nonlinear time-delay systems is still a challenging problem [1]. For instance, in stabilization problems, the transient performance should not be overlooked. There is agreement among researchers that the transient response of adaptive systems is generally not acceptable due to the large initial swings in their performance [2]. Hence, various control approaches were proposed to improve systems' transient stability performance [3].

However, a trade-off between settling time and overshoot persists in most of existing nonlinear control designs [4]. The Composite Nonlinear Feedback (CNF) approach was recently proposed to overcome this problem and improve transient performance by combining linear and nonlinear feedback controllers without switching components [5]. The linear component is implemented to ensure a fast response whereas the nonlinear portion gradually changes the damping ratio as the system's output converges to zero whilst reducing the overshoot produced via the linear component. The CNF method was first introduced in [6] to modify the transient performance of the tracking controller of second-order linear systems with input-saturation. A CNF-based Integral Sliding Mode Control (ISMC) approach was proposed in [7] for fast and accurate robust-tracker and model-follower design of linear uncertain systems subject to time-delays and external disturbances. In [8], a two-term CNF control technique is proposed for nonlinear time-delay systems with input saturations. Reference [9] developed a CNF-based finite-time robust tracker for chaotic systems with external disturbances, Lipschitz nonlinearities and time delays.

Nonholonomic systems are a special class of nonlinear systems with non-integrable constraints on their velocities [10]. In other words, systems with constraints on their velocity those are not derivable from position constraints. Characteristics commonly found in various mechanical systems such as surface vessels, space vehicles, wheeled robots, to name a few. Designing stabilizers or trackers for nonholonomic robotic systems is a challenging problem since this class of systems is not controllable (linearly) around the equilibria and does not guarantee the necessary smooth-feedback stability condition (Brockett theorem) [11]. Hence, smooth-feedback controllers cannot stabilize these systems. Some discontinuous control schemes such as Sliding Mode Control (SMC), hybrid control and time-varying feedback [12] were developed to control this class of dynamical systems. In practice, nonholonomic systems are also prone to time delays. Since certain time delays can be a potential source of instability, they should be taken into consideration at the control design stage [13-15]. Thus, given the complex nature of nonholonomic systems, their stabilization and tracking continues to be an active research topic [16].

In [14], a control approach is suggested to asymptotically stabilize the chained input-delay nonholonomic systems via input-state scaling static gain controller methods. The tracking problem of a chained-form nonholonomic system was also considered in [17] using the *K*-exponential control technique. System stabilization was considered using the Linear Matrix Inequality (LMI) method assuming the system is free of disturbances and time-delays. In [18], output-feedback stability of nonholonomic system with time-delay is addressed. One distinguished feature of [18] is that time delays exist in polynomial nonlinear growing circumstances. The considered system of [18] has low nonlinearity and is free of perturbations. A recursive Terminal Sliding Mode (TSM) strategy was proposed in [19] for the tracking problem of a chained-form nonholonomic system with disturbances. The design ensured that state trajectory is forced to converge to the origin in finite time; however, the proposed design did not take into consideration time delays. The global stabilization problem for nonholonomic systems in chained-form under input delays was investigated in [20]. The approach considered a specific transformation to convert the original system into a delay-free form. However, the approach was limited to constant and time invariant time delays. A control approach that ensures the global asymptotic stability for a class of time delay nonholonomic systems was proposed in [1]. The control design process entailed relaxing the powers of the nonlinear terms and adopting a new Lyapunov-Krasovskii functional. The design along with the performance assessment was focused on the asymptotic stability of the system and neglected its transient behavior. Hence, in this paper, we propose to further expand that work and design a CNF approach that is able to simultaneously improve the system's transient behavior whilst guaranteeing asymptotic stability. Additionally, the simultaneous presence of time-varying time delays and nonlinear disturbances will be considered in this paper, contrary to [1], which only dealt with time delays.

The main contributions of this work are as follows:

- The construction of a new power-integrator-based Lyapunov–Krasovskii functional that takes into consideration system's transient performance and time-delays.
- A CNF control approach to counteract the effect of time delays and ensure both robust stabilization and performance improvement of the system.
- An output control structure that guarantees both steady-state accuracy and improved transient performance despite the time delays and nonlinear disturbances.

The remainder of the paper is organized as follows. The problem under consideration is formulated in section 2. The design procedure for the CNF-based approach for nonholonomic systems with time delays and external disturbances is detailed in section 3. Simulation results illustrating the performance of the proposed approach are given in section 4. Finally, some conclusions are provided in section 5.

2. Problem formulation

Given the following nonholonomic system [1]

$$
\begin{aligned}
&\dot{x}_0(t) = d_0(t)u_0(t)^{q_0} + D_0(t),\\
&\dot{x}_i(t) = d_i(t)x_{i+1}(t)^{p_i}u_0(t)^{qi} + f_i(t, x_0(t), x(t), x(t-\tau(t))), \quad i = 1, 2, \ldots, n-1,\\
&\dot{x}_n(t) = d_n(t)u_1(t)^{p_n} + f_n(t, x_0(t), x(t), x(t-\tau(t))),
\end{aligned}
\tag{1}
$$

where $[x_0(t), x(t)^T]^T = [x_0(t), x_1(t), \ldots, x_n(t)]^T \in R^{n+1}$, $u(t) = [u_0(t), u_1(t)]^T$ denote the states and control inputs, respectively; $\tau(t)$ refer to the state's time-delay; $d_0(t), \ldots, d_n(t)$ represent the time-varying control coefficients; $D_0(t)$ represent the system perturbations; $q_0, \ldots, q_{n-1}$ and $p_0, \ldots, p_{n-1}$ are positive odd integers; and $f_1(.), \ldots, f_n(.)$ are time-delay nonlinear functions. Furthermore, assume the following conditions:

1. For each $k = 0, \ldots, n$, there exist coefficients $c_{k1} \geq 0$ and $c_{k2} \geq 0$ with $c_{k1} \leq d_k(t) \leq c_{k2}$ [1].

2. The delay term $\tau(t)$ fulfills $0 \leq \tau(t) \leq \tau_0$, $\dot{\tau}(t) \leq \gamma \leq 1$ for positive constants τ_0 and γ.

3. For each $i, j = 1, \ldots, n$, there exist continuous non-negative functions $C(x_0(t))$ and $w \geq 0$ with

$$
\begin{aligned}
\left| f_i(t, x_0(t), x(t), x(t-\tau(t))) \right| \leq{}& C(x_0(t)) \sum_{j=1}^{i} \left(\left| x_j(t) \right|^{\frac{s_i + w}{s_j}} + \left| x_j(t-\tau(t)) \right|^{\frac{s_i + w}{s_j}} \right)\\
&+ C(x_0(t)) \sum_{j=1}^{i} \left(\left| x_j(t) \right|^{\frac{1}{p_j p_{j+1} \cdots p_{i-1}}} + \left| x_j(t-\tau(t)) \right|^{\frac{1}{p_j p_{j+1} \cdots p_{i-1}}} \right).
\end{aligned}
\tag{2}
$$

where $p_0 = 1$, $p_j \geq 1$, and $s_i\ (i = 1, \ldots, n)$ are defined as

$$
s_1 = 1, \quad s_i = \frac{s_{i-1} + w}{p_{i-1}}, \quad i = 2, 3, \ldots, n. \text{ [18]}. \tag{3}
$$

Designing the control input $u_1(t)$ entails some transformations in the dynamical equations. When $u_0(t) \neq 0$ for any finite time, the following scaling transformation is considered:

$$\zeta_i(t) = \begin{cases} \dfrac{x_i(t)}{\kappa^{l_i}}, & t_0 - \tau_0 \le t \le t_0 \\ \dfrac{x_i(t)}{\kappa^{l_i} u_0(t)^{h_i}}, & t \ge t_0 \end{cases} \tag{4}$$

$$v_1(t) = \frac{u_1(t)}{\kappa^{l_n}}, \tag{5}$$

where $\kappa \ge 1$ is a coefficient, and l_i and h_i are such that:

$$l_1 = 0, \tag{6}$$

$$l_i = \frac{l_{i-1} + 1}{p_{i-1}}, \quad i = 1, \dots, n, \tag{7}$$

$$h_n = 0\,, \tag{8}$$

$$h_i = q_i + p_i h_{i+1}, \quad i = n-1, \dots, 1. \tag{9}$$

Differentiating Eq. (4), one gets

$$\dot{\zeta}_i(t) = \frac{1}{\kappa^{l_i}} \left(\frac{\dot{x}_i(t) u_0(t)^{h_i} - h_i x_i(t) \dot{u}_0(t) u_0(t)^{h_i - 1}}{u_0\ (t)^{2h_i}} \right), \tag{10}$$

where substituting (1) into (10), one obtains

$$\dot{\zeta}_i(t) = \frac{1}{\kappa^{l_i}}\left(\frac{d_i(t)x_{i+1}(t)^{p_i}u_0(t)^{h_i+q_i} + f_i(.)u_0(t)^{h_i} - h_i x_i(t)\dot{u}_0(t)u_0(t)^{h_i-1}}{u_0(t)^{2h_i}}\right)$$
$$= \frac{1}{\kappa^{l_i}}\left(\frac{d_i(t)x_{i+1}(t)^{p_i}u_0(t)^{q_i} + f_i(.)}{u_0(t)^{h_i}} - \frac{h_i x_i(t)\dot{u}_0(t)}{u_0(t)^{h_i+1}}\right). \tag{11}$$

Now, using (7) and (9), and defining

$$\bar{f}_i(t, x_0, \bar{\zeta}(t), \bar{\zeta}(t-\tau)) = \frac{f_i(t, x_0, \zeta(t), \zeta(t-\tau))}{\kappa^{l_i}u_0^{h_i}(t)} - h_i\zeta_i(t)\frac{\dot{u}_0(t)}{u_0(t)}, \tag{12}$$

one has

$$\dot{\zeta}_i(t) = d_i(t)\kappa\zeta_{i+1}(t)^{p_i} + \bar{f}_i(t, x_0, \bar{\zeta}(t), \bar{\zeta}(t-\tau)). \tag{13}$$

On the other hand, using (4) and (8), one obtains

$$\zeta_n(t) = \frac{x_n(t)}{\kappa^{l_n}}, \tag{14}$$

where differentiating (14) and using (1), one finds

$$\dot{\zeta}_n(t) = \frac{f_n(t, x_0(t), x(t), x(t-\tau(t))) + d_n(t)u_1(t)^{p_n}}{\kappa^{l_n}}. \tag{15}$$

The positive constant l_n can be obtained from (7) as

$$l_n = l_{n+1}p_n - 1. \tag{16}$$

Now, defining $\bar{f}_n = \frac{f_n(.)}{\kappa^{l_n}}$ and using (15) and (16), one achieves

$$\dot{\zeta}_n(t) = \frac{d_n(t)u_1(t)^{p_n}}{\kappa^{l_{n+1}p_n}\kappa^{-1}} + \bar{f}_n(.). \tag{17}$$

