Logic for coalitions with bounded resources *

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Abstract

Recent work on Alternating-Time Temporal Logic and Coalition Logic has allowed the expression of many interesting properties of coalitions and strategies. However there is no natural way of expressing resource requirements in these logics. In this paper we present a Resource-Bounded Coalition Logic ($RBCL$) that has explicit representation of resource bounds in the language. We give a complete and sound axiomatisation of $RBCL$, a procedure for deciding satisfiability of $RBCL$ formulas, and a model-checking algorithm.

1 Introduction

Recent work on Alternating-Time Temporal Logic ATL and Coalition Logic CL, for example, [13, 10, 14, 7, 17], has allowed the expression of many interesting properties of coalitions and strategies. However, there is no natural way of expressing resource requirements in these logics. For example, there is no easy way to verify properties of the form 'a set of agents C can achieve a state of the world satisfying φ under the given resource bound b'. Essentially, this is the *successful coalition under resource bound* problem investigated by Wooldridge and Dunne in [18]. However, unlike Wooldridge and Dunne, we consider multi-shot games where the agents need to perform a *sequence* of actions to achieve the goal. As a motivating example, consider a system of distributed reasoning agents as described

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in [2, 6]. Agents' actions involve inferring new clauses from their knowledge bases and communicating derived clauses to other agents. Clearly these activities require resources such as time (which can be identified with the number of inference steps), memory (the space required to store premises in reasoning, and any intermediate lemmata, which can be measured as the number of clauses in the agents memory at any one time) and communication bandwidth (which can be measured as the number of communicated clauses). Properties of interest for such systems may include 'the set of reasoners C can derive the clause $[p]$ under the resource bound 10 for time, 3 for memory and 2 for communication'. In general, we would like to be able to express properties of systems where the abilities of individual agents and coalitions of agents are constrained by available resources in a non-trivial way.

In this paper we present a sound and complete logic, $RBCL$, in which we can express the costs of (multi-step) strategies and hence coalitional ability under resource bounds in multi-shot games. The logic is sufficiently expressive to formalise, e.g., the decision problems for Coalitional Resource Games discussed in [18] and the properties of resource-bounded communicating reasoners investigated in [2]. We show how to verify properties expressed in $RBCL$ and give a model-checking algorithm for $RBCL$. While there has been work on introducing resource bounds into temporal logic [9], we believe that our contribution presents a significant advance on this work. In [9], a logic RTL* is introduced, which is CTL* extended with quantifiers representing the cost of paths. However using CTL* as a starting point means that only single-agent systems can be analysed in RTL*. The setting of [9] is also different from the one considered in this paper, in that the actions not only consume but also produce resources. As a result, the model-checking problem for RTL* is quite complex, and only partial solutions (e.g. for RTL rather than RTL* where actions only consume resources) are presented in [9]. No axiomatisation of RTL* is given.

This paper is a revised and extended version of [4]. The main differences with respect to [4] are the decidability result for $RBCL$, the model-checking algorithm and proofs of all theorems. In [3] we introduced a logic CLRG for verifying properties of Coalitional Resource Games (with single step strategies) and gave a model-checking algorithm for CLRG, which is essentially a special case of the model-checking algorithm given in this paper.

The work presented in this paper is motivated by our research on epistemic logics for non-omniscient reasoners such as [6] and on belief ascription to non-omniscient reasoners such as [1, 5], which encouraged us to consider general issues of representing resources in logic. However the emphasis and the formal setting of those papers is very different to that considered here. In [6, 5] we consider a syntactic temporal epistemic logic based on CTL which contains an explicit counter modality $cp_i^{=n}$ which is true in a state if agent i has performed n communication actions in the past. The reasoners are assumed to have bounded memory and require time to perform inference steps, but the time and memory resources do not have corresponding counters in the language (time bounds are modelled using temporal operators and memory bounds have a corresponding axiom which says that the agent cannot have more than m different beliefs). In [1] we presented a dynamic logic for describing the observation-inference-action cycle of a non-omniscient agent; resources are not considered, and the emphasis is on correct belief ascription to the agent at different points in the cycle. The present work does not extend and is not technically related to our work in epistemic logic, apart from the latter providing a motivation for introducing resources explicitly in the logic.

The remainder of this paper is organised as follows. In the next section, we introduce a simple formalism based on Coalition Logic [14] extended with resource bounds, $RBCL_1$, which describes single-step strategies. We then motivate multi-step strategies in section 3 and introduce a more complex logic, $RBCL$, which can express multi-step strategies (similarly to the Extended Coalition Logic of [13]). We give a sound and complete axiomatisation of $RBCL$ in section 4 and show that the satisfiability problem for $RBCL$ is decidable in section 5. In section 6 we present a model-checking algorithm for $RBCL$. In section 7 we conclude and outline directions for future work.

2 Formalising single step strategies

We assume a set of agents $A = \{1, \ldots, n\}$ and a set of resources $R = \{1, \ldots, r\}$. Agents can perform actions from a set $\Sigma = \cup_{i \in A} \Sigma_i$, where Σ_i is the set of actions that can be performed by agent i. Each action $a \in \Sigma$ has an associated cost $Res(a)$, which is a vector of costs (assumed to be natural numbers) for each resource in R. A joint action executed by a coalition $C \subseteq A$ is a tuple of actions $a_C = (a_1, \ldots, a_k)$ (for simplicity, we assume, unless otherwise stated, that $C = \{1, \ldots, k\}$ for some $k \leq n$). For the moment, we stipulate that the cost of a joint action a_C is the vector sum of costs of actions in a_C (we generalise the way costs for different resources are combined in section 3). We compare vectors of resources using pointwise vector comparison $\leq$, e.g., for $b = (b_1, \ldots, b_r)$ and $d = (d_1, \ldots, d_r)$, $b \leq d$ iff for each $j \leq r$, $b_j \leq d_j$.

The language of $RBCL_1$ is defined relative to the sets A and R and a set of propositional variables $Prop$. Formulas of $RBCL_1$ are defined as follows:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid [C^b]\varphi$$

where $p \in Prop$, $C \subseteq A$, and $b \in \mathbb{N}^r$. The intuitive meaning of $[C^b]\varphi$ for $C \neq \emptyset$ is that coalition C can enforce the outcome ψ under resource bound b, or, in other words, the agents in C have a strategy costing at most b which enables them to achieve a φ-state no matter what the agents $\bar{C} = A \setminus C$ do. The modality corresponding to the empty coalition is a special case. $[\emptyset^b]\varphi$ means that if the grand coalition A executes any joint action which together costs at most b, then the system will end up in a φ state; that is, φ is unavoidable if A acts within resource bound b. We have chosen this 'non-standard' treatment of the empty coalition modality because it is suggested by the duality of Coalition Logic, $\neg[\emptyset]\varphi \leftrightarrow [A]\neg\varphi$ (we have $\neg[\emptyset^b]\varphi \leftrightarrow [A^b]\neg\varphi$). We do, however, lose other properties, such as, for example, monotonicity involving $\emptyset$: $[\emptyset^b]\varphi \rightarrow [C^b]\varphi$. An alternative would be to treat $\emptyset$ as a normal coalition. However, in this case since the cost of an empty set of actions is always 0, we would have $[\emptyset^b]\varphi \leftrightarrow [\emptyset^d]\varphi$ for any b and d, meaning simply that φ is inevitable in the next state (and we would not have the duality property).

We define models of $RBCL_1$ as transition systems, where in each state agents execute actions in parallel to determine the next state. These are essentially the same as the models for coalition logic with the addition of costs of actions. First we define *resource-bounded action frames* which underlie the models:

Definition 1. *A resource-bounded action (RBA) frame* F *is a tuple* $(A, R, \Sigma = \cup_{i \in A}\Sigma_i, S, T, o, Res)$ *where:*

A *is a non-empty set of agents,*

R *is a non-empty set of resources,*

Σ *is the set of actions agents can perform,*

S *is a non-empty set of states,*

$T : S \times A \rightarrow \wp(\Sigma_i)$ *assigns to each state the set of actions available to agent* i *in this state; this set is always non-empty as it contains an action* $noop$ *with* $Res(noop) = \bar{0} = (0, \ldots, 0)$,

o *is the outcome function which takes a state* s *and a joint action* a_A *and returns the state resulting from the execution of* a_A *by the agents in* s[1]

$Res : \Sigma \rightarrow \mathbb{N}^r$ *is the resource requirement function.*

In the case of joint actions, we generalise the function T as follows: a joint action $a_C \in T(s, C)$ iff $a_i \in T(s, i)$ for all $i \in C$. By $Res(a_C)$ we denote the vector sum of $Res(a_i)$ for $i \in C$.

[1]In order not to impose too many restrictions on the models, we do not require that $o(s, (noop, \ldots, noop)) = s$. If this requirement is added, we need an extra axiom schema $\phi \leftrightarrow [A^{\bar{0}}]\phi$.

Definition 2. *A single-step resource-bounded action (RBA) model M is a pair (F, V) where F is an RBA frame and $V : Prop \rightarrow \wp(S)$ is an assignment function.*

The truth definition for single-step RBA models is as follows:

- $M, s \models p$ *iff* $s \in V(p)$
- $M, s \models \neg\varphi$ *iff* $M, s \not\models \varphi$
- $M, s \models \varphi \wedge \psi$ *iff* $M, s \models \varphi$ *and* $M, s \models \psi$
- $M, s \models [C^b]\varphi$ *for* $C \neq \emptyset$ *iff there is* $a_C \in T(s, C)$ *with* $Res(a_C) \leq b$ *such that for every joint action* $a_{\bar{C}} \in T(s, \bar{C})$ *by the agents not in* C*, the outcome of the resulting tuple of actions executed in* s *satisfies* φ*:* $M, o(s, (a_C, a_{\bar{C}})) \models \varphi$
- $M, s \models [\emptyset^b]\varphi$ *iff the outcome of any joint action* $a_A \in T(s, A)$ *with* $Res(a_A) \leq b$ *executed in* s *satisfies* φ*:* $M, o(s, a_A) \models \varphi$.

The notions of satisfiability and validity are standard. Let us call the set of all formulas valid in single-step RBA models $RBCL_1$ (where 1 refers to considering only one-step strategies, as in Coalition Logic).

Theorem 1. *$RBCL_1$ is completely axiomatised by the following set of axiom schemas and inference rules:*

A0 *All propositional tautologies*

A1 $[C^b]\top$

A2 $\neg[C^b]\bot$

A3 $\neg[\emptyset^b]\varphi \leftrightarrow [A^b]\neg\varphi$

A4 $[C^b](\varphi \wedge \psi) \rightarrow [C^b]\varphi$

A5 $[C^b]\varphi \rightarrow [C^d]\varphi$ *where* $d \geq b$ *if* $C \neq \emptyset$ *or* $d \leq b$ *if* $C = \emptyset$

A6a $[C^b]\varphi \wedge [D^d]\psi \rightarrow [(C \cup D)^{b+d}](\varphi \wedge \psi)$ *where* C *and* D *are both disjoint and non-empty*

A6b $[\emptyset^b]\varphi \wedge [C^b]\psi \rightarrow [C^b](\varphi \wedge \psi)$ *where* C *is either* $\emptyset$ *or* A

MP $\vdash \varphi, \ \vdash \varphi \rightarrow \psi \Rightarrow \vdash \psi$

Equivalence $\vdash \varphi \leftrightarrow \psi \Rightarrow \vdash [C^b]\varphi \leftrightarrow [C^b]\psi$

The notions of derivability and consistency are standard. Note that if we erase the resource superscript in the axiomatisation above, we get the complete axiomatisation of Coalition Logic as given in [14], and a trivial formula resulting from **A5**. The rule of monotonicity (**RM**) is derivable as in Coalition Logic, that is, if $\vdash \varphi \to \psi$, then $\vdash [C^b]\varphi \to [C^b]\psi$.

We omit the completeness proof here as it is a special case of the completeness proof of $RBCL$ given below.

2.1 Example

As an illustration, we show how to express some properties of coalitional resource games from [18] in $RBCL_1$.

A coalitional resource game (CRG) Γ is defined as a tuple $(A, G, R, G_1, \ldots, G_n, en, req)$ where

- $A = \{1, \ldots, n\}$ is a set of agents,
- $G = \{g_1, \ldots, g_m\}$ is a set of goals,
- $R = \{r_1, \ldots, r_t\}$ is a set of resources,
- $G_i \subseteq G$ is the set of goals for agent i,
- $en : A \times R \to \mathbb{N}$ is the resource endowment function (how many units of a given resource is allocated to an agent),
- $req : G \times R \to \mathbb{N}$ is the resource requirement function (how many units of a particular resource is required to achieve a goal). It is assumed that each goal requires a non-zero amount of at least one resource.

In CRGs, the endowment of a coalition is equal to the sum of the endowments of its members: $en(C, r) = \Sigma_{i \in C} en(i, r)$.