Assume

$$l_{n+1} = l_n \tag{18}$$

such that the term $\dot{\zeta}_n(t)$ can be written as

$$\dot{\zeta}_n(t) = d_n(t)\kappa\left(\frac{u_1(t)}{\kappa^{l_n}}\right)^{p_n} + \overline{f}_n(.) \tag{19}$$

where using (5), one obtains

$$\dot{\zeta}_n(t) = d_n(t)\kappa v_1(t)^{p_n} + \overline{f}_n(t, x_0, \overline{\zeta}(t), \overline{\zeta}(t-\tau)). \tag{20}$$

Now, the transformations are introduced as

$$z_i(t) = \zeta_i(t)^{p_1 \cdots p_{i-1}} - \alpha_{i-1}(t)^{p_1 \cdots p_{i-1}}, \tag{21}$$

$$\alpha_{i-1}(t)^{p_1 \cdots p_{i-1}} = -\Im_i\left(z_i(t) + z_i(t)_i^{(s_i+w)/s_i}\right), \qquad i = 1, \ldots, n, \tag{22}$$

where $p_0 = p_1^{-1}$, $\alpha_0(t) = 0$ and $\Im_i > 1$ are some constant values which will be determined later. The Lyapunov candidate functions are chosen as

$$V_j = V_{j1} + V_{j2} + V_{j-1}, \tag{23}$$

with $V_0 = 0$, where V_{j1} and V_{j2} can be described by

$$V_{j1} = \int_{\alpha_{j-1}}^{\zeta_j} (s^{p_1 \cdots p_{j-1}} - \alpha_{j-1}(s)^{p_1 \cdots p_{j-1}})^{2 - \frac{1}{p_1 \cdots p_{j-1}}} ds + \int_{\alpha_{j-1}}^{\zeta_j} (s^{p_1 \cdots p_{j-1}} - \alpha_{j-1}(s)^{p_1 \cdots p_{j-1}})^{\frac{2\sigma - s_{j+1}p_j}{s_j p_{j-1} \cdots p_1}} ds, \tag{24}$$

$$V_{j2} = \frac{\delta_j}{1-\gamma}\kappa^{1-b} \int_{t-\tau(t)}^{t} (z_j(l)^2 + z_j(l)^{\frac{2\sigma}{s_j p_{j-1} \cdots p_1}})dl \tag{25}$$

where δ_j and $\sigma \geq \max_{1 \leq i \leq n}\{s_i + w\}$ are two positive constants.

In what follows, we consider the state-observer proposed in [21]:

$$\zeta_j(t)^{p_{j-1}} = \left(\chi_j(t) + m_{j-1}\zeta_{j-1}(t)\right)^{\frac{s_{j-1}+w}{s_{j-1}}} + \left(\chi_j(t) + m_{j-1}\zeta_{j-1}(t)\right), \tag{26}$$

where m_{j-1} is the constant design parameter and $\chi_j(t)$ is the observer variable. Similar to [21], the reduced-order observer is adopted as

$$\hat{\zeta}_j(t)^{p_{j-1}} = \left(\hat{\chi}_j(t) + m_{j-1}\hat{\zeta}_{j-1}(t)\right)^{\frac{s_{j-1}+w}{s_{j-1}}} + \hat{\chi}_j(t) + m_{j-1}\hat{\zeta}_{j-1}(t), \tag{27}$$

$$\dot{\hat{\chi}}_j(t) = -m_{j-1}\kappa\hat{\zeta}_j(t)^{p_{j-1}}, \tag{28}$$

where $\hat{\zeta}_{j-1}(t)$ and $\hat{\chi}_j(t)$ are the estimations of $\zeta_{j-1}(t)$ and $\chi_j(t)$, respectively.

Lemma 1 [22]. For $1 \le p \in R_{odd}$ and $\forall x, y \in R$, one has

$$\left(|x| + |y|\right)^{1/p} \le |x|^{1/p} + |y|^{1/p} \le 2^{\frac{p-1}{p}}\left(|x| + |y|\right)^{\frac{1}{p}}, \tag{29}$$

$$|x-y|^p \le 2^{p-1}\left|x^p - y^p\right|, \tag{30}$$

$$|x+y|^p \le 2^{p-1}\left|x^p + y^p\right|. \tag{31}$$

Lemma 2 [1, 18]. For the real numbers δ_i guaranteeing $0 \le \delta_1 \le \ldots \le \delta_n$ and some nonnegative functions $c_i(x, y)$, it yields

$$c_1(x,y)|x|^{\delta_1} + c_n(x,y)|x|^{\delta_n} \le \sum_{j=1}^{n} c_j(x,y)|x|^{\delta_j} \le (|x|^{\delta_1} + |x|^{\delta_n})\sum_{j=1}^{n} c_j(x,y). \tag{32}$$

Lemma 3 [1, 18]. For two positive numbers m and n, and a function $a(x, y) > 0$, a function $c(x, y) > 0$ exists so that

$$\left|a(x,y)x^m y^n\right| \le c(x,y)|x|^{m+n} + \frac{n}{m+n}\left(\frac{m}{(m+n)c(x,y)}\right)^{\frac{m}{n}}|a(x,y)|^{\frac{m+n}{n}}|y|^{m+n}, \forall x, y \in R. \tag{33}$$

Lemma 4 [18]. If the initial state $x_0(t_0)$ satisfies $x_0(t_0) \ne 0$, there exist two constants $a > 0$ and $0 \le b \le 1$ for $i = 1,\ldots,n$ such that

$$\left|\overline{f}_i\right| \le a\kappa^{1-b}\sum_{j=1}^{i}\left(\left|z_j(t)\right|^{\frac{1}{p_1p_2\cdots p_{i-1}}}+\left|z_j(t)\right|^{\frac{s_i+w}{s_jp_j\cdots p_1}}+\left|z_j(t-\tau)\right|^{\frac{1}{p_1p_2\cdots p_{i-1}}}+\left|z_j(t-\tau)\right|^{\frac{s_i+w}{s_jp_j\cdots p_1}}\right). \tag{34}$$

Lemma 5 [1]. The subsequent inequalities are satisfied:

$$\begin{aligned} d_{k-1}\kappa\left(z_{k-1}^{2-\frac{1}{p_1\cdots p_{k-2}}}+z_{k-1}^{\frac{2\sigma-w-s_{k-1}}{s_{k-1}p_{k-2}\cdots p_1}}\right)\left(\zeta_k^{p_{k-1}}-\alpha_k^{p_{k-1}}\right)+\frac{\partial V_{k1}}{\partial\zeta_k}\overline{f}_k \\ \le \kappa L_{k1}\left(z_k^2+z_k^{\frac{2\sigma}{s_kp_{k-1}\cdots p_1}}\right)+\frac{\kappa}{2}\sum_{i=1}^{k-1}\left(z_i^2+z_i^{\frac{2\sigma}{s_ip_{i-1}\cdots p_1}}\right) \\ +\frac{\kappa^{1-b}}{2}\sum_{i=1}^{k-1}\left(z_i(t-\tau)^2+z_i(t-\tau)^{\frac{2\sigma}{s_ip_{i-1}\cdots p_1}}\right) \\ +\kappa^{1-b}\left(z_k(t-\tau)^2+z_k^{\frac{2\sigma}{s_kp_{k-1}\cdots p_1}}(t-\tau)\right), \end{aligned} \tag{35}$$

$$\begin{aligned} \sum_{i=1}^{k-1}\frac{\partial V_{k1}}{\partial\zeta_i}(d_i\kappa\zeta_{i+1}^{p_i}+\overline{f}_i) \le \kappa L_{k2}\left(z_k^2+z_k^{\frac{2\sigma}{s_kp_{k-1}\cdots p_1}}\right)+\frac{\kappa}{2}\sum_{i=1}^{k-1}\left(z_i^2+z_i^{\frac{2\sigma}{s_ip_{i-1}\cdots p_1}}\right) \\ +\frac{\kappa^{1-b}}{2}\sum_{i=1}^{k-1}\left(z_i^2(t-\tau)+z_i(t-\tau)^{\frac{2\sigma}{s_ip_{i-1}\cdots p_1}}\right) \end{aligned} \tag{36}$$

where L_{k1} and L_{k2} are two positive constants.

Lemma 6 [23]. The Leibniz rule for differentiating an integral is

$$(\int_u^v f(s)ds)' = v'f(v) - u'f(u). \tag{37}$$

Lemma 7 [22]. Assume that c and d are two positive numbers and the function $\pi(x,y)>0$, we have:

$$|x|^c|y|^d \le \frac{c\,\pi(x,y)|x|^{c+d}}{c+d}+\frac{d\,\pi(x,y)^{-c/d}|y|^{c+d}}{c+d}. \tag{38}$$

3. Proposed approach

The proposed approach aims at implementing the CNF control paradigm for high-order nonholonomic systems with time-delays. The linear feedback input is designed to generate a quick dynamic response with small damping ratio, whereas the nonlinear feedback is designed to improve the damping ratio as system states approach the origin. Thus, it results in simultaneous improvement in both steady-state accuracy and transient performance.

In what follows, we proceed to design the control laws $u_0(t)$ and $u_1(t)$. First, the control law $u_0(t)$ is designed and used to analyze the stability of the state x_0. Then control law $u_1(t)$ is synthesized to ensure the asymptomatic stability and performance improvement of the other states in the presence of time delays and disturbances.

3.1. Design of the control law $u_0(t)$

The linear feedback controller $u_{0L}(t)$ and the nonlinear control law $u_{0N}(t)$ are defined as

$$u_{0L}(t) = -\beta_0 x_0(t), \tag{39}$$

$$u_{0N}(t) = \psi_0 \varphi_0 x_0(t), \tag{40}$$

where combining these two parts, one obtains

$$u_0(t) = u_{0L}(t) + u_{0N}(t) = -\beta_0 x_0(t) + \psi_0 \varphi_0 x_0(t)\,. \tag{41}$$

The nonlinear function ψ_0 in (40) is defined as

$$\psi_0 = -\frac{\rho^2}{d_0(t)\left(\rho\left|\varphi_0 x_0(t)\right| + \Omega_0(x_0(t))\right)}, \tag{42}$$

where ρ is upper bound of external disturbance ($\left|D_0(t)\right| \le \rho$), φ_0 is a positive scalar, and Ω_0 is a positive continuous (uniformly bounded) function with

$$\lim_{t\to\infty} \int \Omega_0(x_0(\tau))\, d\tau \le \bar{\Omega}_0 < \infty \tag{43}$$

where $\bar{\Omega}_0 > 0$. The selection process of $\Omega_0(x_0(t))$ will be discussed in the next subsection.