As an example, we give a simple CRG from [18], where $A = \{1, 2, 3\}$; $G = \{g_1, g_2\}$; $R = \{r_1, r_2\}$; $G_1 = \{g_1\}$, $G_2 = \{g_2\}$, $G_3 = \{g_1, g_2\}$; $en(1, r_1) = 2$, $en(1, r_2) = 0$, $en(2, r_1) = 0$, $en(2, r_2) = 1$, $en(3, r_1) = 1$, $en(3, r_2) = 2$; $req(g_1, r_1) = 3$, $req(g_1, r_2) = 2$, $req(g_2, r_1) = 2$, and $req(g_2, r_2) = 1$. In $RBCL_1$, we can state properties such as the coalition of agents 1 and 3 can achieve g_1 under the resource bound corresponding to the sum of their endowments as given in the example: $[1, 3^{\langle 3,2 \rangle}]g_1$. More generally, a decision problem which is called *coalition C is successful under resource bound b* in [18] can be expressed as

$$[C^b] \bigwedge_{i \in C} \bigvee_{g \in G_i} g.$$

3 Formalising multi-step strategies and arbitrary resource combinators

In this section, we generalise the logic described above in two ways. First, we consider multi-step strategies, as in Extended Coalition Logic with the $[C^*]$ operator [13], or in ATL. The reason for this is that we are interested in the resource requirements of strategies which involve multiple steps. For example, suppose a coalition C can enforce φ in three steps: $[C^{b_1}][C^{b_2}][C^{b_3}]\varphi$. We can deduce from this that the agents have a strategy to achieve φ which costs at most $b_1 + b_2 + b_3$. However expressing the fact in this way is rather clumsy. Even worse, to say that 'C has some strategy which achieves φ in three steps which costs at most b' in $RBCL_1$, we have to use a disjunction over all possible vectors of natural numbers b_1, b_2, b_3 which sum up to b: $\vee_{b_1+b_2+b_3=b}[C^{b_1}][C^{b_2}][C^{b_3}]\varphi$. We therefore change the truth definition of $[C^b]$ to allow us to directly express the existence of a multi-step strategy with cost b.

The second generalisation involves the way in which we calculate the resource requirements of complex actions. We argue that not all resource costs should be combined using simple addition. For example, if one of the resources is time and the agents execute their actions concurrently, then, if each individual action costs one unit of time, the parallel combination of those actions also costs one unit of time. If one of the resources is memory, one can argue that if action a_1 requires n units of memory and action a_2 requires m units of memory, then executing actions a_1 and a_2 sequentially requires $max(n, m)$ units of memory. For generality, we introduce two cost combinators to express how resource requirements are combined in parallel and in sequence. If two actions a_1 and a_2 are performed in parallel, then the cost of executing them is $Res(a_1) \oplus Res(a_2)$ and the cost of executing them sequentially is $Res(a_1) \otimes Res(a_2)$, where $\oplus$ and $\otimes$ may be sum for some resources, and max or some other combinator for others.

In the rest of the paper, we assume that the set of resources R always includes time, that every action costs exactly one unit of time, and that the time cost is the last component of every cost vector. The cost of the $noop$ action is redefined as $(0, \ldots, 0, 1)$. We denote by $t(b)$ the time component of cost vector b. In particular, $t(Res(a)) = 1$ for any $a \in \Sigma$. We realise that this is a significant restriction on the logic, but we rely on it in an essential way in many of the proofs of the technical results (intuitively, it enables us to do induction on resource bounds).

In accordance with the intuitions above, $t(b_1 \otimes b_2) = t(b_1) + t(b_2)$ and $t(b_1 \oplus b_2) = max(t(b_1), t(b_2))$. In the language, only operators $[C^b]$ with $t(b) \geq 1$ are allowed.

3.1 Strategies and multi-step RBA models

Given an RBA frame $F = (A, R, \Sigma, S, T, o, Res)$, a *strategy* for an agent $i \in A$ is a function $f_i : S^+ \to \Sigma_i$ from finite non-empty sequences of states to actions, such that $f_i(\lambda s) = a \in T(s, i)$, where λs is a sequence of states ending in state s. Intuitively, f_i says what action agent i should perform in state s given the previous history of the system. A strategy for a coalition C is a set $F_C = \{f_1, \ldots, f_k\}$ of strategies for each agent.

For a sequence $\lambda = s_0 s_1 \ldots \in S^\omega$, we denote $\lambda[i] = s_i$ and $\lambda[i, j] = s_i \ldots s_j$. The set of possible computations generated by a strategy F_C from a state s_0, $out(s_0, F_C)$, is

$$\{\lambda \mid \lambda[0] = s_0 \ \wedge \ \forall j \geq 0 : \lambda[j+1] \in o^*(\lambda[j], (f_i(\lambda[0, j]))_{i \in C})\}$$

where $o^*(s, a_C) = \{o(s, (a_C, a_{\bar{C}})) \mid a_{\bar{C}} \in T(s, \bar{C})\}$. Now we define the cost of a multi-step strategy. Let $\lambda \in out(s_0, F_C)$. The cost of F_C over a prefix $\lambda[0, m]$ where $m > 0$ is defined inductively as follows:

$cost(\lambda[0, 1], F_C) = \oplus_{i \in C} Res(f_i(\lambda[0]))$, where $Res(f_i(\lambda[0]))$ is the cost of the action of agent i in $\lambda[0]$, and $\oplus_{i \in C}$ is the operator for combining the costs of actions executed in parallel by the agents in C;

$cost(\lambda[0, m], F_C) = cost(\lambda[0, m-1], F_C) \otimes (\oplus_{i \in C} Res\ (f_i(\lambda[0, m-1])))$ for $m > 1$; this is the cost of the previous $m - 1$ steps in the strategy combined sequentially with the cost of the mth step.

Definition 3. *A multi-step resource-bounded action model M is a pair (F, V) where F is an RBA frame, and $V : Prop \to \wp(S)$ is an assignment function, and the truth definition for the $[C^b]$ modality is*

- $M, s \models [C^b]\varphi$ *for* $C \neq \emptyset$ *iff there is a strategy* F_C *such that for all* $\lambda \in out(s, F_C)$*, there exists* $m > 0$ *such that* $cost(\lambda[0, m], F_C) \leq b$ *and* $M, \lambda[m] \models \varphi$*,*
- $M, s \models [\emptyset^b]\varphi$ *iff for all strategies* F_A*, computations* $\lambda \in out(s, F_A)$*, and* $m > 0$ *such that* $cost(\lambda[0, m], F_A) \leq b$*,* $M, \lambda[m] \models \varphi$*.*

Note that under this definition, the meaning of $[C^b]\varphi$ (for non-empty C) becomes as follows: C has a multi-step strategy to bring about C, and the cost of this strategy is less than b. The meaning of $[\emptyset^b]\varphi$ is that the outcome of *any* strategy of the grand coalition A which costs less than b, satisfies φ.

The set of all formulas valid in multi-step RBA models will be denoted by $RBCL$.

3.2 Example

As an illustration, we show how we can express properties of coalitions of resource-bounded reasoners from [2].

In [2], a temporal epistemic logic for resource-bounded reasoners was presented. The states in the models of the logic correspond to tuples of local states of the agents - intuitively the sets of formulas the agents believe or the contents of their memories. Belief operators B_i for each agent i are interpreted syntactically, that is $B_i\phi$ if and only if the formula ϕ is in agent i's memory. Transitions between states correspond to each agent performing, in parallel, one of the following actions: applying a rule of inference (resolution) to two formulas in its memory, 'reading' a formula from the agent's knowledge base into the agent's memory, or 'copying' a formula from another agent's memory (the latter is a very simple model of communication). The resources of interest are: time (the number of steps the system performed), memory (the maximal number of formulas that must be held simultaneously in the agent's local state) and communication (the number of 'copy' actions).

For example, the derivation below can be modelled as a transition system consisting of two agents, 1 and 2, where the agents' knowledge bases contains all formulas of the form $\sim A_1 \vee \sim A_2$ (where $\sim A_i$ is either $\neg A_i$ or A_i), and in the initial state the memory of both agents is empty. The first transition consists of agent 1 executing the action **Read**$(A_1 \vee A_2)$ and agent 2 executing the action **Read**$(A_1 \vee \neg A_2)$. This transition leads to a state where agent 1's memory contains $(A_1 \vee A_2)$ and agent 2's memory contains $(A_1 \vee \neg A_2)$, etc.:

	Agent 1		Agent 2	
#	Configuration	Op.	Configuration	Op.
1	$\{\}$		$\{\}$	
2	$\{A_1 \vee A_2\}$	**Read**	$\{A_1 \vee \neg A_2\}$	**Read**
3	$\{A_1 \vee A_2, \neg A_1 \vee A_2\}$	**Read**	$\{\neg A_1 \vee \neg A_2, A_1 \vee \neg A_2\}$	**Read**
4	$\{A_1 \vee A_2, A_2\}$	**Infer**	$\{\neg A_2, A_1 \vee \neg A_2\}$	**Infer**
5	$\{A_1 \vee \neg A_2, A_2\}$	**Read**	$\{\neg A_2, A_2\}$	**Copy**
6	$\{A_1, A_2\}$	**Infer**	$\{\{\}, A_2\}$	**Infer**

Figure 1: Example derivation using resolution with two agents

The logic defined in [2] did not have any way of expressing coalitional abilities of agents. However in $RBCL$ we can express that, for example, reasoners 1 and 2 can derive an empty clause within resource bounds 4 for memory, 1 for communication, and 5 for time: $[\{1,2\}^{(4,1,5)}]\ B_2\bot$ (where $B_2\bot$ means that $\bot$ is in agent 2's configuration.

3.3 Effectivity structures

To prove completeness of $RBCL$ it is easier to work with an alternative semantics, given not in terms of multi-step RBA models, but in terms of effectivity structures. These are closely related to RBA models, and we will show that effectivity structures satisfying some natural properties give rise to an alternative semantics for $RBCL$.

Let $\wp(A)^{\mathbb{B}} = \{C^b \mid C \subseteq A,\ b \in \mathbb{N}^r,\ t(b) \geq 1\}$. Intuitively, this is the set of all possible coalitions with all possible resource allocations. An *effectivity structure* (for a set of states S and a set of agents A) is a function $E : S \rightarrow (\wp(A)^{\mathbb{B}} \rightarrow \wp(\wp(S)))$ which describes, for each state in S, which properties of the world (corresponding to subsets of S) a coalition C can enforce under resource bound b.

Given an RBA frame F, *the effectivity structure corresponding to F* is defined as follows:

- for any $C \neq \emptyset$ and $X \subseteq S$, $X \in E(s)(C^b)$ iff there exists a strategy F_C such that for all $\lambda \in out(s, F_C)$, there exists $m > 0$ such that $cost(\lambda[0,m], F_C) \leq b$ and $\lambda[m] \in X$;
- $X \in E(s)(\emptyset^b)$ iff for all strategies F_A, sequences of states $\lambda \in out(s, F_A)$, and $m > 0$ such that $cost(\lambda[0,m], F_A) \leq b$, we have $\lambda[m] \in X$.

In other words, $X \in E(s)(C^b)$, where C is not the empty coalition, means that the coalition C has a strategy to bring about X within the bound b. $X \in E(s)(\emptyset^b)$ means that all strategies for the grand coalition which cost less b always result in a state in X, i.e., X is inevitable.

3.4 Example

In this section, we illustrate how effectivity structures are connected to action frames. The same example will be used in section 3.6 to show how an effectivity structure satisfying certain conditions gives rise to an action frame. Consider the following RBA frame $F = (A, R, \Sigma, S, T, o, Res)$, where the set of agents $A = \{1, 2, 3\}$, R consists of two resources (the second of which is time), the set of actions $\Sigma = \{a_1, a_2\}$ where a_1 is the $noop$ action, $S = \{s_1, s_2\}$, T and o are shown in Figure 2 and resource requirements of actions are $Res(a_1) = (0, 1)$ and $Res(a_2) = (1, 1)$:

Intuitively, in s_1, if agents 1 and 2 perform the same action, then the system stays in s_1, and if they perform different actions, the system will move to state s_2.

The corresponding effectivity structure E is as follows (where b is any resource bound which is at least equal to $(2, 1)$, and C and d are any coalition and any

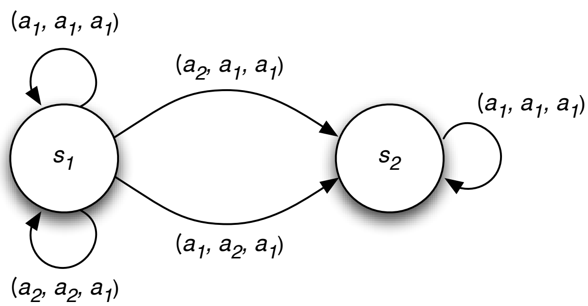

Figure 2: Example action frame F

resource bound):

$$
\begin{array}{rcl}
E(s_1)(\emptyset^{(0,1)}) & = & \{\{s_1\}, S\} \\
E(s_1)(\emptyset^{(1,1)}) = E(s_1)(\emptyset^b) & = & \{S\} \\
E(s_1)(\{1\}^{(0,1)}) = E(s_1)(\{1\}^{(1,1)}) = E(s_1)(\{1\}^b) & = & \{S\} \\
E(s_1)(\{2\}^{(0,1)}) = E(s_1)(\{2\}^{(1,1)}) = E(s_1)(\{2\}^b) & = & \{S\} \\
E(s_1)(\{3\}^{(0,1)}) = E(s_1)(\{3\}^{(1,1)}) = E(s_1)(\{3\}^b) & = & \{S\} \\
E(s_1)(\{1,2\}^{(0,1)}) & = & \{\{s_1\}, S\} \\
E(s_1)(\{1,2\}^{(1,1)}) = E(s_1)(\{1,2\}^b) & = & \{\{s_1\}, \{s_2\}, S\} \\
E(s_1)(\{1,3\}^{(0,1)}) = E(s_1)(\{1,3\}^{(1,1)}) = E(s_1)(\{1,3\}^b) & = & \{S\} \\
E(s_1)(\{2,3\}^{(0,1)}) = E(s_1)(\{2,3\}^{(1,1)}) = E(s_1)(\{2,3\}^b) & = & \{S\} \\
E(s_1)(\{1,2,3\}^{(0,1)}) & = & \{\{s_1\}, S\} \\
E(s_1)(\{1,2,3\}^{(1,1)}) = E(s_1)(\{1,2,3\}^b) & = & \{\{s_1\}, \{s_2\}, S\} \\
E(s_2)(C^d) & = & \{\{s_2\}, S\}
\end{array}
$$

3.5 Characterising effectivity in RBA frames

Every RBA frame gives rise to an effectivity structure, but the reverse does not hold. In this section, we characterise properties which an effectivity structure should satisfy to be an effectivity structure corresponding to an RBA frame. Following Pauly in [14], who introduced the term ‘playable’ for effectivity structures in Coalition Logic, we call such effectivity structures RB-playable, where RB stands for resource-bounded.

Below we state some useful properties of RB-playable effectivity structures. These are very similar - with the exception of the resource bound - to the properties of playable effectivity structures listed in [14], and are given the same names:

- An effectivity structure E is *outcome monotonic* iff

$$X \in E(s)(C^b) \Rightarrow X' \in E(s)(C^b) \text{ for all } X' \supseteq X$$

- An effectivity structure E is *coalition monotonic* iff

$$X \in E(s)(C^b) \Rightarrow X \in E(s)(D^b)$$

  where $C \neq \emptyset$ and $C \subseteq D$; and

$$X \in E(s)(\emptyset^b) \Rightarrow X \in E(s)(A^b)$$

- An effectivity structure E is *A-maximal* iff

$$X \notin E(s)(\emptyset^b) \Rightarrow \overline{X} \in E(s)(A^b)$$

- An effectivity structure E is *A-minimal* iff

$$X \in E(s)(A^b) \wedge Y \notin E(s)(A^b) \Rightarrow X \setminus Y \in E(s)(A^b)$$

  Note that *A-minimality* is not listed in [14], but its analogue is derivable.

- An effectivity structure E is *regular* iff for all coalitions C which are neither empty nor equal to A

$$X \in E(s)(C^b) \Rightarrow \overline{X} \notin E(s)(\overline{C}^{b'}) \text{ for all } t(b) = t(b') = 1$$

  In the case where the time component is greater than one, we also have a similar property to regularity but only for A. An effectivity structure E is *A-regular* iff $X \in E(s)(A^b) \Rightarrow \overline{X} \notin E(s)(\emptyset^b)$.

- An effectivity structure E is *super-additive* iff for all b and d with $t(b) = t(d) = 1$, and $C \cap D = \emptyset$:

  - If $C \neq \emptyset$ and $D \neq \emptyset$, $X_1 \in E(s)(C^b)$ and $X_2 \in E(s)(D^d) \Rightarrow$

$$X_1 \cap X_2 \in E(s)((C \cup D)^{b \oplus d})$$

  - If $C = \emptyset$ and $D = \emptyset$ or A, $X_1 \in E(s)(\emptyset^d)$ and $X_2 \in E(s)(D^d)$ then $X_1 \cap X_2 \in E(s)(D^d)$

We have two different cases in the definition of super-additivity because in $\emptyset^b$, b is not the resource bound for the coalition it annotates but for its complement. Therefore, it should not be possible to sum up the bounds as in the case when both coalitions C and D are non-empty. Notice that super-additivity requires the time component of both resource bounds to be equal to 1. If time components are not equal, this property might not be true. Consider for example the case when agent 1 can bring about p by executing one action, and agent 2 can bring about $\neg p$ by executing a sequence of two actions: $V(p) \in E(s)(\{1\}^{b_1})$, where $t(b_1) = 1$, and $\overline{V(p)} \in E(s)(\{2\}^{b_2})$, where $t(b_2) = 2$. It should not be possible to conclude that together, agents 1 and 2 can bring about $p \wedge \neg p$ by expending $b_1 \oplus b_2$ resources, or that $\emptyset \in E(s)(\{1, 2\}^{b_1 \oplus b_2})$. Intuitively, if agent 2 can enforce $\neg p$ after two steps, then it must be that agent 1 cannot keep enforcing p for longer than 1 step. However, from the super-additivity property for $t(b) = t(d) = 1$, we can derive the same property for $t(b) = t(d) \geq n$.

We also have the following more general property for the case when one of the coalitions is empty:

- An effectivity structure E is *general super-additive* iff it is super-additive and

  $$X_1 \in E(s)(\emptyset^b) \text{ and } X_2 \in E(s)(C^b) \Rightarrow X_1 \cap X_2 \in E(s)(C^b)$$

  where C is either empty or the grand coalition.

We also have properties corresponding to sequential composition of strategies:

- An effectivity structure E is *super-transitive* iff for all $C \neq \emptyset$

  $$\{s' \in S \mid X \in E(s')(C^{b_2})\} \in E(s)(C^{b_1}) \Rightarrow X \in E(s)(C^{b_1 \otimes b_2})$$

  (if a set of states where X is obtainable under b_2 can be enforced under b_1, then X can be enforced by the combined strategy under $b_1 \otimes b_2$).

- An effectivity structure E is *transitive* iff for any b with $t(b) > 1$ and $C \neq \emptyset$: $X \in E(s)(C^b) \Rightarrow \exists b' < b : X \in E(s)(C^{b'})$ (X can be achieved under a tighter bound b') or $\exists b_1 \otimes b_2 = b : \{s' \in S \mid X \in E(s')(C^{b_2})\} \in E(s)(C^{b_1})$ (X can be achieved by combining two strategies costing b_1 and b_2 such that $b_1 \otimes b_2 = b$).

Finally, the following property is specific to resource bounds:

- An effectivity structure E is *bound-monotonic* iff
  $X \in E(s)(C^b) \Rightarrow X \in E(s)(C^d)$ for all $d \geq b$ if $C \neq \emptyset$ or $d \leq b$ if $C = \emptyset$.

Bound-monotonicity is a very natural property which states that if a non-empty coalition can achieve something under bound b, then it can achieve it with more resources. For $C = \emptyset$, this property means that if an outcome cannot be avoided when the grand coalition is restricted to strategies which cost at most b, then it cannot be avoided if A uses fewer resources (hence has fewer strategies available).

It is easy to prove that the properties above are true for any effectivity structure obtained from a RBA frame. Conversely, RB-playable effectivity structures defined below are effectivity structures of an RBA frame.

Definition 4. *An effectivity structure $E : S \rightarrow (\wp(A)^{\mathbb{B}} \rightarrow \wp(\wp(S)))$ is RB-playable iff, for every $s \in S$, E has the following properties:*

1. *For all $C^b \in \wp(A)^{\mathbb{B}}$, $S \in E(s)(C^b)$*
2. *For all $C^b \in \wp(A)^{\mathbb{B}}$, $\emptyset \notin E(s)(C^b)$*
3. *Outcome-monotonicity*
4. *A-maximality*
5. *A-regularity*
6. *Super-additivity*
7. *Super-transitivity*
8. *Transitivity*
9. *Bound-monotonicity*

It can be shown that RB-playability implies the other properties listed above.

Lemma 1. *Let E be a RB-playable effectivity structure, then E has the following properties:*

1. *Coalition monotonicity*
2. *A-minimality*
3. *Regularity*
4. *General super-additivity*

In the following, we provide the proof of the above lemma. First general super-additivity is proved by induction on resource bounds using super-additivity. The proofs of the other properties are based on general super-additivity.

Proof. By super-transitivity, we have that, for any b and $b_1 \otimes b_2 = b$

$$\{s' \mid X \in E(s')(A^{b_2})\} \in E(s)(A^{b_1}) \Rightarrow X \in E(s)(A^{b_1 \otimes b_2})$$

Hence,

$$X \notin E(s)(A^{b_1 \otimes b_2}) \Rightarrow \{s' \mid X \in E(s')(A^{b_2})\} \notin E(s)(A^{b_1})$$

By A-regularity and A-maximality, we have $\overline{X} \in E(s)(\emptyset^{b_1 \otimes b_2}) \Rightarrow X \notin E(s)(A^{b_1 \otimes b_2})$ and $\{s' \mid X \in E(s')(A^{b_2})\} \notin E(s)(A^{b_1}) \Rightarrow \{s' \mid \overline{X} \in E(s')(\emptyset^{b_2})\} \in E(s)(\emptyset^{b_1})$, respectively. Therefore,

$$X \in E(s)(\emptyset^{b_1 \otimes b_2}) \Rightarrow \{s' \mid X \in E(s')(\emptyset^{b_2})\} \in E(s)(\emptyset^{b_1}) \tag{1}$$

We now prove general super-additivity by induction on time component of b. The base case follows directly from super-additivity. Let $X \in E(s)(\emptyset^b)$ where time component of b is greater than 1. Assume that $Y \in E(s)(C^b)$ where C is either $\emptyset$ or A. If $Y \in E(s)(C^{b'})$ for some $b' < b$, then bound-monotonicity for the empty coalition and the induction hypothesis imply that $X \cap Y \in E(s)(C^{b'})$. Hence, bound-monotonicity implies $X \cap Y \in E(s)(C^b)$. If $Y \notin E(s)(C^{b'})$ for all such b', we have that there exist b_1 and b_2 such that $b_1 \otimes b_2 = b$ and

$$\{s' \mid Y \in E(s')(C^{b_2})\} \in E(s)(C^{b_1})$$

which follows from transitivity when $C = A$ or from (1) with arbitrary $b_1 \oplus b_2 = b$ when $C = \emptyset$. Note that we also have $\{s' \mid X \in E(s')(\emptyset^{b_2})\} \in E(s)(\emptyset^{b_1})$. Applying the induction hypothesis twice together with outcome-monotonicity, we have the following result:

$$\{s' \mid X \cap Y \in E(s')(C^{b_2})\} \in E(s)(C^{b_1})$$

Therefore, super-transitivity implies that $X \cap Y \in E(s)(C^b)$.

1. Assume that $X \in E(s)(\emptyset^b)$. By RB-playability, we have $S \in E(s)(A^b)$. Applying general super-additivity, we obtain $X \in E(s)(A^b)$.

   Let $\emptyset \neq C \subset A$, we prove by induction on the time component of b that $X \in E(s)(C^b) \Rightarrow X \in E(s)(D^b)$ for any $D \supset C$.

In the base case, when time component of b is equal to 1, let $C' = D \setminus C$. We have $S \in E(s)(C'^{(0,\ldots,0,1)})$, thus super-additivity implies that $X = X \cap S \in E(s)(D^b)$.

Let us assume that time component of b is greater than 1. If $X \in E(s)(C^{b'})$ for some $b' < b$, then it is obvious by the induction hypothesis that $X \in E(s)(D^{b'})$. Hence, bound-monotonicity implies that $X \in E(s)(D^b)$. If $X \notin E(s)(C^{b'})$ for any such b', then we have by transitivity that there exists $b_1 \otimes b_2 = b$ such that

$$\{s' \mid X \in E(s')(C^{b_2})\} \in E(s)(C^{b_1})$$

By the induction hypothesis, we have

$$\{s' \mid X \in E(s')(C^{b_2})\} \in E(s)(D^{b_1})$$

and

$$\{s' \mid X \in E(s')(C^{b_2})\} \subseteq \{s' \mid X \in E(s')(D^{b_2})\}$$

Thus, outcome-monotonicity implies that

$$\{s' \mid X \in E(s')(D^{b_2})\} \in E(s)(D^{b_1})$$

Therefore, we have by super-transitivity that $X \in E(s)(D^b)$.

2. Assume that $X \in E(s)(A^b)$ and $Y \notin E(s)(A^b)$. By A-maximality, we have $\overline{Y} \in E(s)(\emptyset^b)$. Therefore, general super-additivity implies that $X \cap \overline{Y} \in E(s)(A^b)$.

3. Assume that $\emptyset \neq C \subset A$ and $X \in E(s)(C^b)$ where the time component of b is equal to 1. Furthermore, assume by contradiction that $\overline{X} \in E(s)(\overline{C}^{b'})$, where the time component of b' is also equal to 1. Applying super-additivity, we have $X \cap \overline{X} \in E(s)(A^{b \oplus b'})$ which contradicts the fact that E is RB-playable. Therefore, E is regular.

□

Theorem 2. *An effectivity structure is RB-playable iff it is the effectivity structure of some RBA frame.*

Proof. It is easy to check that effectivity structures obtained from RBA frames satisfy all properties of RB-playability. In order to prove the other direction for a given RB-playable effectivity structure E, we need to construct a RBA frame such that its effectivity structure is identical to E.

Let E be an RB-playable effectivity structure. The construction of the RBA frame is similar to that in Coalition Logic extended with costs for actions. First, we define the set of possible actions for each agent at each state $s \in S$ with their associated costs Res. Then the construction is completed by defining the outcome function o.