Now, considering $q_0 = 1$ in (1) and constructing the Lyapunov function as

$$V_0 = 0.5 P_0 x_0^2, \tag{44}$$

the time-derivative of V_0 is obtained as

$$\begin{aligned} \dot{V}_0 &= P_0 x_0 (d_0(t) u_0(t) + D_0) \\ &= -Q x_0^2 + 2 d_0(t) \psi_0 \varphi_0^2 x_0^2 + 2\varphi_0 x_0 D_0, \end{aligned} \tag{45}$$

where $Q = 2\beta_0 d_0(t) \varphi_0$. Then, using ρ as the upper bound of external disturbance, Eq. (45) can be written as

$$\dot{V}_0 \le -Q x_0^2 + 2\psi_0 d_0(t) \left(\varphi_0 x_0(t)\right)^2 + 2\rho \left|\varphi_0 x_0(t)\right|. \tag{46}$$

The value of Q is positive; hence, the first term in the last function is negative. In order to have a negative derivation of the Lyapunov function, the nonlinear function ψ_0 is chosen as (42).

3.2. Design of the control law $u_1(t)$

For designing the virtual controllers in the next section, the nonlinear functions ψ_1 and ψ_{k-1} are defined as

$$\psi_1 = -\frac{d_1(t)\kappa\varphi_1^{-1}\left(\left|\zeta_2\right| + \Im_1\bar{z}_1(t)\right)^2}{d_1(t)\kappa\left|\zeta_2\right|\left|\bar{z}_1(t)\right| + \kappa\Im_1 d_1(t)\bar{z}_1(t)^2 + \Omega_1} \tag{47}$$

$$\psi_{k-1} = -\frac{\Lambda}{\Pi} \tag{48}$$

with

$$\begin{aligned}\Lambda &= \left(d_{k-1}(t)\kappa\right)\left(\varphi_{k-1}\right)^{-\frac{1}{p_1\cdots p_{k-2}}}\bar{z}_{k-1}(t)^{-\frac{1}{p_1\cdots p_{k-2}}} \\ &\quad\times\left(z_{k-1}(t)^{2-\frac{1}{p_1\cdots p_{k-2}}} + z_{k-1}(t)^{\frac{\sigma}{s_{k-1}p_{k-2}\cdots p_1}}\right)\left(\left|\zeta_k^{p_{k-1}}\right| + \Im_{k-1}\bar{z}_{k-1}(t)^{\frac{1}{p_1\cdots p_{k-2}}}\right)^2\end{aligned} \tag{49}$$

$$\begin{aligned}\Pi &= \Omega_{k-1} + d_{k-1}(t)\kappa\Im_{k-1}\left(z_{k-1}(t)^{2-\frac{1}{p_1\cdots p_{k-2}}} + z_{k-1}(t)^{\frac{\sigma}{s_{k-1}p_{k-2}\cdots p_1}}\right)\bar{z}_{k-1}(t)^{\frac{1}{p_1\cdots p_{k-2}}} \\ &\quad + d_{k-1}(t)\kappa\left|\zeta_k^{p_{k-1}}\right|\left|z_{k-1}(t)^{2-\frac{1}{p_1\cdots p_{k-2}}} + z_{k-1}(t)^{\frac{\sigma}{s_{k-1}p_{k-2}\cdots p_1}}\right|\end{aligned} \tag{50}$$

where φ_1 and φ_{k-1} are positive values, Ω_1 and Ω_{k-1} are positive uniform continuous bounded functions and $\bar{z}_i(t) = z_i(t) + z_i(t)^{\frac{\sigma}{s_i}}$, $i = 1,\ldots,n$ which satisfy

$$\lim_{t\to\infty}\int\Omega_1(x_1(\tau))d\tau \le \bar{\Omega}_1 < \infty \tag{51}$$

$$\lim_{t\to\infty}\int\Omega_{k-1}(x_{k-1}(\tau))d\tau \le \bar{\Omega}_{k-1} < \infty \tag{52}$$

where $\bar{\Omega}_1$ and $\bar{\Omega}_{k-1}$ are two positive constants. The selection procedure of $\Omega_1(x_1(t))$ and $\Omega_{k-1}(x_{k-1}(t))$ will be discussed later.

Theorem 1: For the time-delay nonholonomic system (1), considering the CNF control law (41) and output-feedback controller as

$$u_1(t) = -\kappa^{l_n} \Im_n^{\frac{1}{p_1 \cdots p_n}} \left(\hat{z}_n(t) + \hat{z}_n(t)^{\frac{\sigma}{s_n}} \right)^{\frac{1}{p_1 \cdots p_n}} . \tag{53}$$

Then, using the CNF virtual controllers $\bar{\alpha}_i(t)$, the control laws $u_0(t)$ and $u_1(t)$ guarantee that the system states are bounded for any initial condition $[x_0(t_0), x_1(t_0), \ldots, x_n(t_0)] \in R^n$ and $\lim_{t \to +\infty} x_i(t) = 0, \; i = 0, \ldots, n$.

Proof: Assume $x_0(t_0) \neq 0$. The proofs of the stability analysis for the subsystems $x_0(t)$ and $x_1(t), \ldots, x_n(t)$ are presented in the following procedure:

Step I. Stability analysis and controller design for subsystem x_0

Substituting $q_0 = 1$ in (1), one obtains

$$\dot{x}_0(t) = D_0(t) + d_0(t)u_0(t) . \tag{54}$$

Construct the Lyapunov functional as

$$V_0(x_0(t)) = 0.5\varphi_0 x_0(t)^2 , \tag{55}$$

where differentiating (55) and using (54), one attains

$$\dot{V}_0(x_0(t)) = \varphi_0 x_0(t)\left(d_0(t)u_0(t) + D_0(t)\right) . \tag{56}$$

Substitution (41) into (56), one can achieve

$$\dot{V}_0(x_0(t)) = -Qx_0(t)^2 + \psi_0 d_0(t)\left(\varphi_0 x_0(t)\right)^2 + \varphi_0 x_0(t) D_0(t) . \tag{57}$$

where $Q = \beta_0 \varphi_0 d_0(t)$. From (57), one obtains

$$\dot{V}_0(x_0(t)) \leq -Qx_0(t)^2 + \psi_0 d_0(t)\left|\varphi_0 x_0(t)\right|^2 + \rho\left|\varphi_0 x_0(t)\right| \tag{58}$$

where substituting the nonlinear function (42) into (58), one has

$$\dot{V}_0(x_0(t)) \le -Qx_0(t)^2 + \frac{\rho\Omega_0(x_0(t))\left|\varphi_0 x_0(t)\right|}{\Omega_0(x_0(t)) + \rho\left|\varphi_0 x_0(t)\right|}. \tag{59}$$

Considering the fact that

$$0 \le \frac{\nu\Omega_0(x_0(t))}{\nu + \Omega_0(x_0(t))} \le \Omega_0(x_0(t)), \tag{60}$$

the last term of Eq. (59) is less than $\Omega_0(x_0(t))$, and hence one obtains

$$\dot{V}_0(x_0(t)) \le -Q\left|x_0(t)\right|^2 + \Omega_0(x_0(t)). \tag{61}$$

Besides, there exist positive coefficients β_1 and β_2 so that for every $t \ge t_0$ one gets

$$\beta_1\left|x_0(t)\right|^2 \le V_0(x_0(t)) \le \beta_2\left|x_0(t)\right|^2. \tag{62}$$

From (61) and (62), it follows that

$$\begin{aligned} 0 \le \beta_1\left|x_0(t)\right|^2 &\le V_0(x_0(t)) = V_0(x_0(t_0)) + \int_{t_0}^{t}\dot{V}_0(x_0(\tau))d\tau \\ &\le \beta_2\left|x_0(t_0)\right|^2 - \int_{t_0}^{t}Q\left|x_0(\tau)\right|^2 d\tau + 2\int_{t_0}^{t}\Omega_0(x_0(\tau))d\tau, \end{aligned} \tag{63}$$

Since $Q > 0$, one can obtain $-\int_{t_0}^{t}Q\left|x_0(\tau)\right|^2 d\tau < 0$; then, it follows from (63) that

$$0 \le \beta_1\left|x_0(t)\right|^2 \le \beta_2\left|x_0(t_0)\right|^2 + 2\int_{t_0}^{t}\Omega_0(x_0(\tau))d\tau. \tag{64}$$

Now, notice that for $t \ge t_0$, one obtains

$$\sup_{t\in[t_0,\infty)}\int_{t_0}^{t}\Omega_0(x_0(\tau))d\tau \le \bar{\Omega}_0 \tag{65}$$

where from (64) and (65), one attains

$$0 \le \beta_1 \left|x_0(t)\right|^2 \le \beta_2 \left|x_0(t_0)\right|^2 + 2\bar{\Omega}_0 . \tag{66}$$

Furthermore, taking the limit of the last term of (63) as time goes to infinity follows that

$$0 \le \beta_2 \left|x_0(t_0)\right|^2 - \lim_{t\to\infty} \int_{t_0}^{t} Q \left|x_0(\tau)\right|^2 d\tau + 2 \lim_{t\to\infty} \int_{t_0}^{t} \Omega_0(x_0(\tau)) d\tau \tag{67}$$

where from (65) and (67), one can obtain

$$\lim_{t\to\infty} \int Q \left|x_0(\tau)\right|^2 d\tau \le \beta_2 \left|x_0(t_0)\right|^2 + 2\bar{\Omega}_0 . \tag{68}$$

It is confirmed from (66) that $x_0(t)$ is (uniformly) bounded. Because $x_0(t)$ is a continuous signal, the term $Q\left|x_0(t)\right|^2$ in (68) is also (uniformly) continuous. Using Barbalat lemma [24] on (68) gives

$$\lim_{t\to\infty} Q \left|x_0(t)\right|^2 = 0 . \tag{69}$$

Since Q is positive, one obtains

$$\lim_{t\to\infty} \left|x_0(t)\right| = 0 . \tag{70}$$

Step II. Stability analysis and controller design for the x-subsystem

Since $x_0(t_0) \ne 0$, one can obtain from (41) that $u_0(t) \ne 0$ for any $t < \infty$. Then, using (4) and (5), the *x*-subsystem (1) is transformed into ζ -subsystem (13) and (20). In what follows, for simplification of the controller design, one can assume that $w = q / p$, where q and p are even and odd integers, respectively. First, a state-feedback control law is designed assuming that all the states are measurable; then, a suitable state-observer is constructed to design an output feedback controller.