In order to make the proof easier to follow, we first provide an informal sketch of the argument, which follows closely the argument in [14]. The main task in defining the RBA frame is to define the actions available to each agent at a particular state. We construct these actions in a way that makes it easier to define costs of actions and the outcome function later. Each action for an agent is a triple (g, t, h) where:

- g is a function which defines the preferred set of outcomes for each coalition in which the agent participates and is willing to contribute a certain amount of resources (then, the cost of this action is this amount of resources). Given the actions of all agents, the component g of those actions will define the coalitions in which the agents participate, hence also the preferred set of outcomes for each agent.

- t is a natural number which is used to determine which agent has the power to decide the outcome.

- when we know which agent has the power to decide the outcome and its preferred set of outcomes, h is a function which picks a single outcome among those in the preferred set.

In the following, we present in detail how actions and outcomes of actions are defined. We present a worked example in section 3.6.

For every $i \in A$ and bound b such that $t(b) = 1$, we define $\mathcal{C}_i^b = \{C^d \mid i \in C \wedge t(d) = 1 \wedge d \geq b\}$ which is the set of all coalitions in which i may participate and contribute b amount of resources. Note that for all actions $t(b)$ is always 1.

For every $s \in S$, we define

$$\Gamma(s, i) = \{g^b_{(s,i)} : \mathcal{C}_i^b \rightarrow \wp(S) \mid g^b_{(s,i)}(C^d) \in E(s)(C^d)\}$$

$\Gamma(s, i)$ is the set of option functions for an agent i at state s. Each option function in $\Gamma(s, i)$ is a mapping $g^b_{(s,i)}$ in which b is a resource bound such that $t(b) = 1$. $g^b_{(s,i)}$ determines the outcome when agent i agrees to participate in a coalition. How an agent agrees to participate in a coalition will be specified later when we define the outcome function.

Let $H = \{h : \wp(S) \rightarrow S \mid h(X) \in X\}$ be the set of choice functions, that is, if an agent has the power to decide the outcome, it will use some h function to do so. We then define the set of available actions for an agent i at a state s as follows:

$$T(s,i) = \Gamma(s,i) \times \mathbb{N} \times H$$

Each action is a triple $(g^b_{(s,i)}, t, h)$ consisting of an option function $g^b_{(s,i)}$, an index t (a natural number) and a choice function h. Informally, option functions determine how the agents group together to form coalitions and then which outcome options they will choose. The index determines which agent has the power to decide the outcome based on its associated h function. We set $Res((g^b_{(s,i)}, t, h)) = b$. Note that for any action, we have $t(Res((g^b_{(s,i)}, t, h))) = 1$.

Let $\Sigma_i = \bigcup_{s\in S} T(s,i)$. We now define the outcome of a joint action $\sigma \in \Sigma_A$ at a state s. Assume that $\sigma = \{(g^{b_i}_{(s,i)}, t_i, h_i) \mid i = 1,\ldots,n\}$ where $t(b_i) = 1$ for all $i \in A$. For any coalition $C \subseteq A$, let $b_C = \oplus_{i\in C} b_i$ and $g = (g^{b_i}_{(s,i)})_{i\in A}$. We denote by $P(g,C)$ the coarsest partition $\langle C_1,\ldots,C_m\rangle$ of C such that:

$$\forall l \leq m\ \forall i,j \in C_l : g^{b_i}_{(s,i)}(C^{b_C}) = g^{b_j}_{(s,j)}(C^{b_C})$$

We define how coalitions are formed based on g as follows:

$$\begin{array}{rcl}
P_0(g) & = & \langle A\rangle \\
P_1(g) & = & \langle P(g,A)\rangle = \langle C_{1,1},\ldots,C_{1,k_1}\rangle \\
P_2(g) & = & \langle P(g,C_{1,1}),\ldots,P(g,C_{1,k_1})\rangle \\
 & & = \langle C_{2,1},\ldots,C_{2,k_2}\rangle \\
 & \vdots & \\
P_\eta(g) & = & \langle C_{\eta,1},\ldots,C_{\eta,k_\eta}\rangle
\end{array}$$

As A is finite, the above computation reaches some η such that $P_\eta(g) = P_{\eta+1}(g)$. Let $P(g) = P_\eta(g)$ be the partition which shows how agents are grouped into coalitions.

For technical convenience, let $E^o(s)(A^b)$ denote the collection of minimal sets in $E(s)(A^b)$. By A-minimality, it is easy to show that $E^o(s)(A^b)$ contains only singletons. In other words, by outcome-monotonicity, we have $X \in E(s)(A^b)$ if and only if $X \supseteq X^o$ for some $X^o \in E^o(s)(A^b)$. By regularity, we have $X \in E(s)(\emptyset^b)$ if and only if $X \supseteq \cup E^o(s)(A^b)$ where $\cup E^o(s)(A^b) = \cup_{X^o \in E^o(s)(A^b)} X^o$.

Assume that $P(g) = \langle C_1,\ldots,C_m\rangle$. For convenience, let $g(C_l) = g^{b_i}_{(s,i)}({C_l}^{b_{C_l}})$ for some $i \in C_l$ where $l \leq m$.

We define $G(g) = \bigcap\limits_{l\leq m} g(C_l) \cap (\cup E^o(s)(A^{b_A}))$. By super-additivity and the fact that $\emptyset \notin E(s)(A^{b_A})$ as E is RB-playable, it is straightforward to show that $G(g) \neq \emptyset$.

Let $t_0 = (\sum_{i \in A} t_i \mod n) + 1$. The outcome function is defined as follows: $o(s, \sigma) = h_{t_0}(G(g))$. Let E_F be the effectivity structure of the frame constructed above. We claim that $E = E_F$.

Firstly, we show the left-to-right inclusion by induction on bounds. In the base case, assume $X \in E(s)(C^b)$ where $t(b) = 1$. Choose the actions for agents in $C = \{1, \ldots, k\}$ as follows,

$$\begin{array}{rcl} a_1 & = & (g_1^b, t_1, h_1) \\ a_2 & = & (g_2^{\overline{0}}, t_2, h_2) \\ & \vdots & \\ a_k & = & (g_k^{\overline{0}}, t_k, h_k) \end{array}$$

where $g_1^b(D^d) = g_i^{\overline{0}}(D^d) = X$ for all $i = 2, \ldots, k$, $D \supseteq C$, $d \geq b$. Notice that the choices of g_1^b, $g_2^{\overline{0}}$, ..., $g_k^{\overline{0}}$ must exist because of bound-monotonicity and coalition-monotonicity. Moreover, the choices of t_i and h_i, where $i = 1, \ldots, k$, are arbitrary. Let $\sigma_C = \{(g_1^b, t_1, h_1), (g_2^{\overline{0}}, t_2, h_2), \ldots, (g_k^{\overline{0}}, t_k, h_k)\}$.

Let $\sigma_{\overline{C}}$ be an arbitrary joint action for $\overline{C}$. Let $\sigma = (\sigma_C, \sigma_{\overline{C}})$ and let g be the set of the option functions from σ. By the choice of σ_C, C must be a subset of a partition C_l in $P(g)$. Then, we have

$$o(s, \sigma) = h_{t_0}(G(g)) \in G(g) \subseteq g(C_l) = X$$

Hence, $X \in E_F(s)(C^b)$.

For the induction step, let $X \in E(s)(C^b)$ where $t(b) > 1$. If $X \in E(s)(C^{b'})$ for some $b' < b$, by the induction hypothesis, we have $X \in E_F(s)(C^{b'})$. Therefore, bound-monotonicity implies that $X \in E_F(s)(C^b)$.

If $X \notin E(s)(C^{b'})$ for any $b' < b$, by transitivity there are $b_1 \otimes b_2 = b$ such that

$$\{s' \mid X \in E(s')(C^{b_2})\} \in E(s)(C^{b_1})$$

By the induction hypothesis, we have

$$\{s' \mid X \in E(s')(C^{b_2})\} \in E_F(s)(C^{b_1})$$

and

$$\{s' \mid X \in E(s')(C^{b_2})\} \subseteq \{s' \mid X \in E_F(s')(C^{b_2})\}$$

By outcome-monotonicity, we have

$$\{s' \mid X \in E_F(s')(C^{b_2})\} \in E_F(s)(C^{b_1})$$

Hence, by super-transitivity $X \in E_F(s)(C^b)$.

For the other direction, we consider two cases in which $C = A$ and $C \subset A$. Assume that $X \notin E(s)(A^b)$. By A-maximality, we obtain $\overline{X} \in E(s)(\emptyset^b)$. However, the previous proof implies that $\overline{X} \in E_F(s)(\emptyset^b)$. As E_F is RB-playable, by regularity we have $X \notin E_F(s)(A^b)$.

For the case of $C \subset A$, the proof is done by induction on bounds. Assume that $X \notin E(s)(C^b)$ where $t(b) = 1$ and $C \subset A$, i.e. there is $i_0 \in A \setminus C$. Let $\sigma_C = \{(g^{b_i}_{(s,i)}, t_i, h_i) \mid i \in C\}$ be a joint action for C such that $Res(\sigma_C) \leq b$. We choose a strategy $\sigma_{\overline{C}} = \{(g^{b_i}_{(s,i)}, t_i, h_i) \mid i \in \overline{C}\}$ for $\overline{C}$ such that:

- $b_i = \overline{0}$ for all $i > k$
- $g^{b_i}_{(s,i)}(D^d) = S$ for all $i \in \overline{C}$, $D \supseteq \overline{C}$, $d \geq b_i$
- $(\sum_{i \in A} t_i \mod n) + 1 = i_0$
- h_i for $i \neq i_0$ is arbitrary, we will select h_{i_0} shortly

As before, let $\sigma = (\sigma_C, \sigma_{\overline{C}})$ and g be the collection of option functions in σ. We use notation $b_D = \oplus_{i \in D} b_i$ for any $D \subseteq A$.

By the choice of option functions in $\sigma_{\overline{C}}$, it follows that $\overline{C}$ is the subset of some partition C_l of $P(g)$. For other partitions, super-additivity shows that $G(g) \in E(s)(\overline{C_l}^{b_{\overline{C_l}}})$. By coalition-monotonicity and bound-monotonicity, we have that $G(g) \in E(s)(C^b)$. As $X \notin E(s)(C^b)$, it follows that $G(g) \not\subseteq X$ by outcome-monotonicity, i.e. there is some $s_0 \in G(g) \setminus X$. Select h_{i_0} such that $h_{i_0}(G(g)) = s_0$, then

$$o(s, \sigma) = h_{i_0}(G(g)) = s_0 \notin X$$

Hence, $X \notin E_F(s)(C^b)$.

In the induction step, assume that $X \notin E(s)(C^b)$ where $t(b) > 1$. Bound-monotonicity shows that for all $b' \leq b$, $X \notin E(s)(C^{b'})$ and super-transitivity implies that for all $b_1 \otimes b_2 = b$,

$$\{s' \mid X \in E(s')(C^{b_2})\} \notin E(s)(C^{b_1})$$

By the induction hypothesis, we have that for all $b' < b$, $X \notin E_F(s)(C^{b'})$ and for all $b_1 \otimes b_2 = b$,

$$\{s' \mid X \in E(s')(C^{b_2})\} \notin E_F(s)(C^{b_1})$$

and $\{s' \mid X \in E(s')(C^{b_2})\} = \{s' \mid X \in E_F(s')(C^{b_2})\}$. Then, $\{s' \mid X \in E_F(s')(C^{b_2})\} \notin E_F(s)(C^{b_1})$. Therefore, transitivity implies that $X \notin E_F(s)(C^b)$. □

3.6 Example

We illustrate the construction of an action frame given in the proof above on the example of the effectivity structure from section 3.4.

Observe that the set $\Gamma(s,i)$ for each s and i is infinite, if only because of the infinitely many possible resource bounds. Below is an element $g_1^{(1,1)}$ of $\Gamma(s_1,1)$ which will be used in our example:

- $g_1^{(1,1)}(\{1,2,3\}^d) = \{s_2\}$ for any $d \geq (2,1)$. This is a valid definition for an element of $\Gamma(s_1,1)$ because $\{s_2\} \in E(s_1)(\{1,2,3\}^{(2,1)})$.
- $g_1^{(1,1)}(\{1,2\}^d) = \{s_2\}$ for any $d \geq (2,1)$.
- $g_1^{(1,1)}(C^d) = S$ for any other C and $d \geq (1,1)$.

Similarly, $\Gamma(s_1,2)$ contains $g_2^{(1,1)}$ such that

- $g_2^{(1,1)}(\{1,2,3\}^d) = \{s_2\}$ for any $d \geq (2,1)$.
- $g_2^{(1,1)}(\{1,2\}^d) = \{s_1\}$ for any $d \geq (2,1)$.
- $g_2^{(1,1)}(C^d) = S$ for any other C and $d \geq (1,1)$.

and $\Gamma(s_1,3)$ contains $g_3^{(0,1)}$ such that

- $g_3^{(0,1)}(\{1,2,3\}^d) = \{s_1\}$ for any $d \geq (2,1)$.
- $g_3^{(0,1)}(C^d) = S$ for any other C and $d \geq (1,1)$.

The set H contains all possible functions h from subsets of S to elements of S, with the condition that $h(X) \in X$. We will consider two elements of H, $h(\{s_i\}) = h'(\{s_i\}) = s_i$, $h(S) = s_1$ and $h'(S) = s_2$. The set of actions for each agent i, $\Gamma(s,i) \times \mathbb{N} \times H$, is infinite, both because $\Gamma(s,i)$ is infinite and because of $\mathbb{N}$. We will show how to determine outcomes of joint actions for just one example triple of actions, $(g_1^{(1,1)}, 1, h)$ for agent 1, $(g_2^{(1,1)}, 5, h)$ for agent 2 and $(g_3^{(0,1)}, 8, h')$ for agent 3.

Let $g = (g_1^{(1,1)}, g_2^{(1,1)}, g_3^{(0,1)})$, we have the following coarsest partition $P(g,C)$:

- $P(g,\{1,2,3\}) = \langle\{1,2\},\{3\}\rangle$
- $P(g,\{1,2\}) = \langle\{1\},\{2\}\rangle$
- $P(g,\{1,3\}) = \langle\{1,3\}\rangle$

- $P(g, \{2,3\}) = \langle\{2,3\}\rangle$
- $P(g, \{1\}) = \langle\{1\}\rangle$
- $P(g, \{2\}) = \langle\{2\}\rangle$
- $P(g, \{3\}) = \langle\{3\}\rangle$

We apply P repeatedly as follows.

$$\begin{array}{lcl} P_0(g) & = & \{1,2,3\} \\ P_1(g) & = & P(g,\{1,2,3\}) = \langle\{1,2\},\{3\}\rangle \\ P_2(g) & = & \langle P(g,\{1,2\}), P(g,\{3\})\rangle = \langle\{1\},\{2\},\{3\}\rangle \\ P_3(g) & = & \langle P(g,\{1\}), P(g,\{2\}), P(g,\{3\})\rangle = \langle\{1\},\{2\},\{3\}\rangle \end{array}$$

Then, we have,

- $g(\{1\}) = g_1^{(1,1)}(\{1\}^{(1,1)}) = S$
- $g(\{2\}) = g_2^{(1,1)}(\{2\}^{(0,1)}) = S$
- $g(\{3\}) = g_3^{(0,1)}(\{3\}^{(0,1)}) = S$

Note that given our E, $\cup E^o(s_1)(\{1,2,3\}^{(2,1)}) = \{s_1\} \cup \{s_2\} = S$. Then, $G(g) = g(\{1\}) \cap g(\{2\}) \cap g(\{3\}) \cap \cup E^o(s_1)(\{1,2,3\}^{(1,1)}) = S$. We have $t_0 = (1+5+8) \bmod 3 + 1 = 3$. Then, we choose the choice function from the action of agent 3 which is h'. Therefore, the outcome of the joint action which we consider in this example is $h'(G(g)) = s_2$.

4 Axiomatisation of $RBCL$

In this section we define models based on RB-playable effectivity structures, and give a complete axiomatisation for the set of validities in those models.

Definition 5. *A resource-bounded effectivity model $M = (S, E, V)$ is a triple consisting of a non-empty set of states, a RB-playable effectivity structure and a valuation function $V : Prop \to \wp(S)$. The truth definition for $[C^b]$ modalities is as follows:*

- $M, s \models [C^b]\varphi$ *iff* $\varphi^M \in E(s)(C^b)$ *where* $\varphi^M = \{s' \mid M, s' \models \varphi\}$

Notice that in the above definition, we do not define the truth for $[C^b]$ modalities in two separate cases, one for non-empty coalitions C and one for empty coalitions. This is because the two cases are covered by the RB-playable effectivity structure E, - see the correspondence of effectivity structures to RBA frames in Section 3.3 for more details.

For convenience, we also extend the definition of the function V for a given model $M = (S, E, V)$ as follows, $V(\varphi) = \{s \in S \mid M, s \models \varphi\}$.

Theorem 3. *The sets of formulas valid in multi-step RBA models and in resource-bounded effectivity models are equal.*

This follows from the correspondence between RBA frames and RB-playable effectivity structures, and the correspondence between the two truth definitions. Therefore the next result also provides an axiomatisation for $RBCL$.

Theorem 4. *The following set of axiom schemas and inference rules provides a sound and complete axiomatisation of the set of validities over all resource-bounded effectivity models:*

A0 *All propositional tautologies*

A1 $[C^b]\top$

A2 $\neg[C^b]\bot$

A3 $\neg[\emptyset^b]\varphi \leftrightarrow [A^b]\neg\varphi$

A4 $[C^b](\varphi \wedge \psi) \rightarrow [C^b]\varphi$

A5 $[C^b]\varphi \rightarrow [C^d]\varphi$ *where* $d \geq b$ *if* $C \neq \emptyset$ *or* $d \leq b$ *if* $C = \emptyset$

A6a $[C^b]\varphi \wedge [D^d]\psi \rightarrow [(C \cup D)^{b\oplus d}](\varphi \wedge \psi)$ *where* C *and* D *are both non-empty and disjoint, and* $t(b) = t(d) = 1$

A6b $[\emptyset^b]\varphi \wedge [C^b]\psi \rightarrow [C^b](\varphi \wedge \psi)$ *where* C *is either* $\emptyset$ *or* A

A7 $[C^{b_1}][C^{b_2}]\varphi \rightarrow [C^{b_1 \otimes b_2}]\varphi$ *for* $C \neq \emptyset$

A8 $[C^b]\varphi \rightarrow \bigvee_{b'<b}[C^{b'}]\varphi \vee \bigvee_{b_1 \otimes b_2 = b}[C^{b_1}][C^{b_2}]\varphi$ *for all* $C \neq \emptyset$

MP $\vdash \varphi,\ \vdash \varphi \rightarrow \psi \Rightarrow \vdash \psi$

Equivalence $\vdash \varphi \leftrightarrow \psi \Rightarrow \vdash [C^b]\varphi \leftrightarrow [C^b]\psi$

Proof. Notice that for axiom **A8**, when $b = (0, \ldots, 0, 1)$, it simply has the form $[C^b]\varphi \to \top$ which is a propositional tautology.

The proof of soundness is straightforward. We prove completeness by constructing a canonical model. Let us denote by $\vdash_\Lambda$ derivability in the axiom system above. Let S^Λ be the set of all Λ-maximally consistent sets. For any formula φ, we denote $\tilde{\varphi} = \{s \in S^\Lambda \mid \varphi \in s\}$. Then, we define the canonical valuation function $V^\Lambda(p) = \tilde{p}$.

We define the canonical effectivity structure E^Λ by induction on b as follows:

- For all b such that $t(b) = 1$ and $C \neq A$, $X \in E^\Lambda(s)(C^b)$ iff $\exists \tilde{\varphi} \subseteq X$: $[C^b]\varphi \in s$. $X \in E^\Lambda(s)(A^b)$ iff $\overline{X} \notin E^\Lambda(s)(\emptyset^b)$.
- For all b such that $t(b) > 1$ and $C \neq \emptyset$, $X \in E^\Lambda(s)(C^b)$ iff $X \in E^\Lambda(s)(C^{b'})$ for some $b' < b$ or there are $b_1 \otimes b_2 = b$ such that $\{s' \mid X \in E^\Lambda(s')(C^{b_2})\} \in E^\Lambda(s)(C^{b_1})$. $X \in E^\Lambda(s)(\emptyset^b)$ iff $\overline{X} \notin E^\Lambda(s)(A^b)$.

The following property $(*)$ is crucial for the proof:

$$(*) \quad \tilde{\varphi} \in E^\Lambda(s)(C^b) \text{ iff } [C^b]\varphi \in s$$

We prove it by induction on the bounds. In the base case, assume that $\tilde{\varphi} \in E^\Lambda(s)(C^b)$ for some $t(b) = 1$. For $C \neq A$, $\tilde{\varphi} \in E^\Lambda(s)(C^b)$ iff $\exists \tilde{\psi} \subseteq \tilde{\varphi}$: $[C^b]\psi \in s$. By $\vdash_\Lambda \psi \to \varphi$ and **RM**, it is implied that $[C^b]\varphi \in s$. In the inverse direction, $[C^b]\varphi \in s$ implies directly that $\tilde{\varphi} \in E^\Lambda(s)(C^b)$ by the definition of E^Λ.

If $C = A$, we have $\tilde{\varphi} \in E^\Lambda(s)(A^b)$ iff $\widetilde{\neg\varphi} \notin E^\Lambda(s)(\emptyset^b)$ iff $\neg[\emptyset^b]\neg\varphi \in s$ (as just proved) iff $[A^b]\varphi \in s$ (by axiom $A3$).

For the induction step, assume that $\tilde{\varphi} \in E^\Lambda(s)(C^b)$ where $t(b) > 1$. For $C \neq \emptyset$, there are two cases to consider. (1) $\tilde{\varphi} \in E^\Lambda(s)(C^{b'})$ for some $b' < b$. By the induction hypothesis, we have $[C^{b'}]\varphi \in s$. Then, axiom **A5** implies that $[C^b]\varphi \in s$. (2) There are $b_1 \otimes b_2 = b$ such that

$$\{s' \mid \tilde{\varphi} \in E^\Lambda(s')(C^{b_2})\} \in E^\Lambda(s)(C^{b_1}).$$

Let $\psi = [C^{b_2}]\varphi$, by the induction hypothesis, we have $\tilde{\psi} = \{s' \mid \tilde{\varphi} \in E^\Lambda(s')(C^{b_2})\}$, thus, $\tilde{\psi} \in E^\Lambda(s)(C^{b_1})$. Again, the induction hypothesis gives us $[C^{b_1}][C^{b_2}]\varphi \in s$. Therefore, by axiom **A7**, we have $[C^b]\varphi \in s$.

For the inverse direction, assume that $[C^b]\varphi \in s$ for some $t(b) > 1$. By axiom **A8**, there are two cases to consider. If $[C^{b'}]\varphi \in s$ for some $b' < b$, then the induction hypothesis implies that $\tilde{\varphi} \in E^\Lambda(s)(C^{b'})$. Hence, by the definition of E^Λ, we have $\tilde{\varphi} \in E^\Lambda(s)(C^b)$. In the second case, there are $b_1 \otimes b_2 = b$ such that $[C^{b_1}][C^{b_2}]\varphi \in s$. Similar to the proof above, let $\psi = [C^{b_2}]\varphi$, the induction hypothesis implies that $\tilde{\psi} \in E^\Lambda(s)(C^{b_1})$. As we have that $\tilde{\psi} = \{s' \mid \tilde{\varphi} \in E^\Lambda(s')(C^{b_2})\}$, this shows

$$\{s' \mid \tilde{\varphi} \in E^{\Lambda}(s')(C^{b_2})\} \in E^{\Lambda}(s)(C^{b_1}).$$

By the definition of E^{Λ}, we obtain $\tilde{\varphi} \in E^{\Lambda}(s)(C^b)$.

If $C = \emptyset$, we have $\tilde{\varphi} \in E^{\Lambda}(s)(\emptyset^b)$ iff $\tilde{\neg\varphi} \notin E^{\Lambda}(s)(A^b)$ iff $\neg[A^b]\neg\varphi \in s$ (as just proved) iff $[\emptyset^b]\varphi \in s$ (by axiom **A3**).

The proof that E^{Λ} is RB-playable is straightforward given the property $(*)$, the definition of E^{Λ} and the axioms of Λ.

1. As $[C^b]\top \in s$ for all $s \in S^{\Lambda}$, we have by (*) that $S^{\Lambda} = \tilde{\top} \in E^{\Lambda}(s)(C^b)$.

2. Similarly, $[C^b]\bot \notin s$ for all $s \in S^{\Lambda}$ implies by (*) that that $\emptyset = \tilde{\bot} \notin E^{\Lambda}(s)(C^b)$.

3. We prove outcome-monotonicity by induction on bounds. Assume that $X \in E^{\Lambda}(s)(C^b)$.

   - If $t(b) = 1$ and $C \neq A$, $X \in E^{\Lambda}(s)(C^b)$ iff there exists φ such that $\tilde{\varphi} \subseteq X$ and $[C^b]\varphi \in s$. Hence, for all $X' \supseteq X$, we have that $\tilde{\varphi} \subseteq X'$. This implies by the definition of E^{Λ} that $X' \in E^{\Lambda}(s)(C^b)$
   - If $t(b) = 1$, $X \in E^{\Lambda}(s)(A^b)$ iff $\overline{X} \notin E^{\Lambda}(s)(\emptyset^b)$. Let $X' \supseteq X$, then $\overline{X'} \subseteq \overline{X}$. Assume by contradiction that $X' \notin E^{\Lambda}(s)(A^b)$. Then, $\overline{X'} \in E^{\Lambda}(s)(\emptyset^b)$. As $\overline{X'} \subseteq \overline{X}$, this implies that $\overline{X} \in E^{\Lambda}(s)(\emptyset^b)$ which is a contradiction.
   - If $t(b) > 1$ and $C \neq \emptyset$. If $X \in E^{\Lambda}(s)(C^{b'})$ for some $b' < b$, the induction hypothesis shows that $X' \in E^{\Lambda}(s)(C^{b'})$ for all $X' \supseteq X$. Then, by the definition of E^{Λ} we have $X' \in E^{\Lambda}(C^b)(s)$. Assume $X \notin E^{\Lambda}(s)(C^{b'})$ for all $b' < b$. By the definition of E^{Λ}, there are b_1, b_2 such that $b_1 \otimes b_2 = b$ and

     $$\{s' \mid X \in E^{\Lambda}(s')(C^{b_2})\} \in E^{\Lambda}(s)(C^{b_1})$$

     Let $X' \supseteq X$, by the induction hypothesis we have

     $$\begin{aligned}&\{s' \mid X \in E^{\Lambda}(s')(C^{b_2})\} \subseteq \{s' \mid X' \in E^{\Lambda}(s')(C^{b_2})\}\\ &\Rightarrow \{s' \mid X' \in E^{\Lambda}(s')(C^{b_2})\} \in E^{\Lambda}(s)(C^{b_1})\end{aligned}$$

     By the definition of E^{Λ}, we have $X' \in E^{\Lambda}(s)(C^b)$.
   - If $t(b) > 1$, $X \in E^{\Lambda}(s)(\emptyset^b)$ iff $\overline{X} \notin E^{\Lambda}(s)(A^b)$. Let $X' \supseteq X$ and assume by contradiction that $X' \notin E^{\Lambda}(s)(\emptyset^b)$. This implies that $\overline{X'} \in E^{\Lambda}(s)(A^b)$. By the previous proof, we have $\overline{X} \in E^{\Lambda}(s)(A^b)$ as $\overline{X'} \subseteq \overline{X}$, which is a contradiction.