The Lyapunov-Krasovskii functional is constructed as

$$V_1 = V_{11}(z_1(t)) + V_{12}(z_1(t), z_1(t-\tau)) \ , \tag{71}$$

the time-derivative of (71) fulfills

$$\dot{V}_1 = \dot{V}_{11}(z_1(t)) + \dot{V}_{12}(z_1(t), z_1(t-\tau)) \tag{72}$$

where using the time-derivatives of (24) and (25), it follows that

$$\begin{aligned} \dot{V}_1 = \dot{\zeta}_1 & \left(\left(\zeta_1(t)^{p_1} - \alpha_0(t)^{p_1} \right)^{2-\frac{1}{p_1}} + \left(\zeta_1(t)^{p_1} - \alpha_0(t)^{p_1} \right)^{\frac{2\sigma - s_2 p_1}{s_1}} \right) \\ & + \frac{\delta_1}{1-\gamma} \kappa^{1-b} \left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) - \frac{\delta_1}{1-\gamma} \kappa^{1-b} \left(1 - \dot{\tau}(t) \right) \left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right). \end{aligned} \tag{73}$$

From (13) and (21), one obtains

$$\begin{aligned} \dot{V}_1 = & \left(d_1(t) \kappa \zeta_2(t)^{p_1} + \overline{f}_1(t, x_0, \overline{\zeta}(t), \overline{\zeta}(t-\tau)) \right) \left(z_1(t)^{2-\frac{1}{p_1}} + z_1(t)^{\frac{2\sigma - s_2 p_1}{s_1}} \right) \\ & + \frac{\delta_1}{1-\gamma} \kappa^{1-b} \left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) - \frac{\delta_1}{1-\gamma} \kappa^{1-b} \left(1 - \dot{\tau}(t) \right) \left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right), \end{aligned} \tag{74}$$

where using Assumption 2, one gets

$$\begin{aligned} \dot{V}_1 \le & \left(d_1(t) \kappa \zeta_2(t)^{p_1} + \overline{f}_1(t, x_0, \overline{\zeta}(t), \overline{\zeta}(t-\tau)) \right) \left(z_1(t)^{2-\frac{1}{p_1}} + z_1(t)^{\frac{2\sigma - s_2 p_1}{s_1}} \right) \\ & + \frac{\delta_1}{1-\gamma} \kappa^{1-b} \left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) - \frac{\delta_1}{1-\gamma} \kappa^{1-b} \left(1 - \gamma \right) \left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right) \\ \le & \ d_1(t) \kappa \zeta_2(t)^{p_1} \left(z_1(t)^{2-\frac{1}{p_1}} + z_1(t)^{\frac{2\sigma - s_2 p_1}{s_1}} \right) \\ & + \left| \overline{f}_1(t, x_0, \overline{\zeta}(t), \overline{\zeta}(t-\tau)) \right| \left| z_1(t)^{2-\frac{1}{p_1}} + z_1(t)^{\frac{2\sigma - s_2 p_1}{s_1}} \right| \\ & + \frac{\delta_1}{1-\gamma} \kappa^{1-b} \left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) - \delta_1 \kappa^{1-b} \left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right). \end{aligned} \tag{75}$$

From Lemma 4, one deduces that

$$\begin{aligned}\dot{V}_1 &\le d_1(t)\kappa\zeta_2(t)^{p_1}\left(z_1(t)^{2-\frac{1}{p_1}} + z_1(t)^{\frac{2\sigma-s_2p_1}{s_1p_1}}\right) \\ &+ a\kappa^{1-b}\left(\left|z_1(t)\right|^{\frac{1}{p_1}} + \left|z_1(t)\right|^{\frac{s_1+w}{s_1p_1}} + \left|z_1(t-\tau)\right|^{\frac{1}{p_1}} + \left|z_1(t-\tau)\right|^{\frac{s_1+w}{s_1p_1}}\right)\left|z_1(t)^{2-\frac{1}{p_1}} + z_1(t)^{\frac{2\sigma-s_2p_1}{s_1p_1}}\right| \\ &+ \frac{\delta_1}{1-\gamma}\kappa^{1-b}\left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}}\right) - \delta_1\kappa^{1-b}\left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}}\right).\end{aligned} \tag{76}$$

According to Cauchy Lemma [25], one obtains

$$\left|z_1(t)^{2-\frac{1}{p_1}} + z_1(t)^{\frac{2\sigma-s_2p_1}{s_1p_1}}\right| \le \left(\left|z_1(t)\right|^{2-\frac{1}{p_1}} + \left|z_1(t)\right|^{\frac{2\sigma-s_2p_1}{s_1p_1}}\right) \tag{77}$$

where from (76) and (77), one obtains

$$\begin{aligned}\dot{V}_1 &\le d_1(t)\kappa\zeta_2(t)^{p_1}\left(z_1(t)^{2-\frac{1}{p_1}} + z_1(t)^{\frac{2\sigma-s_2p_1}{s_1p_1}}\right) \\ &+ \frac{\delta_1}{1-\gamma}\kappa^{1-b}\left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}}\right) - \delta_1\kappa^{1-b}\left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}}\right) \\ &+ a\kappa^{1-b}\left\{\left|z_1(t)\right|^2 + \left|z_1(t)\right|^{\frac{1}{p_1}+\frac{2\sigma-s_2p_1}{s_1p_1}} + \left|z_1(t)\right|^{2-\frac{1}{p_1}+\frac{s_1+w}{s_1p_1}} + \left|z_1(t)\right|^{\frac{s_1+w}{s_1p_1}+\frac{2\sigma-s_2p_1}{s_1p_1}} + \left|z_1(t)\right|^{2-\frac{1}{p_1}}\left|z_1(t-\tau)\right|^{\frac{1}{p_1}}\right. \\ &\left.+ \left|z_1(t)\right|^{\frac{2\sigma-s_2p_1}{s_1p_1}}\left|z_1(t-\tau)\right|^{\frac{1}{p_1}} + \left|z_1(t)\right|^{2-\frac{1}{p_1}}\left|z_1(t-\tau)\right|^{\frac{s_1+w}{s_1p_1}} + \left|z_1(t)\right|^{\frac{2\sigma-s_2p_1}{s_1p_1}}\left|z_1(t-\tau)\right|^{\frac{s_1+w}{s_1p_1}}\right\}.\end{aligned} \tag{78}$$

From (3), the following inequality is obtained for $i = 2$:

$$s_2 p_1 \le \sigma \tag{79}$$

where $\sigma = \max_{1\le i\le n}\{s_i + w\}$. It follows from (78) and (79) that

$$\dot{V}_1 \le d_1(t)\kappa\zeta_2(t)^{p_1}\left(z_1(t)^{2-\frac{1}{p_1}} + z_1(t)^{\frac{\sigma}{s_1 p_1}} \right)$$
$$+\frac{\delta_1}{1-\gamma}\kappa^{1-b}\left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) - \delta_1\kappa^{1-b}\left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right)$$
$$+a\kappa^{1-b}\left\{ \left|z_1(t)\right|^2 + \left|z_1(t)\right|^{\frac{1}{p_1}+\frac{\sigma}{s_1 p_1}} + \left|z_1(t)\right|^{2-\frac{1}{p_1}+\frac{\sigma}{s_1 p_1}} + \left|z_1(t)\right|^{\frac{2\sigma}{s_1 p_1}} + \left|z_1(t)\right|^{2-\frac{1}{p_1}}\left|z_1(t-\tau)\right|^{\frac{1}{p_1}} \right. \tag{80}$$
$$\left. + \left|z_1(t)\right|^{\frac{\sigma}{s_1 p_1}}\left|z_1(t-\tau)\right|^{\frac{1}{p_1}} + \left|z_1(t)\right|^{2-\frac{1}{p_1}}\left|z_1(t-\tau)\right|^{\frac{\sigma}{s_1 p_1}} + \left|z_1(t)\right|^{\frac{\sigma}{s_1 p_1}}\left|z_1(t-\tau)\right|^{\frac{\sigma}{s_1 p_1}} \right\}.$$

Now, from Lemma 7 and considering $\pi = p_1 = 1$, one attains

$$\dot{V}_1 \le d_1(t)\kappa\zeta_2(t)\left(z_1(t)^{\frac{\sigma}{s_1}} + z_1(t) \right) + \frac{\delta_1}{1-\gamma}\kappa^{1-b}\left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right)$$
$$-\delta_1\kappa^{1-b}\left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right) + a\kappa^{1-b}\left\{ 1.5\left|z_1(t)\right|^2 + 2\left|z_1(t)\right|^{1+\frac{\sigma}{s_1}} + 1.5\left|z_1(t)\right|^{\frac{2\sigma}{s_1}} \right. \tag{81}$$
$$\left. + \frac{\left|z_1(t-\tau)\right|^2}{2} + \frac{\frac{\sigma}{s_1}\left|z_1(t)\right|^{\frac{\sigma}{s_1}+1}}{\frac{\sigma}{s_1}+1} + \frac{\left|z_1(t-\tau)\right|^{\frac{\sigma}{s_1}+1}}{\frac{\sigma}{s_1}+1} + \frac{\left|z_1(t)\right|^{\frac{\sigma}{s_1}+1}}{\frac{\sigma}{s_1}+1} + \frac{\frac{\sigma}{s_1}\left|z_1(t-\tau)\right|^{\frac{\sigma}{s_1}+1}}{\frac{\sigma}{s_1}+1} + 0.5\left|z_1(t-\tau)\right|^{\frac{2\sigma}{s_1}} \right\}.$$

Eq. (81) can be simplified as

$$\dot{V}_1 \le \left(z_1(t) + z_1(t)^{\frac{\sigma}{s_1}} \right) d_1(t)\kappa\zeta_2(t)$$
$$+\left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right)\frac{\delta_1}{1-\gamma}\kappa^{1-b} - \delta_1\kappa^{1-b}\left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right) \tag{82}$$
$$+a\kappa^{1-b}\frac{3}{2}\left(\left|z_1(t)\right| + \left|z_1(t)\right|^{\frac{\sigma}{s_1}} \right)^2 + a\kappa^{1-b}\frac{1}{2}\left(\left|z_1(t-\tau)\right| + \left|z_1(t-\tau)\right|^{\frac{\sigma}{s_1}} \right)^2,$$

where using Lemma 1, one gets

$$\dot{V}_1 \le \left(z_1(t) + z_1(t)^{\frac{\sigma}{s_1}} \right) d_1(t)\kappa\zeta_2(t) \tag{83}$$
$$+\left(\frac{\delta_1}{1-\gamma}\kappa^{1-b} + 3a\kappa^{1-b} \right)\left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) - \left(\delta_1 - a\right)\kappa^{1-b}\left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right).$$

Eq. (83) can be rewritten with the virtual controller $\alpha_1(t)$ as

$$
\begin{aligned}
\dot{V}_1 \le & -n\kappa \left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) \\
& - (\delta_1 - a)\kappa^{1-b} \left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right) + d_1(t)\kappa \left(z_1(t) + z_1(t)^{\frac{2\sigma - s_1 - w}{s_1}} \right)(\zeta_2(t) - \alpha_1(t)),
\end{aligned}
\tag{84}
$$

with $\alpha_1(t)$ which is defined as

$$
\alpha_1(t) \le -\left(\frac{\delta_1 + 3a}{2d_1(t)(1-\gamma)} \kappa^{-b} + n \right) \overline{z}_1(t), \tag{85}
$$

where $\Im_1 = \dfrac{\delta_1 + 3a}{2d_1(t)(1-\gamma)} \kappa^{-b} + n$, and if the condition $\delta_1 \ge a$ is satisfied, then $\dot{V}_1 \le 0$ is obtained.

Similarly, for $k = 2, \ldots, n$, to proceed the controller design, there exist a candidate Lyapunov-Krasovskii function V_{k-1} as (23) and a virtual control law α_{k-1} with

$$
\begin{aligned}
\dot{V}_{k-1} \le & -(n-k+2)\kappa \sum_{i=1}^{k-1} \left(z_i(t)^2 + z_i(t)^{\frac{2\sigma}{s_i p_{i-1} \cdots p_1}} \right) - \kappa^{1-b} \sum_{i=1}^{k-1} (\delta_i - k + i) \left(z_i(t-\tau)^2 + z_i(t-\tau)^{\frac{2\sigma}{s_i p_{i-1} \cdots p_1}} \right) \\
& + d_{k-1}(t)\kappa \left(z_{k-1}(t)^{2 - \frac{1}{p_1 \cdots p_{k-2}}} + z_{k-1}(t)^{\frac{\sigma}{s_{k-1} p_{k-2} \cdots p_1}} \right) (\zeta_k(t)^{p_{k-1}} - \alpha_{k-1}(t)^{p_{k-1}})
\end{aligned}
\tag{86}
$$

where δ_i's are positive constants. Then, choosing $V_k = V_{k-1} + V_{k1} + V_{k2}$, the time-derivative of V_k is attained as

$$
\frac{dV_k}{dt} = \frac{dV_{k-1}}{dt} + \frac{dV_{k1}}{dt} + \frac{dV_{k2}}{dt}. \tag{87}
$$

Since V_{k1} is a functional of $\zeta_1, \ldots, \zeta_k$, it follows from (13) and (24) that

$$\begin{aligned}\frac{dV_{k1}}{dt} &= \frac{\partial V_{k1}}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial t} + \ldots + \frac{\partial V_{k1}}{\partial \zeta_{k-1}}\frac{\partial \zeta_{k-1}}{\partial t} + \frac{\partial V_{k1}}{\partial \zeta_k}\frac{\partial \zeta_k}{\partial t} \\ &= \frac{\partial V_{k1}}{\partial \zeta_1}\dot{\zeta}_1 + \ldots + \frac{\partial V_{k1}}{\partial \zeta_{k-1}}\dot{\zeta}_{k-1} + \frac{\partial V_{k1}}{\partial \zeta_k}\dot{\zeta}_k = \sum_{i=1}^{k}\frac{\partial V_{k1}}{\partial \zeta_i}\dot{\zeta}_i \\ &= \frac{\partial V_{k1}}{\partial \zeta_k}\bar{f}_k(.) + d_k(t)\kappa\frac{\partial V_{k1}}{\partial \zeta_k}\zeta_{k+1}^{p_k} + \sum_{i=1}^{k-1}\frac{\partial V_{k1}}{\partial \zeta_i}\left(d_i(t)\kappa\zeta_{i+1}^{p_i} + \bar{f}_i(.)\right).\end{aligned} \tag{88}$$

Now, using (25), (86)-(88) and Lemma 1, one obtains

$$\begin{aligned}\dot{V}_k \le& -(n-k+2)\kappa\sum_{i=1}^{k-1}\left(z_i(t)^2 + z_i(t)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) + d_{k-1}(t)\kappa\left(z_{k-1}(t)^{2-\frac{1}{p_1\cdots p_{k-2}}} + z_{k-1}(t)^{\frac{\sigma}{s_{k-1}p_{k-2}\cdots p_1}}\right)(\zeta_k(t)^{p_{k-1}} - \alpha_{k-1}(t)^{p_{k-1}}) \\ &- \kappa^{1-b}\sum_{i=1}^{k-1}(\delta_i - k + i)\left(z_i(t-\tau)^2 + z_i(t-\tau)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) + \frac{\delta_k}{1-\gamma}\kappa^{1-b}\left(z_k(t)^2 + z_k(t)^{\frac{2\sigma}{s_k p_{k-1}\cdots p_1}}\right) + \frac{\partial V_{k1}}{\partial \zeta_k}\bar{f}_k(.) \\ &- \delta_k\kappa^{1-b}\left(z_k(t-\tau)^2 + z_k(t-\tau)^{\frac{2\sigma}{s_k p_{k-1}\cdots p_1}}\right) + d_k(t)\kappa\frac{\partial V_{k1}}{\partial \zeta_k}\zeta_{k+1}(t)^{p_k} + \sum_{i=1}^{k-1}\frac{\partial V_{k1}}{\partial \zeta_i}\left(d_i(t)\kappa\zeta_{i+1}(t)^{p_i} + \bar{f}_i(.)\right).\end{aligned} \tag{89}$$

or equivalently

$$\begin{aligned}\dot{V}_k \le& -(n-k+2)\kappa\sum_{i=1}^{k-1}\left(z_i(t)^2 + z_i(t)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) \\ &+ d_{k-1}(t)\kappa\left(z_{k-1}(t)^{2-\frac{1}{p_1\cdots p_{k-2}}} + z_{k-1}(t)^{\frac{\sigma}{s_{k-1}p_{k-2}\cdots p_1}}\right)(\zeta_k(t)^{p_{k-1}} - \alpha_{k-1}(t)^{p_{k-1}}) \\ &- \kappa^{1-b}\sum_{i=1}^{k}(\delta_i - k + i)\left(z_i(t-\tau)^2 + z_i(t-\tau)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) + \frac{\delta_k}{1-\gamma}\kappa^{1-b}\left(z_k(t)^2 + z_k(t)^{\frac{2\sigma}{s_k p_{k-1}\cdots p_1}}\right) \\ &+ \frac{\partial V_{k1}}{\partial \zeta_k}\bar{f}_k(.) + d_k(t)\kappa\frac{\partial V_{k1}}{\partial \zeta_k}\zeta_{k+1}(t)^{p_k} + \sum_{i=1}^{k-1}\frac{\partial V_{k1}}{\partial \zeta_i}\left(d_i(t)\kappa\zeta_{i+1}(t)^{p_i} + \bar{f}_i(.)\right).\end{aligned} \tag{90}$$

Now, using Lemma 5, it follows that

$$\dot{V}_k \le -(n-k+2)\kappa\sum_{i=1}^{k-1}\left(z_i(t)^2 + z_i(t)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) + \frac{\delta_k}{1-\gamma}\kappa^{1-b}\left(z_k(t)^2 + z_k(t)^{\frac{2\sigma}{s_k p_{k-1}\cdots p_1}}\right)$$
$$-\kappa^{1-b}\sum_{i=1}^{k}(\delta_i - k + i)\left(z_i(t-\tau)^2 + z_i(t-\tau)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) + \kappa^{1-b}\sum_{i=1}^{k}\left(z_i(t-\tau)^2 + z_i(t-\tau)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) \tag{91}$$
$$+\kappa L_k\left(z_k(t)^2 + z_k(t)^{\frac{2\sigma}{s_k p_{k-1}\cdots p_1}}\right) + \kappa\sum_{i=1}^{k-1}\left(z_i(t)^2 + z_i(t)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) + d_k(t)\kappa\zeta_{k+1}(t)^{p_k}\frac{\partial V_{k1}}{\partial \zeta_k}$$
$$+ d_k(t)\kappa\alpha_k(t)^{p_k}\frac{\partial V_{k1}}{\partial \zeta_k} - d_k(t)\kappa\alpha_k(t)^{p_k}\frac{\partial V_{k1}}{\partial \zeta_k}$$

where $L_k = L_{k1} + L_{k2}$. Eq. (91) can be rewritten as

$$\dot{V}_k \le -(n-k+1)\kappa\sum_{i=1}^{k-1}\left(z_i(t)^2 + z_i(t)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) + \frac{\delta_k}{1-\gamma}\kappa^{1-b}\left(z_k(t)^2 + z_k(t)^{\frac{2\sigma}{s_k p_{k-1}\cdots p_1}}\right)$$
$$-\kappa^{1-b}\sum_{i=1}^{k}(\delta_i - k + i - 1)\left(z_i(t-\tau)^2 + z_i(t-\tau)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) \tag{92}$$
$$+\kappa L_k\left(z_k(t)^2 + z_k(t)^{\frac{2\sigma}{s_k p_{k-1}\cdots p_1}}\right) + d_k(t)\kappa\left(\zeta_{k+1}(t)^{p_k} - \alpha_k(t)^{p_k}\right)\frac{\partial V_{k1}}{\partial \zeta_k} + d_k(t)\kappa\alpha_k(t)^{p_k}\frac{\partial V_{k1}}{\partial \zeta_k}.$$

For obtaining the Lyapunov stability condition ($\dot{V}_k \le 0$), the virtual controller is deduced as

$$\alpha_k^{p_k} = -\left(d_k(t)\frac{\partial V_{k1}}{\partial \zeta_k}\right)^{-1}\left(n - k + L_k + 1 + \frac{\delta_k}{1-\gamma}\kappa^{-b}\right)\left(z_k(t)^2 + z_k(t)^{\frac{2\sigma}{s_k p_{k-1}\cdots p_1}}\right), \tag{93}$$

where substituting derivative of (24) (with respect to ζ_k) into (93), one obtains

$$\alpha_k^{p_k} = -\left(\frac{n-k+L_k+1}{d_k(t)} + \frac{\delta_k}{(1-\gamma)d_k(t)}\kappa^{-b}\right)\left(z_k(t)^{\frac{1}{p_{k-1}\cdots p_1}} + z_k(t)^{\frac{\sigma}{s_k p_{k-1}\cdots p_1}}\right). \tag{94}$$

From Eq. (94) and Lemma 1, one achieves

$$\alpha_k^{p_k} \geq -\left(\frac{n-k+L_k+1}{c_{k1}} + \frac{\delta_k}{(1-\gamma)c_{k1}}\kappa^{-b}\right)\left(z_k(t) + z_k(t)^{\frac{\sigma}{s_k}}\right)^{\frac{1}{p_{k-1}\cdots p_1}}, \tag{95}$$

or equivalently

$$\alpha_k \geq -\left(\frac{n-k+L_k+1}{c_{k1}} + \frac{\delta_k}{(1-\gamma)c_{k1}}\kappa^{-b}\right)^{\frac{1}{p_k}}\left(z_k(t)^{\frac{\sigma}{s_k}} + z_k(t)\right)^{\frac{1}{p_k\cdots p_1}}, \tag{96}$$

which results that substituting (96) into (92), one finds $\dot{V}_k \leq 0$.

$$\begin{aligned}\dot{V}_k \leq& -(n-k+1)\kappa\sum_{i=1}^{k}\left(z_i(t)^2 + z_i(t)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) + d_k(t)\kappa\left(z_k(t)^{2-\frac{1}{p_{k-1}\cdots p_1}} + z_i(t)^{\frac{2\sigma-w-s_k}{s_k}}\right)\left(\zeta_{k+1}(t)^{p_k} - \alpha_k(t)^{p_k}\right)\\ &-\kappa^{1-b}\sum_{i=1}^{k}(\delta_i - n - 1 + i)\left(z_i(t-\tau)^2 + z_i(t-\tau)^{\frac{2\sigma}{s_i p_{k-1}\cdots p_1}}\right)\end{aligned} \tag{97}$$

When k=n, the time-derivative of Lyapunov-Krasovskii functional is attained from (92) as

$$\begin{aligned}\dot{V}_n \leq& -\kappa\sum_{i=1}^{n}\left(z_i(t)^2 + z_i(t)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right) + d_n(t)\kappa\left(z_n(t)^{2-\frac{1}{p_1\cdots p_{n-1}}} + z_n(t)^{\frac{2\sigma-w-s_n}{s_n}}\right)\left(v_1(t)^{p_n} - \alpha_n^{p_n}\right)\\ &-\kappa^{1-b}\sum_{i=1}^{n}(\delta_i - n - 1 + i)\left(z_i(t-\tau)^2 + z_i(t-\tau)^{\frac{2\sigma}{s_i p_{i-1}\cdots p_1}}\right),\end{aligned} \tag{98}$$

where taking $v_1 = \alpha_n$, the Lyapunov stability condition $\dot{V}_n \leq 0$ is ensured.