4. A-maximality follows directly from the definition of E^Λ for A when $t(b) = 1$ and for $\emptyset$ when $t(b) > 1$.

5. Similarly, A-regularity also follows directly from the definition of E^Λ for A when $t(b) = 1$ and for $\emptyset$ when $t(b) > 1$.

6. In order to show super-additivity, we consider the following three cases. Let $t(b) = t(d) = 1$, $C \cap D = \emptyset$, $X \in E^\Lambda(s)(C^b)$ and $Y \in E^\Lambda(s)(D^d)$.

   - If both C and D are not empty by the definition of E^Λ, we have that there are φ and ψ such that $\tilde{\varphi} \subseteq X$, $\tilde{\psi} \subseteq Y$, $[C^b]\varphi$ and $[D^d]\psi \in s$. According to axiom **A6a**, we have $[(C \cup D)^{b \oplus d}](\varphi \wedge \psi) \in s$. Obviously, $\tilde{\varphi} \cap \tilde{\psi} \subseteq X \cap Y$, hence $X \cap Y \in E^\Lambda(s)((C \cup D)^{b \oplus d})$.
   - If $C = \emptyset$, $b = d$ and $D = \emptyset$, the proof is similar to the one above except that axiom **A6b** gives us $[D^d](\varphi \wedge \psi) \in s$. Hence, $X \cap Y \in E^\Lambda(s)(D^d)$.
   - If $C = \emptyset$, $b = d$ and $D = A$, we need to show that $X \cap Y \in E^\Lambda(A^b)(s)$. Assume to the contrary that $X \cap Y \notin E^\Lambda(A^b)(s)$, then A-maximality, which has been proved above, implies that $\overline{X \cap Y} \in E^\Lambda(\emptyset^b)(s)$. Then, by the previous case of super-additivity, we have $X \cap \overline{Y} \in E^\Lambda(\emptyset^b)(s)$. As we already showed outcome-monotonicity, $\overline{Y} \in E^\Lambda(\emptyset^b)(s)$. However, by A-regularity, we have $Y \notin E^\Lambda(A^b)(s)$ which is a contradiction.

7. Super-transitivity follows directly from the definition of E^Λ when $t(b) > 1$.

8. Similarly, transitivity follows directly from the definition of E^Λ when $t(b) > 1$.

9. Finally, we show that E^Λ is indeed bound-monotonic. Let us assume that $X \in E^\Lambda(s)(C^b)$.

   - If $t(b) = 1$ and $C \neq A$, $X \in E^\Lambda(s)(C^b)$ iff there exists φ such that $\tilde{\varphi} \subseteq X$ and $[C^b]\varphi \in s$. By axiom **A5**, we have that if $C \neq \emptyset$, then for any $d \geq b$, $[C^d]\varphi \in s$. For $C = \emptyset$, $[\emptyset^d]\varphi \in s$ for any $d \leq b$. Then, by the definition of E^Λ, $X \in E^\Lambda(s)(C^d)$.
   - If $t(b) = 1$ and $C = A$, $X \in E^\Lambda(s)(A^b)$ iff $\overline{X} \notin E^\Lambda(s)(\emptyset^b)$. Then, axiom **A5** implies that $\overline{X} \notin E^\Lambda(s)(\emptyset^d)$ for any $d \geq b$. Once again, by the definition of E^Λ, we have $X \in E^\Lambda(s)(A^d)$.
   - If $t(b) > 1$ and $C \neq \emptyset$, it is straightforward from the definition of E^Λ that $X \in E^\Lambda(s)(C^d)$ for any $d \geq b$.

- If $t(b) > 1$ and $C = \emptyset$, $X \in E^\Lambda(s)(\emptyset^b)$ iff $\overline{X} \notin E^\Lambda(s)(A^b)$. By the proof of the previous case, we have $\overline{X} \notin E^\Lambda(s)(A^d)$ for any $d \leq b$. Hence, $X \in E^\Lambda(s)(\emptyset^d)$.

Since we have already shown (*), the following truth lemma is straightforward:

$$(**) \quad M^\Lambda, s \models \varphi \text{ iff } \varphi \in s$$

As usual, we show (**) by induction on the structure of φ. The cases for propositional variables and usual Boolean connectives are trivial and are omitted.

- If $\varphi = [C^b]\psi$, then,

$$\begin{aligned} M^\Lambda, s \models [C^b]\psi &\Leftrightarrow \psi^{M^\Lambda} \in E^\Lambda(s)(C^b) \\ &\Leftrightarrow \tilde{\psi} \in E^\Lambda(s)(C^b) \quad \text{by the induction hypothesis} \\ &\Leftrightarrow [C^b]\psi \in s \quad \text{by (*)} \end{aligned}$$

From (**), it is obvious that for any consistent formula φ, there is a state $s \in S^\Lambda$ such that $\varphi \in s$, hence $M^\Lambda, s \models \varphi$. In other words, φ is satisfiable. We complete the proof of completeness as follows. Let $\nvdash_\Lambda \varphi$, i.e. $\neg\varphi$ is consistent. Hence, $\neg\varphi$ is satisfiable. Therefore, φ is not valid.

□

5 Satisfiability

In this section we show that the satisfiability problem for $RBCL$ is decidable by providing an algorithm which determines the satisfiability of a given formula φ. Similar to Coalition Logic, our algorithm is developed by adopting the approach presented in [16]. In outline, the algorithm tries to guess a valuation satisfying certain conditions for an extended set of subformulas of φ. Such a valuation is used to construct a model for φ, or in other words, assure the satisfiability of φ.

Given a formula φ, we define a set $sub(\varphi)$ inductively as follows.

- $sub(p) = \{p\}$ for any propositional variable p
- $sub(\neg\psi) = \{\neg\psi\} \cup sub(\psi)$
- $sub(\psi_1 \vee \psi_2) = \{\psi_1 \vee \psi_2\} \cup sub(\psi_1) \cup sub(\psi_2)$
- $sub([C^b]\psi) = \{[C^b]\psi\} \cup sub(\psi)$ for $t(b) = 1$ and $C \neq A$
- $sub([A^b]\psi) = \{[A^b]\psi\} \cup sub(\neg[\emptyset^b]\neg\psi)$ for $t(b) = 1$

- $sub([C^b]\psi) = \{[C^b]\psi\} \cup \bigcup_{b'<b} sub([C^{b'}]\psi) \cup \bigcup_{b_1 \otimes b_2 = b} sub([C^{b_1}][C^{b_2}]\psi)$ for $t(b) > 1$ and $C \neq \emptyset$
- $sub([\emptyset^b]\psi) = \{[\emptyset^b]\psi\} \cup sub(\neg[A^b]\neg\psi)$ for $t(b) > 1$

It is easy to show that $sub(\varphi)$ is finite. Then, we define the closure $cl(\varphi)$ of a given formula φ as follows.

$$\begin{aligned} cl(\varphi) \quad = \quad & \{\psi, \neg\psi \mid \psi \in sub(\varphi)\} \cup \\ & \{[\emptyset^b]\neg\psi, \neg[\emptyset^b]\neg\psi \mid [A^b]\psi \in sub(\varphi)\} \cup \\ & \{[A^b]\neg\psi, \neg[A^b]\neg\psi \mid [\emptyset^b]\psi \in sub(\varphi)\} \end{aligned}$$

Notice that we identify $\neg\neg\psi$ with ψ. Moreover, we denote by $\bar{0}$ the smallest bound of which all components are 0 except for the time component which is 1. We have the following definition of valuations.

Definition 6. *A valuation for a given formula φ is a mapping $v : cl(\varphi) \to \{0, 1\}$ which satisfies the following conditions:*

1. $v(\varphi) = 1$
2. $v(\top) = 1$
3. $v(\neg\psi) = 1 - v(\psi)$
4. $v(\psi_1 \vee \psi_2) = \max(v(\psi_1), v(\psi_2))$
5. $v([\emptyset^b]\psi) = v(\neg[A^b]\neg\psi)$
6. $v([C^b]\psi) \leq v([C^d]\psi)$ *where* $b \leq d$ *if* $C \neq \emptyset$ *or* $b \geq d$ *otherwise*
7. $v([C^b]\psi) = \max\{\bigcup_{b'<b}\{v([C^{b'}]\psi)\} \cup \bigcup_{b_1 \otimes b_2 = b}\{v([C^{b_1}][C^{b_2}]\psi)\}\}$ *where* $t(b) > 1$ *and* $C \neq \emptyset$

In the following lemma, we determine when such a valuation qualifies as a starting point to build a model for φ. The idea of the proof is similar to [14] which in turn builds on [16] but is made somewhat more complicated by the presence of resource bounds and the need to treat resource bounds for the empty coalition differently (for example, with respect to monotonicity).

Lemma 2. *A formula φ is satisfiable if and only if there exists a valuation v for φ such that*

1. *If there are $[C_1^{b_1}]\psi_1, \ldots, [C_k^{b_k}]\psi_k \in cl(\varphi)$ for some $k > 0$ such that:*
    - $t(b_j) = 1$ *for all* $j \leq k$

- *$C_1, \ldots, C_k$ are pairwise disjoint*
- *for any $[C_j^{b_j}]\psi_j$ such that $C_j = \emptyset$, $b_j \geq \oplus_{C_{j'} \neq \emptyset} b_{j'}$*
- *$v([C_j^{b_j}]\psi_j) = 1$ for all $j \leq k$*

then $\wedge_{j \leq k} \psi_j$ is satisfiable.

2. *If there are $[C_1^{b_1}]\psi_1, \ldots, [C_k^{b_k}]\psi_k \in cl(\varphi)$ for some $k > 0$ such that:*

   - *$t(b_j) = 1$ for all $j \leq k$*
   - *$C_1, \ldots, C_{k-1}$ are pairwise disjoint and all non-empty*
   - *$\bigcup_{j<k} C_j \subseteq C_k$*
   - *$\oplus_{j<k} b_j = b_k$*
   - *$v([C_j^{b_j}]\psi_j) = 1$ for all $j < k$*
   - *$v([C_k^{b_k}]\psi_k) = 0$*

   then $\wedge_{j<k} \psi_j \wedge \neg \psi_k$ is satisfiable.

Proof. Firstly, we prove the left-to-right direction by defining a valuation based on a model satisfying the formula φ. In particular, let us assume that φ is satisfiable by a model $M = (S, E, V)$ at some state $s \in S$. As before, for convenience, we extend the function V to arbitrary formulas so that $V(\phi) = \{s \in S \mid M, s \models \phi\}$.

We define a valuation v for $cl(\varphi)$ as follows:

$$v(\psi) = \begin{cases} 1 & \text{if } M, s \models \psi \\ 0 & \text{otherwise} \end{cases}$$

Based on the definition of the semantics for $RBCL$, it is straightforward to show that v satisfies all the conditions listed in Definition 6. What remains is to prove that it also has the two properties listed in the lemma.

1. Assume that there are $[C_1^{b_1}]\psi_1, \ldots, [C_k^{b_k}]\psi_k \in cl(\varphi)$ for some $k > 0$ such that:

   - $t(b_j) = 1$ for all $j \leq k$
   - $C_1, \ldots, C_k$ are pairwise disjoint
   - for any $[C_j^{b_j}]\psi_j$ such that $C_j = \emptyset$, $b_j \geq \oplus_{C_{j'} \neq \emptyset} b_{j'}$
   - $v([C_j^{b_j}]\psi_j) = 1$ for all $j \leq k$

That is $M, s \models [C_j^{b_j}]\psi_j$ for all $j \leq k$.

If there is some non-empty C_j then, by super-additivity, we have that $M, s \models [C^b](\wedge_{j \leq k, C_j \neq \emptyset} \psi_j)$ where $C = \bigcup_{j \leq k} C_j$ and $b = \oplus_{j \leq k, C_j \neq \emptyset} b_j$. By coalition monotonicity, we have that $M, s \models [A^b](\wedge_{j \leq k, C_j \neq \emptyset} \psi_j)$. Furthermore, super-additivity implies that, for all $C_j = \emptyset$, $M, s \models [A^b](\wedge_{j \leq k} \psi_j)$. Because of playability, $\emptyset \notin E(s)(A^b)$, thus $V(\wedge_{j \leq k} \psi_j) \neq \emptyset$. Therefore, there exists $s' \in V(\wedge_{j \leq k} \psi_j)$ and it is straightforward that $M, s' \models \wedge_{j \leq k} \psi_j$.