Now, from (5) and (96), the designed output-feedback controller is obtained as (53) and all the state trajectories of the system are globally bounded, i.e. $\lim_{t\to+\infty} z(t) = 0$. Then, from (20) and (21), and considering $u_0(t) \neq 0$, one obtains: $\lim_{t\to+\infty} x(t) = 0$.

In what follows, the CNF-based virtual control function $\bar{\alpha}_1(t)$ is presented as

$$\overline{\alpha}_1(t) = -\Im_1 \overline{z}_1(t) - \psi_1 \varphi_1 \overline{z}_1(t), \tag{99}$$

where $\varphi_1 > 0$. Substituting (99) into (84) follows that

$$\begin{aligned} \dot{V}_1 \le & -n\kappa\left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) - (\delta_1 - a)\kappa^{1-b}\left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right) \\ & + \kappa d_1(t)\left|\zeta_2\right|\left|\overline{z}_1(t)\right| + \kappa\Im_1 d_1(t)\overline{z}_1(t)^2 + \kappa\psi_1\varphi_1 d_1(t)\overline{z}_1(t)^2. \end{aligned} \tag{100}$$

Substituting the nonlinear function (47) into (100) and considering (60), one obtains

$$\begin{aligned} \dot{V}_1 \le & -n\kappa\left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) - (\delta_1 - a)\kappa^{1-b}\left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right) \\ & + \frac{\left(d_1(t)\kappa\left|\zeta_2\right|\left|\overline{z}_1(t)\right| + \kappa\Im_1 d_1(t)\overline{z}_1(t)^2 \right)\Omega_1}{d_1(t)\kappa\left|\zeta_2\right|\left|\overline{z}_1(t)\right| + \kappa\Im_1 d_1(t)\overline{z}_1(t)^2 + \Omega_1} \\ \le & -n\kappa\left(z_1(t)^2 + z_1(t)^{\frac{2\sigma}{s_1}} \right) - (\delta_1 - a)\kappa^{1-b}\left(z_1(t-\tau)^2 + z_1(t-\tau)^{\frac{2\sigma}{s_1}} \right) + \Omega_1 \\ \le & -n\kappa\left|z_1(t)\right|^2 - (\delta_1 - a)^{1-b}\left|z_1(t-\tau)\right|^2 + \Omega_1. \end{aligned} \tag{101}$$

There exist positive constants r_{11}, r_{12}, g_{11}, g_{12} such that for every $t \ge t_0$, it follows:

$$r_{11}\left|z_1(t)\right|^2 \le V_{11} \le r_{12}\left|z_1(t)\right|^2, \tag{102}$$

$$g_{11}\left|z_1(t-\tau)\right|^2 \le V_{12} \le g_{12}\left|z_1(t-\tau)\right|^2, \tag{103}$$

where from (23), (102) and (103), one can obtain

$$r_{11}\left|z_1(t)\right|^2 + g_{11}\left|z_1(t-\tau)\right|^2 \le V_1 \le r_{12}\left|z_1(t)\right|^2 + g_{12}\left|z_1(t-\tau)\right|^2. \tag{104}$$

From (101) and (104), it follows that

$$0 \le r_{11}\left|z_1(t)\right|^2 + g_{11}\left|z_1(t-\tau)\right|^2 \le V_1 = V_1(z_1(t_0), z_1(t_0-\tau)) + \int_{t_0}^{t} \dot{V}_1(z_1(\iota), z_1(\iota-\tau))d\iota$$

$$\le r_{12}\left|z_1(t_0)\right|^2 + g_{12}\left|z_1(t_0-\tau)\right|^2 - \int_{t_0}^{t} n\kappa\left|z_1(\iota)\right|^2 d\iota \tag{105}$$

$$-\int_{t_0}^{t}(\delta_1 - a)\left|z_1(t-\iota)\right|^2 d\iota + \int_{t_0}^{t}\Omega_1(z_1(\iota), z_1(\iota-\tau))d\iota.$$

Because $\delta_1 - a$ and $n\kappa$ are both positive, then the expressions $-\int_{t_0}^{t} n\kappa\left|z_1(\iota)\right|^2 d\iota$ and $-\int_{t_0}^{t}(\delta_1 - a)\left|z_1(t-\iota)\right|^2 d\iota$ are negative; hence

$$0 \le r_{11}\left|z_1(t)\right|^2 + g_{11}\left|z_1(t-\tau)\right|^2 \le r_{12}\left|z_1(t_0)\right|^2 + g_{12}\left|z_1(t_0-\tau)\right|^2 + \int_{t_0}^{t}\Omega_1(z_1(\iota), z_1(\iota-\tau))d\iota\,. \tag{106}$$

Now, notice that for $t \ge t_0$, one gets

$$\sup_{t\in[t_0,\infty)} \int_{t_0}^{t}\Omega_1(z_1(\iota), z_1(\iota-\tau))d\iota \le \overline{\Omega}_1, \tag{107}$$

where from (106) and (107), one attains

$$0 \le r_{11}\left|z_1(t)\right|^2 + g_{11}\left|z_1(t-\tau)\right|^2 \le r_{12}\left|z_1(t_0)\right|^2 + g_{12}\left|z_1(t_0-\tau)\right|^2 + \overline{\Omega}_1\,. \tag{108}$$

On the other hand, taking the limit of the last term of (105) as time goes to infinity, one achieves

$$\begin{aligned} 0 \le\ & r_{12}\left|z_1(t_0)\right|^2 + g_{12}\left|z_1(t_0-\tau)\right|^2 - \lim_{t\to\infty}\int_{t_0}^{t} n\kappa\left|z_1(\iota)\right|^2 d\iota \\ & -\lim_{t\to\infty}\int_{t_0}^{t}(\delta_1 - a)\left|z_1(t-\iota)\right|^2 d\iota + \lim_{t\to\infty}\int_{t_0}^{t}\Omega_1(z_1(\iota), z_1(\iota-\tau))d\iota \end{aligned} \tag{109}$$

where from (107) and (109), one can find

$$(\delta_1 - a)\int_{t_0}^{t} \left| z_1(\iota - \tau) \right|^2 d\iota + n\kappa \int_{t_0}^{t} \left| z_1(\iota) \right|^2 \le r_{12} \left| z_1(t_0) \right|^2 + g_{12} \left| z_1(t_0 - \tau) \right|^2 + \bar{\Omega}_1 . \quad (110)$$

It can be deduced from (110) that $z_1(t)$ and $z_1(t-\tau)$ are uniformly bounded. Since $z_1(t)$ and $z_1(t-\tau)$ are continuous, the terms $n\kappa \int_{t_0}^{t} \left| z_1(\iota) \right|^2 d\iota$ and $(\delta_1 - a)\int_{t_0}^{t} \left| z_1(\iota - \tau) \right|^2 d\iota$ in (110) are also (uniformly) continuous. If Barbalat lemma is applied to (110), it gives

$$(\delta_1 - a)\lim_{t\to\infty} \left| z_1(\iota - \tau) \right|^2 + n\kappa \lim_{t\to\infty} \left| z_1(\iota) \right|^2 = 0 \quad (111)$$

where since $n\kappa$ and $\delta_1 - a$ are positive, it results that

$$\lim_{t\to\infty} \left| z_1(\iota) \right| = 0 , \quad (112)$$

$$\lim_{t\to\infty} \left| z_1(\iota - \tau) \right| = 0 . \quad (113)$$

The same procedure for the stability analysis can be considered for $z_i(t),\ z_i(t-\tau),\ i = 2, \ldots, n$. □

3.3. Procedure for selecting Ω

The main function of the nonlinear term $\Omega(x_i(t))$ is to accelerate the settling time and hence improve the system's speed of response. When the norms of the system states are small, a significant amount is contributed to the linear control signal. The selection procedure of a suitable nonlinear function $\Omega(x_i(t))$ is the central problem of the CNF control design. The function $\Omega(x_i(t))$ is required to be selected such that:

(1) Since $\Omega(x_i(t))$ acts over the absolute value of $x_i(t)$ and should satisfy (107), then it follows that $\Omega\left(x_i(t)\right) = \Omega\left(-x_i(t)\right) \ge 0$.

(2) When the states $[x_0(t), \ldots, x_n(t)]^T$ are far away from origin, $\Omega(x_i(t))$ will have high value such that the term $|\psi(x_i(t))|$ will become small, hence making the contribution of the nonlinear portion insignificant.

(3) When the states converge to zero, $\Omega(x_i(t))$ converges to a low value such that $|\psi(x_i(t))|$ will be large, thus increasing the significance of the nonlinear portion of the control design.

The nonlinear portion $\Omega(x_i(t))$ is not unique and one can define it in numerous procedures. In this article, the function $\Omega(x_i(t))$ is described in exponential form as

$$\Omega(x_i(t)) = \beta \exp\left(-\frac{\alpha}{|x_i(t)|}\right) + \varepsilon , \tag{114}$$

$$\varepsilon = \varepsilon_0 \exp(-\tau_0 t) , \tag{115}$$

where α , β , ε_0 and τ_0 are some positive tuning coefficients. The function $\Omega(x_i(t))$ reaches the maximum amount $(\beta + \varepsilon_0)$ when $|x_i(t)|$ increases and approaches to minimum amount (zero) while $|x_i(t)|$ converges to the origin.

The design procedure can be illustrated using the flowchart illustrated in Fig.1.

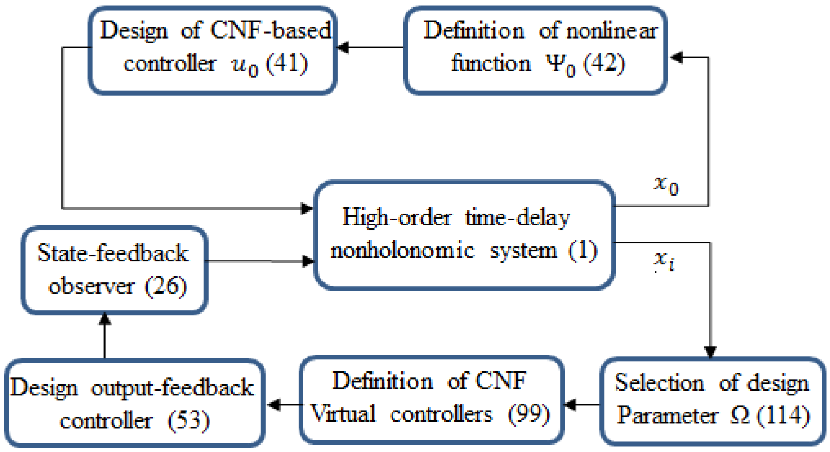

Fig.1. Flowchart of the proposed approach

4. Simulation results

Example. Consider the following nonholonomic system with time delays and nonlinear disturbances [1]:

$$\begin{aligned}
\dot{x}_0(t) &= (1+0.5\sin t)u_0(t),\\
\dot{x}_1(t) &= x_2(t)^3u_0(t) + x_0(t)^2x_1(t-\tau_1(t))^2 + \bar{d}_1(t,x(t),x(t-\tau)),\\
\dot{x}_2(t) &= u_1(t) + x_0(t)x_2(t-\tau_2(t))^2 + \bar{d}_2(t,x(t),x(t-\tau)),
\end{aligned} \tag{116}$$

where $[x_0, x_1, x_2]^T$ are system states; $\bar{d}_1(.)$ and $\bar{d}_2(.)$ are external disturbances; $u_0(t)$ and $u_1(t)$ are control inputs; $\tau_1(t)$ and $\tau_2(t)$ are time-varying delays. This example is simulated in MATLAB® Simulink® (2016b) and run on an Intel® Core i7- 6700K processor with 2 GB of memory.

The following coefficients were considered: $\beta_0 = 1$, $p_1 = 1$, $p_2 = 3$, $s_1 = 1$, $w = 1$, $s_2 = 1$, $m_1 = 1$, $\kappa = 2.1$, $\Im_1 = 2$, $\Im_2 = 3$, $n = 2$, $\rho = 3$, $\Omega_0 = 2$, $\varphi_0 = 1$, $\varphi_1 = 1$ and

$\Omega_1 = 2$. The initial states are taken as $x_0(0) = 0.5$, $x_1(0) = -0.3$, $x_2(0) = -0.7$ and $\hat{\chi}_2(0) = -1.2$. (41), (53) and (99), yield the following control inputs:

$$u_0(t) = -\left(1 + \frac{9}{(1 + 0.5\sin t)(3|x_0(t)| + 2)}\right)x_0(t), \tag{117}$$

$$u_1(t) = -(2.1 \times 2)^{\frac{1}{3}}\left(\hat{z}_2 + \hat{z}_2{}^2\right)^{\frac{1}{3}}, \tag{118}$$

with $\hat{z}_2 = (\hat{\chi}_2 + \zeta_1)(\hat{\chi}_2 + \zeta_1 + 1) + (\psi_1 + 2)(\hat{z}_1 + \hat{z}_1{}^2)$, where $\hat{z}_1 = \zeta_1$, $\dot{\hat{\chi}}_2(t) = -2.1\hat{\zeta}_2$, $\zeta_1(t) = \frac{x_1(t)}{u_0(t)}$, $\hat{\zeta}_2(t) = z_2(t) - 2\overline{z}_1(t) - \psi_1\overline{z}_1(t)$ and $\psi_1 = -\frac{\left(|\zeta_2| + 3\overline{z}_1(t)\right)^2}{|\zeta_2||\overline{z}_1(t)| + 3\overline{z}_1(t)^2 + 0.952}$.

Two cases were considered in assessing the performance of the proposed control framework. Additionally, a comparison analysis to the approach proposed in [1] was also carried over.

Case 1: Performance in the presence of time-varying time delays:

In this case, the system is subjected to the following time-varying time delays: $\tau_1(t) = \tau_2(t) = 0.5\sin t$. The time histories of the states obtained in this case are ilustrated in Fig. 2. Note that the propose approach is able to properly mitigate the effect of time delays. Note also that the proposed approach is able to steer the states to the origin faster than the approach proposed in [1]. The control inputs for both approaches are depicted in Fig. 3. Note that the proposed controller requires less control effort compared to the approach proposed in [1]. The observer's dynamics are highlighted in Fig. 4. As can be seen from Fig. 4, the proposed observer has faster low-frequency responses compared to the approach proposed in [1].

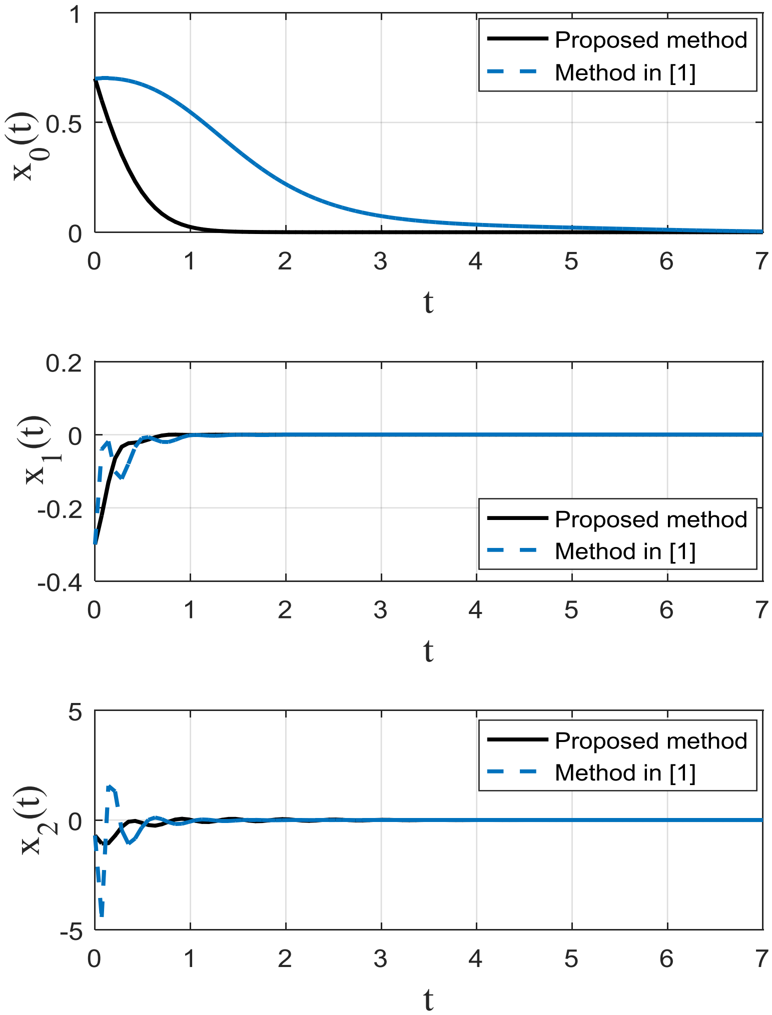

Fig. 2: Dynamics of the system's states.

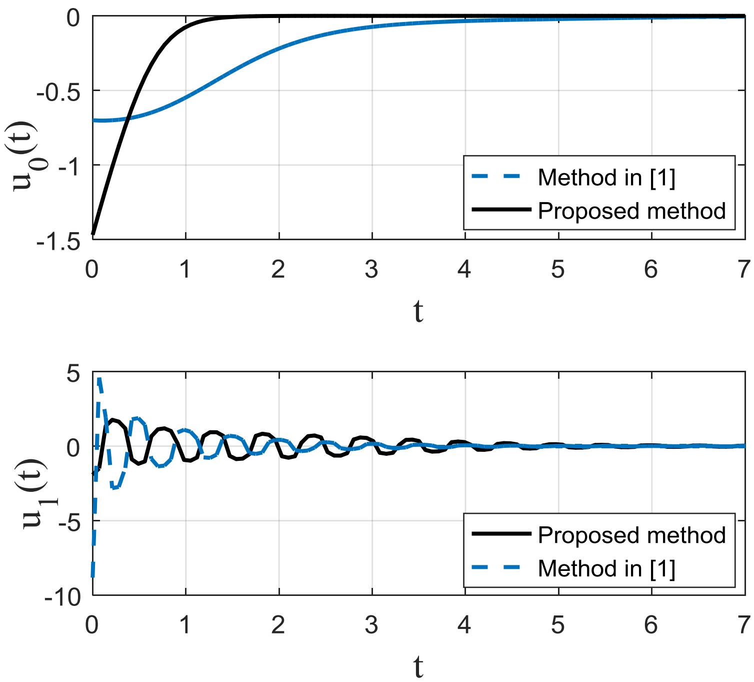

Fig. 3: Time histories of the control inputs.

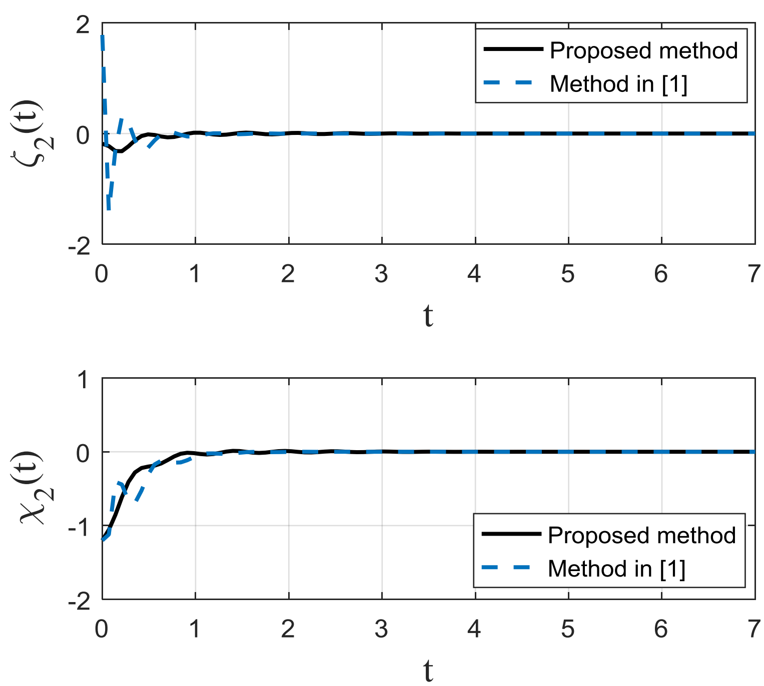

Fig. 4: Time trajectories of observer's variables.