If there is no non-empty C_j then super-additivity gives us directly that $M, s \models [\emptyset^b](\wedge_{j \leq k} \psi_j)$ in which $b = \min\{b_j \mid j \leq k\}$. Applying the same argument for playability, we have that there exists $s' \in V(\wedge_{j \leq k} \psi_j)$ and it is straightforward that $M, s' \models \wedge_{j \leq k} \psi_j$.

2. Assume that there are $[C_1^{b_1}]\psi_1, \ldots, [C_k^{b_k}]\psi_k \in cl(\varphi)$ for some $k > 1$ such that:

   - $t(b_j) = 1$ for all $j \leq k$
   - $C_1, \ldots, C_{k-1}$ are pairwise disjoint and all non-empty
   - $\bigcup_{j<k} C_j \subseteq C_k$
   - $\oplus_{j<k} b_j \leq b_k$
   - $v([C_j^{b_j}]\psi_j) = 1$ for all $j < k$
   - $v([C_k^{b_k}]\psi_k) = 0$

   That is, $M, s \models [C_j^{b_j}]\psi_j$ for all $j < k$ and $M, s \not\models [C_k^{b_k}]\psi_k$. By super-additivity, we have that $M, s \models [C^b](\wedge_{j<k} \psi_j)$ where $C = \bigcup_{j<k} C_j$ and $b = \oplus_{j<k} b_j$. That is, $V(\wedge_{j<k} \psi_j) \in E(s)(C^b)$. By coalition monotonicity and bound monotonicity, we have $V(\wedge_{j<k} \psi_j) \in E(s)(C_k^{b_k})$. Moreover, we already have $M, s \not\models [C_k^{b_k}]\psi_k$, thus, $V(\psi_k) \notin E(s)(C_k^{b_k})$. Then outcome monotonicity implies that $V(\psi_k) \not\supseteq V(\wedge_{j<k} \psi_j)$. Since $V(\wedge_{j<k} \psi_j) \neq \emptyset$, there must exist $s' \in V(\wedge_{j<k} \psi_j) \setminus V(\psi_k)$ and it is straightforward that $M, s' \models \wedge_{j<k} \psi_j \wedge \neg \psi_k$.

   In the case where $k = 1$, the proof is slightly different from above, as we do not have the set $V(\wedge_{j<k} \psi_j)$. However, we make the use of the first requirement of playability which states that $S \in E(s)(C_k^{b_k})$, therefore, $V(\psi_k) \neq S$. Hence, there also exists $s' \in S \setminus V(\psi_k)$ and it is obvious that $M, s' \models \neg \psi_k$.

Let us now prove the right-to-left direction of the lemma. The idea is that we construct a model satisfying the formula φ by collecting models which witness the satisfaction of the formulas in the two conditions of the lemma. That is, for any tuple $([C_j^{b_j}]\psi_j)_{j\leq k}$ of $cl(\varphi)$ which corresponds to one of the two conditions of the lemma, as $\wedge_{i\leq k}\psi_j$ (or $\wedge_{i<k}\psi_j \wedge \neg\psi_k$) is satisfiable, there is a model M' which satisfies $\wedge_{i\leq k}\psi_j$ (or $\wedge_{i<k}\psi_j \wedge \neg\psi_k$) at some state s' of M'. The model M we construct to satisfy φ will be the union of all such witnessing models M' together with a new state s_0 at which φ will be satisfied. We define the assignment function and the effectivity structure at a state of M by using the valuation function if the state is s_0 or the assignment function and the effectivity structures of the witness models otherwise. After constructing the model M, we also have to show that the effectivity structure of M is RB-playable so that M then is a model for φ. In the following, we detail the construction of M.

For each tuple of formulas $([C_j^{b_j}]\psi_j)_{j\leq k}$ of $cl(\varphi)$ which corresponds to one of two cases in the lemma, there is a model which satisfies its corresponding formula in which is either $\wedge_{i\leq k}\psi_j$ or $\wedge_{i<k}\psi_j \wedge \neg\psi_k$. Let $M_1, \dots, M_n$ be the enumeration of the above witnessing models in which $M_i = (S_i, E_i, V_i)$ such that, without loss of generality, all S_i's are assumed to be pairwise disjoint.

We construct a model $M = (S, E, V)$ as follows. The set of states S is the set $\bigcup_{i\leq n} S_i \cup \{s_0\}$ where s_0 is a new state. In order to define V, we firstly introduce a mapping $V_0 : cl(\varphi) \to \wp(\{s_0\})$ in which

$$V_0(\psi) = \begin{cases} \{s_0\} & \text{if } v(\psi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Then, we define an assignment $U : cl(\varphi) \to \wp(S)$ by $U(\psi) = \bigcup_{i=0,\dots,n} V_i(\psi)$. Note that by construction, we have $U(\neg\psi) = S \setminus U(\psi)$, $U(\psi_1 \vee \psi_2) = U(\psi_1) \cup U(\psi_2)$. Now we define the mapping V for M by the projection of U on the set of propositional variables p, that is, $V(p) = U(p)$ (without loss of generality, we can assume that all propositional variables are contained in $cl(\varphi)$).

Finally, we define the effectivity structure E in a way which is similar to that in the completeness proof.

For $C \neq A$ and b such that $t(b) = 1$, we put $X \subseteq S$ in $E(s)(C^b)$ if and only if $X = S$ or there are $[C_1^{b_1}]\psi_1, \dots, [C_k^{b_k}]\psi_k \in cl(\varphi)$ for some $k > 0$ such that:

- $t(b_j) = 1$ for all $j \leq k$
- $C_1, \dots, C_k$ are pairwise disjoint, and all non-empty if C is non-empty
- $\bigcup_{j\leq k} C_j \subseteq C$
- $\oplus_{j\leq k} b_j \leq b$ if $C \neq \emptyset$ or $b \leq b_j$ for all $j \leq k$ otherwise
- $\bigcap_{j\leq k} U(\psi_j) \subseteq X$ for all $j \leq k$

- $v([C_j^{b_j}]\psi_j) = 1$ for all $j \leq k$ if $s = s_0$
- $M_i, s \models [C_j^{b_j}]\psi_j$ for all $j \leq k$ if $s \in S_i$ for some $i \leq n$

For $t(b) = 1$, $X \in E(s)(A^b)$ if and only if $\overline{X} \notin E(s)(\emptyset^b)$. For the case when $t(b) > 1$ and $C \neq \emptyset$, we define $E(s)(C^b)$ inductively as follows: $X \in E(s)(C^b)$ iff one of the following conditions hold:

1. There is $b' < b$ such that $X \in E(s)(C^b)$
2. There are $b_1 \oplus b_2 = b$ such that $\{s' \mid X \in E(s')(C^{b_2})\} \in E(s)(C^{b_1})$

Then, we define for $t(b) > 1$, $X \in E(s)(\emptyset^b)$ iff $\overline{X} \notin E(s)(A^b)$.

Before proving that the model M which we have just constructed is indeed a model for φ, we must show that E is an RB-playable effectivity structure.

Claim 1. *The effectivity structure E is RB-playable.*

Proof.
- We show the first two properties of RB-playability by induction on bounds.

  Let $t(b) = 1$ and $C \neq A$. The definition of E implies directly that $S \in E(s)(C^b)$.

  Moreover, $S \in E(s)(\emptyset^b)$ implies that $\emptyset \notin E(s)(A^b)$ also by the definition of E.

  Let $t(b) = 1$ and $C \neq A$. Assume to the contrary that $\emptyset \in E(s_0)(C^b)$. Hence, there are $[C_1^{b_1}]\psi_1, \ldots, [C_k^{b_k}]\psi_k \in cl(\varphi)$ for some $k > 0$ such that:

  - $t(b_j) = 1$ for all $j \leq k$
  - $C_1, \ldots, C_k$ are pairwise disjoint, and all non-empty if C is not empty
  - $\bigcup_{j \leq k} C_j \subseteq C$
  - $\oplus_{j \leq k} b_j \leq b$ if $C \neq \emptyset$ or $b \leq b_j$ for all $j \leq k$ otherwise
  - $\bigcap_{j \leq k} U(\psi_j) \subseteq X$ for all $j \leq k$
  - $v([C_j^{b_j}]\psi_j) = 1$ for all $j \leq k$ if $s = s_0$
  - $M_i, s \models [C_j^{b_j}]\psi_j$ for all $j \leq k$ if $s \in S_i$ for some $i \leq n$

  Then $\wedge_{j \leq k}\psi_j \equiv \bot$, which contradicts the first condition of the lemma where $\bot$ is required to be satisfiable.

  Similarly to the case when $s \neq s_0$, we can show that $\emptyset \notin E(s)(C^b)$ for $C \neq A$.

Then $\emptyset \notin E(s)(\emptyset^b)$ implies that $S \in E(s)(A^b)$, again by the definition of E.

In the induction step, let $t(b) > 1$ and $C \neq \emptyset$, we directly have that $S \in E(s)(C^b)$ as $S \in E(s)(C^{b'})$ for any $b' < b$ and $t(b') = 1$. $S \in E(s)(A^b)$ also implies that $\emptyset \notin E(s)(\emptyset^b)$. Moreover, by the induction hypothesis, we have that $\emptyset \notin E(s)(C^{b'})$ for any $b' < b$. Furthermore, for any $b_1 \otimes b_2 = b$, we have that $\{s' \mid \emptyset \in E(s')(C^{b_2})\} = \emptyset$ and $\emptyset \notin E(s)(C^{b_1})$ also because of the induction hypothesis. By the definition of E, it follows that $\emptyset \notin E(s)(C^b)$. Once agent, $\emptyset \notin E(s)(A^b)$ implies that $S \in E(s)(\emptyset^b)$.

- Let us now show outcome monotonicity.

  Let $t(b) = 1$ and $C \neq A$. Assume that $X \in E(s)(C^b)$ in which $X \subset S$. By the definition of E, there are $[C_1^{b_1}]\psi_1, \ldots, [C_k^{b_k}]\psi_k \in cl(\varphi)$ for some $k > 0$ such that:

  - $t(b_j) = 1$ for all $j \leq k$
  - $C_1, \ldots, C_k$ are pairwise disjoint, and all non-empty if C is not empty
  - $\bigcup_{j \leq k} C_j \subseteq C$
  - $\oplus_{j \leq k} b_j \leq b$ if $C \neq \emptyset$ or $b \leq b_j$ for all $j \leq k$ otherwise
  - $\bigcap_{j \leq k} U(\psi_j) \subseteq X$ for all $j \leq k$
  - $v([C_j^{b_j}]\psi_j) = 1$ for all $j \leq k$ if $s = s_0$
  - $M_i, s \models [C_j^{b_j}]\psi_j$ for all $j \leq k$ if $s \in S_i$ for some $i \leq n$

  It is straightforward that for any $X' \supseteq X$, we have $\bigcap_{j \leq k} U(\psi_j) \subseteq X \subseteq X'$. Hence, $X' \in E(s)(C^b)$.

  In the case of the grand coalition, assume that $X \in E(s)(A^b)$. By the definition of E, we have $\bar{X} \notin E(s)(\emptyset^b)$. Assume to the contrary that $X' \notin E(s)(A^b)$ for some $X' \supseteq X$. It follows that $\bar{X}' \subseteq \bar{X}$. $X' \notin E(s)(A^b)$ implies that $\bar{X}' \in E(s)(\emptyset^b)$, hence $\bar{X} \in E(s)(\emptyset^b)$, which is a contradiction.

  Now we provide a proof of outcome monotonicity for the case when $t(b) > 1$. It is easy to notice that it is similar to the proof of completeness of $RBCL$.

  Let $t(b) > 1$ and $C \neq \emptyset$. If $X \in E(s)(C^{b'})$ for some $b' < b$, the induction hypothesis shows that $X' \in E(s)(C^{b'})$ for all $X' \supseteq X$. Then by the definition of E, we have $X' \in E(C^b)(s)$. Assume $X \notin E^(s)(C^{b'})$ for all $b' < b$. By the definition of E, there are b_1, b_2 such that $b_1 \otimes b_2 = b$ and

  $$\{s' \mid X \in E(s')(C^{b_2})\} \in E(s)(C^{b_1})$$

Let $X' \supseteq X$. By the induction hypothesis we have

$$\{s' \mid X \in E(s')(C^{b_2})\} \subseteq \{s' \mid X' \in E(s')(C^{b_2})\}$$
$$\Rightarrow \{s' \mid X' \in E(s')(C^{b_2})\} \in E(s)(C^{b_1})$$

By the definition of E, we have $X' \in E(s)(C^b)$.
If $t(b) > 1$, $X \in E(s)(\emptyset^b)$ iff $\overline{X} \notin E(s)(A^b)$. Let $X' \supseteq X$, assume by contradiction that $X' \notin E(s)(\emptyset^b)$. This implies that $\overline{X'} \in E(s)(A^b)$. By the previous proof, we have $\overline{X} \in E(s)(A^b)$ as $\overline{X'} \subseteq \overline{X}$, which is a contradiction.