To further assess the performance of both controller, we consider the Integral of Absolute-value of Error (IAE) as performance index:

$$I_1(\Xi) = \int_0^t |\Xi|\, dt \tag{119}$$

where Ξ is a signal in time domain. The obtained values of IAE (I_1) and settling time (T_s) are depicted in Table 1 for both the proposed approach and the one outlines in [1]. Note that the proposed approach yields smaller IAE and settling time values than the approach depicted in [1]. For instance, the improvements of IAE and settling time values for the state x_0 using the proposed controller are 73.2% and 72.46%, respectively; whereas, the improvements of IAE values for the controller signals u_0 and u_1 are 35.86% and 13.9%, respectively.

Table 1: Performance indices (IAE and settling time values)

	$I_1(x_0)$	$I_1(x_1)$	$I_1(x_2)$	$I_1(u_0)$	$I_1(u_1)$	$T_s(x_0)$	$T_s(x_1)$	$T_s(x_2)$
Method in [1]	1.2891	0.0559	0.8157	1.2891	3.2872	6.9 s	1.4 s	2.6 s
Proposed method	0.3457	0.0638	0.3504	0.8268	2.8302	1.9 s	0.8 s	1.4 s

Case 2: Performance in the simultaneous presence of time-varying time delays and nonlinear disturbances:

To further assess the robustness of the proposed approach, we subject the system to both time-varying delays and disturbances. Hence, we consider $\tau_1(t) = 2 - 2\sin t$, $\tau_2(t) = 1 + 0.5\cos t$, $\overline{d}_1 = 1.5 x_0(t)^2$, $\overline{d}_2 = 2.1\left(x_0(t) + x_1(t)\right)$ and $x_0(0) = 0.6$, $x_1(0) = -0.5$, $x_2(0) = -1.1$ and $\hat{\chi}_2(0) = -1.6$. The obtained dynamics for the states, control signals and observer variables, in this case, are depicted Fig. 5 through Fig. 7, respectively. Note from Fig. 5 that the system states controlled with the proposed approach exhibit lower overshoot and settling time than the approach highlighted in [1]. Fig. 6 shows that the proposed control inputs are faster than the controllers of [1]. Fig. 7 illustrates that the proposed observers have lower frequency responses and smaller settling time compared to the approach proposed in [1]. Hence, the proposed approach is faster and more effective at controlling nonholonomic systems with time delays and disturbances that the approach proposed in [1].

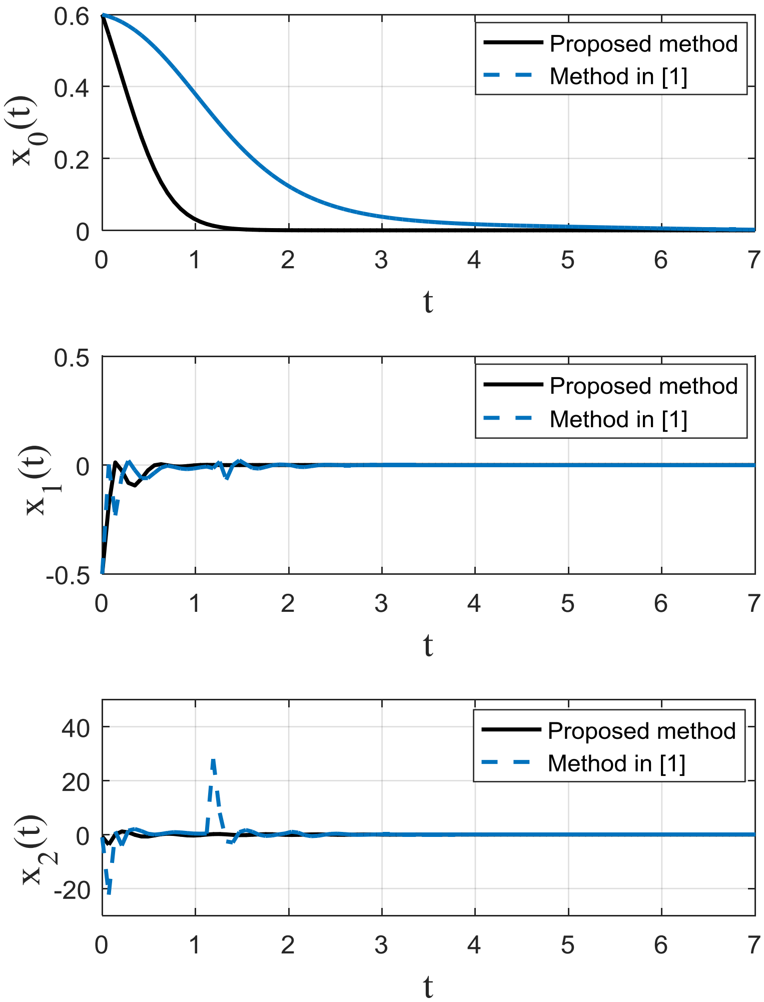

Fig. 5: Time responses of system states.

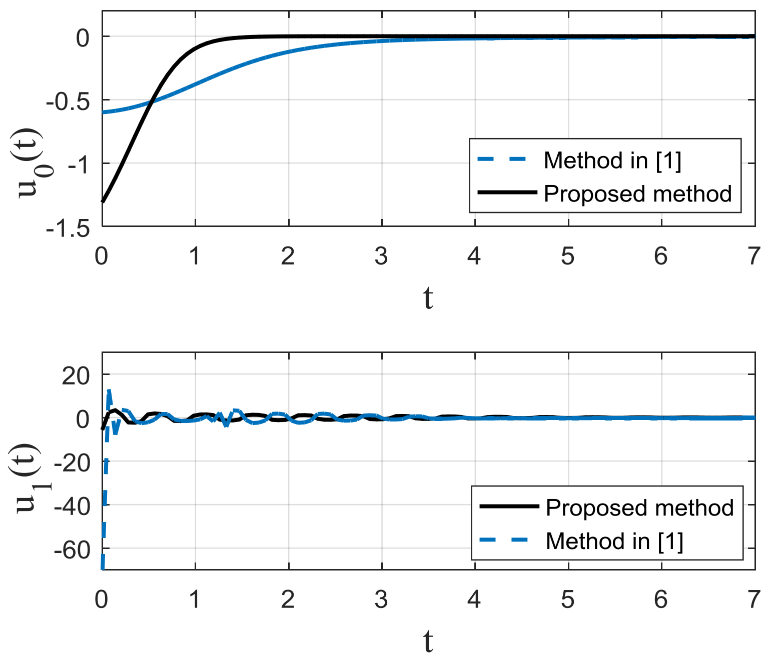

Fig. 6: Time responses of controller signals.

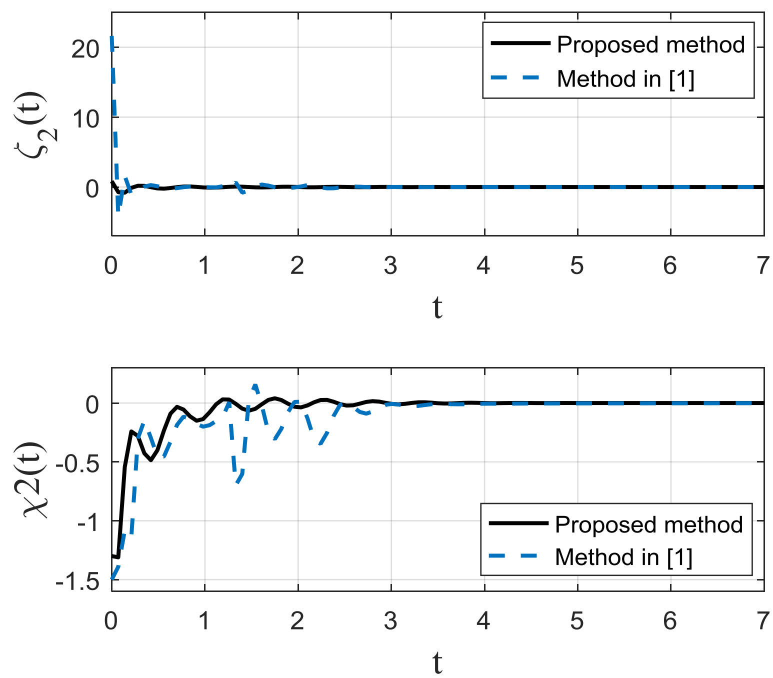

Fig. 7: Time trajectories of observer variables.

A comparison between the IAE and settling time values obtained using the proposed approach compared to those obtained using the approach in [1] are illustrated in Table 2.

Table 2: The values of performance indices

	$I_1(x_0)$	$I_1(x_1)$	$I_1(x_2)$	$I_1(u_0)$	$I_1(u_1)$	$T_s(x_0)$	$T_s(x_1)$	$T_s(x_2)$
Method in [1]	1.2891	0.0705	2.0008	1.2891	6.4690	7 *s*	2.5 *s*	5.3 *s*
Proposed approach	0.3457	0.0638	0.2987	0.8268	3.1612	1.2 *s*	0.8 *s*	1.2 *s*

Note the reduction in IAE and settling time values when using the proposed control approach compared to the method in [1]. More specifically, the improvements of IAE values for the x_0, x_1, x_2, u_0 and u_1 are 73.18%, 9.5%, 85.07%, 35.86% and 51.13%, respectively, wheras the improvements of settling times for the states x_0, x_1 and x_2 using the proposed controller are 82.85%, 68% and 77.35%, respectively.

It is concluded from these simulations and analytical results that the proposed control approach exhibits robust performance in the simultaneous presence of time-varying delays and external disturbances.

5. Conclusion

This paper proposed a control paradigm based on the composite nonlinear feedback control approach for nonholonomic systems with time-delays and external disturbances. To stabilize such systems and ensure the asymptotic convergence of the state trajectories to zero in the presence of external disturbances and time-varying time delays both linear and nonlinear feedback terms are synthesized. The linear term is designed to generate a quick dynamic response with small damping ratio, whereas the nonlinear feedback law is used to improve the damping ratio as system states approach the target reference. A robust stabilizer was synthesized to ensure the global asymptotic stability of nonholonomic systems by constructing a power-integrator-based Lyapunov–Krasovskii functional. Simulation results showed simultaneous improvement in both steady-state accuracy and transient performance. Comparison of IAE and settling times to the results of the method proposed in [1] showed also superior performance. Thus, the proposed approach is very effective at stabilizing highly complex nonholonomic and under-actuated systems. Extending the results to high-order nonholonomic systems with multiple time-varying input-delays will be the focus of our future work.

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