- A-maximality and regularity follow directly from the definition of E for $\emptyset$ when $t(b) = 1$ and also $t(b) > 1$. Therefore, we omit the proof here.
- Let us now prove super-additivity. Let $t(b) = t(d) = 1$, $C \cap D = \emptyset$ with $X \in E(s)(C^b)$ and $Y \in E(s)(D^d)$.
  - If both C and D are non-empty. Assume that both X and Y are not equal to S. By the definition of E, we have that there are $[C_1^{b_1}]\psi_1, \dots, [C_{k_C}^{b_{k_C}}]\psi_{k_C} \in cl(\varphi)$ and $[D_1^{d_1}]\psi'_1, \dots, [D_{k_D}^{d_{k_D}}]\psi'_{k_D} \in cl(\varphi)$ for some $k_C > 0$ and $k_D > 0$ such that:
    - $t(b_j) = 1$ for all $j \leq k_C$
    - $t(d_j) = 1$ for all $j \leq k_D$
    - $C_1, \dots, C_{k_C}$ are pairwise disjoint, and all non-empty
    - $D_1, \dots, D_{k_D}$ are pairwise disjoint, and all non-empty
    - $\bigcup_{j \leq k_C} C_j \subseteq C$
    - $\bigcup_{j \leq k_D} D_j \subseteq D$
    - $\oplus_{j \leq k_C} b_j \leq b$
    - $\oplus_{j \leq k_D} d_j \leq d$
    - $\bigcap_{j \leq k_C} U(\psi_j) \subseteq X$ for all $j \leq k_C$
    - $\bigcap_{j \leq k_D} U(\psi'_j) \subseteq Y$ for all $j \leq k_D$
    - $v([C_j^{b_j}]\psi_j) = 1$ for all $j \leq k_C$ if $s = s_0$
    - $v([D_j^{d_j}]\psi'_j) = 1$ for all $j \leq k_D$ if $s = s_0$
    - $M_i, s \models [C_j^{b_j}]\psi_j$ for all $j \leq k_C$ if $s \in S_i$ for some $i \leq n$
    - $M_i, s \models [D_j^{d_j}]\psi'_j$ for all $j \leq k_D$ if $s \in S_i$ for some $i \leq n$

    Then, it is straightforward that $X \cap Y \supseteq \bigcap_{j \leq k_C} U(\psi_j) \cap \bigcap_{j \leq k_D} U(\psi'_j)$.
    It follows that $X \cap Y \in E((C \cup D)^{b \oplus d})$.
    In the case when Y is S, the proof is similar to above, using $C \subseteq C \cup D$ and $b \leq b \oplus d$.

– If $C = D = \emptyset$ and $b = d$, we apply the same argument as the case above.

– If $C = \emptyset, b = d$ and $D = A$, we need to show that $X \cap Y \in E(A^b)(s)$. Assume to the contrary that $X \cap Y \notin E(A^b)(s)$, then A-maximality, which has been proved above, implies that $\overline{X \cap Y} \in E(\emptyset^b)(s)$. Then, by the previous case of super-additivity, we have $X \cap \overline{Y} \in E(\emptyset^b)(s)$. As we already showed outcome-monotonicity, $\overline{Y} \in E(\emptyset^b)(s)$. However, by A-regularity, we have $Y \notin E(A^b)(s)$ which is a contradiction.

- Super-transitivity follows directly from the definition of E when $t(b) > 1$.
- Similarly, transitivity follows directly from the definition of E when $t(b) > 1$.

□

Therefore, E is RB-playable. In order to show that M satisfies φ, we prove the following two claims.

Claim 2. *For any $[C^b]\psi \in cl(\varphi)$, $U(\psi) \in E(s)(C^b)$ iff $v([C^b]\psi) = 1$ if $s = s_0$ or $M_i, s \models [C^b]\psi$ if $s \in S_i$ for some $i \leq n$.*

Proof. The direction from right to left is straightforward according to the definition of E. Hence, we provide here only a proof for the other direction.

The case where $U(\psi) = S$ is trivial and we ignore it here.

We prove the claim by induction on the resource bounds.

Let $t(b) = 1$ and $C \neq A$. As $U(\psi) \in E(s)(C^b)$, there are $[C_1^{b_1}]\psi_1, \dots, [C_k^{b_k}]\psi_k \in cl(\varphi)$ for some $k > 0$ such that:

- $t(b_j) = 1$ for all $j \leq k$
- $C_1, \dots, C_k$ are pairwise disjoint, and all non-empty if C is not empty
- $\bigcup_{j \leq k} C_j \subseteq C$
- $\oplus_{j \leq k} b_j \leq b$ if $C \neq \emptyset$ or $b \leq b_j$ for all $j \leq k$ otherwise
- $\bigcap_{j \leq k} U(\psi_j) \subseteq U(\psi)$ for all $j \leq k$
- $v([C_j^{b_j}]\psi_j) = 1$ for all $j \leq k$ if $s = s_0$
- $M_i, s \models [C_j^{b_j}]\psi_j$ for all $j \leq k$ if $s \in S_i$ for some $i \leq n$

Suppose $s \in S_i$ for some $i \leq n$. As $M_i, s \models [C_j^{b_j}]\psi_j$ for all $j \leq k$, superadditivity implies that $M_i, s \models [\bigcup_{j \leq k} C_j^{\oplus_{j \leq k} b_j}](\wedge_{j \leq k}\psi_j)$ if $C \neq \emptyset$ or directly, $M_i, s \models [C^b](\wedge_{j \leq k}\psi_j)$ otherwise. In the former case, coalition monotonicity gives us $M_i, s \models [C^b](\wedge_{j \leq k}\psi_j)$. Then, in both cases, we can conclude by outcome-monotonicity that $M_i, s \models [C^b](\psi)$.

When $s = s_0$, assume by contradiction that $v([C^b]\psi) = 0$. Then there is a witnessing model M_i and $s' \in S_i$ such that $M_i, s' \models \wedge_{j \leq k}\psi_i \wedge \neg\psi$, which contradicts the fact that $\bigcap_{j \leq k} U(\psi_j) \subseteq U(\psi)$.

Let $t(b) = 1$ and $C = A$. By the definition of E, $U(\psi) \in E(s)(A^b)$ iff $U(\neg\psi) \notin E(s)(\emptyset^b)$. By the proof above, $U(\neg\psi) \notin E(s)(\emptyset^b)$ iff $v([\emptyset^b]\neg\psi) = 0$ if $s = s_0$ or $M_i, s \not\models [\emptyset^b]\neg\psi$ if $s \in S_i$ for some $i \leq n$. By the definition of v, we have that $v([\emptyset^b]\neg\psi) = 0$ implies $v([A^b]\psi) = 1$. Moreover, by A-maximality, we also have $M_i, s \not\models [\emptyset^b]\neg\psi$ implies that $M_i, s \models [A^b]\psi$.

Let $t(b) > 1$ and $C \neq \emptyset$. We have that $U(\psi) \in E(s)(C^b)$ iff $U(\psi) \in E(s)(C^{b'})$ for some $b' < b$ or there are $b_1 \otimes b_2 = b$ such that $\{s' \mid U(\psi) \in E(s)(C^{b_2})\} \in E(s)(C^{b_1})$. If $U(\psi) \in E(s)(C^{b'})$, by the induction hypothesis, $v([C^{b'}]\psi) = 1$ if $s = s_0$ or $M_i, s \models [C^{b'}]\psi$ if $s \in S_i$ for some $i \leq n$. By the definition of v, $v([C^{b'}]\psi) = 1$ implies that $v([C^b]\psi) = 1$. By RB-playability, $M_i, s \models [C^{b'}]\psi$ implies $M_i, s \models [C^b]\psi$.

If there are $b_1 \otimes b_2 = b$ such that $U([C^{b_2}]\psi) = \{s' \mid U(\psi) \in E(s)(C^{b_2})\} \in E(s)(C^{b_1})$, by the induction hypothesis, $v([C^{b_1}][C^{b_2}]\psi) = 1$ if $s = s_0$ or $M_i, s \models [C^{b_1}][C^{b_2}]\psi$ if $s \in S_i$ for some $i \leq n$. By the definition of v, $v([C^{b_1}][C^{b_2}]\psi) = 1$ implies that $v([C^b]\psi) = 1$. By RB-playability, $M_i, s \models [C^{b_1}][C^{b_2}]\psi$ implies $M_i, s \models [C^b]\psi$.

Let $t(b) > 1$ and $C = \emptyset$. By the definition of E, $U(\psi) \in E(s)(\emptyset^b)$ iff $U(\neg\psi) \notin E(s)(A^b)$. By the proof above, $U(\neg\psi) \notin E(s)(A^b)$ iff $v([A^b]\neg\psi) = 0$ if $s = s_0$ or $M_i, s \not\models [A^b]\neg\psi$ if $s \in S_i$ for some $i \leq n$. By the definition of v, we have that $v([A^b]\neg\psi) = 0$ implies $v([\emptyset^b]\psi) = 1$. Moreover, by A-maximality, we also have $M_i, s \not\models [A^b]\neg\psi$ implies that $M_i, s \models [\emptyset^b]\psi$. □

Claim 3. *V and U agree on $cl(\varphi)$.*

Proof. In the base case, the proof is trivial as according to the definition of V, they already agree on the set of propositions in $cl(\varphi)$. The proof for propositional connectives is also straightforward as we know that $U(\neg\psi) = S \backslash U(\psi)$ and $U(\psi_1 \vee \psi_2) = U(\psi_1) \cup U(\psi_2)$, and similarly for V. For the case of $[C^b]\psi$, the proof is by induction on the resource bounds.

Assume that $s \in U([C^b]\psi)$, then by the definition of U, $v([C^b]\psi) = 1$ if $s = s_0$ or $M_i, s \models [C^b]\psi$ if $s \in S_i$ for some $i \leq n$.

If $t(b) = 1$ and $C \neq A$, then in both above cases, by the definition of E, we have that $U(\psi) \in E(s)(C^b)$. By the induction hypothesis, $U(\psi) = V(\psi)$, hence $V(\psi) \in E(s)(C^b)$ and therefore $M, s \models [C^b]\psi$.

If $t(b) = 1$ and $C = A$, then we have $v([\emptyset^b]\neg\psi) = 0$ if $s = s_0$ or $M_i, s \not\models [\emptyset^b]\neg\psi$ if $s \in S_i$ for some $i \leq n$. In both cases, by the definition of E, we have that $U(\neg\psi) \notin E(s)(\emptyset^b)$, otherwise, $U(\neg\psi) \in E(s)(\emptyset^b)$ will contradict Claim 2. Hence, $U(\psi) = V(\psi) \in E(s)(A^b)$ because E is RB-playable. Then $M, s \models [A^b]\psi$ and $s \in V([A^b]\psi)$.

Assume $t(b) > 1$ and $C \neq \emptyset$. If $s = s_0$ and $v([C^b]\psi) = 1$, then either $v([C^{b'}]\psi) = 1$ for some $b' < b$, or there are $b_1 \otimes b_2 = b$ such that $v([C^{b_1}][C^{b_2}]\psi) = 1$. In both cases, by the induction hypothesis together with the definition of E, we have that $s \in V([C^b]\psi)$. Similarly, if $s \in S_i$ and $M_i, s \models [C^b]\psi$, either $M_i, s \models [C^{b'}]\psi$ or $M_i, s \models [C^{b_1}][C^{b_2}]\psi$. Again, in both cases, by the induction hypothesis together with the definition of E, we imply that $s \in V([C^b]\psi)$.

If $t(b) > 1$ and $C = \emptyset$, then we have $v([A^b]\neg\psi) = 0$ if $s = s_0$ or $M_i, s \not\models [A^b]\neg\psi$ if $s \in S_i$ for some $i \leq n$. In both cases, by the definition of E, we have that $U(\neg\psi) \notin E(s)(A^b)$, otherwise, $U(\neg\psi) \in E(s)(A^b)$ will contradict Claim 2. Hence $U(\psi) = V(\psi) \in E(s)(\emptyset^b)$ because E is RB-playable. Then, $M, s \models [\emptyset^b]\psi$. Therefore $s \in V([\emptyset^b]\psi)$.

Assume that $s \in V([C^b]\psi)$, that is $M, s \models [C^b]\psi$. Therefore, $V(\psi) = U(\psi) \in E(s)(C^b)$. By the above claim, we have $v([C^b]\psi) = 1$ if $s = s_0$ or $M_i, s \models [C^b]\psi$ if $s \in S_i$ for some $i \leq n$. In both cases, the definition of U gives us $s \in U([C^b]\psi)$. □

Finally, we complete the proof for Lemma 2. Since $v(\varphi) = 1$, $s_0 \in U(\varphi)$. Therefore, by Claim 3, we have $s_0 \in V(\varphi)$, hence, $M, s_0 \models \varphi$. In other words, φ is satisfiable.

□

We conclude this section by stating a complexity result for $RBCL$.

Theorem 5. *The problem whether a formula ϕ of $RBCL$ is satisfiable is in PSPACE($|cl(\phi)|$).*

Proof. The satisfiability problem of $RBCL$ is PSPACE-hard; this can be easily shown by reducting the satisfiability problem of the modal logic K (which is PSPACE-complete [12]) to that of $RBCL$. Consider the following reduction function tr from the language of K to the language of $RBCL$ with a single agent and a single resource (time). We set $tr(p) = p$, tr commutes with the booleans, and $tr(\Diamond\phi) = [\{1\}^1]tr(\phi)$. It is easy to see that a modal formula ϕ is K-satisfiable if, and only if, $tr(\phi)$ is $RBCL$-satisfiable (accessibility relation between states s

and s' in the K-model for ϕ corresponds to s' being an outcome of some action by agent 1 in state s in the $RBCL$ action model).

Next we are going to show that the satisfiability problem is in PSPACE($|cl(\phi)|$) by giving an algorithm which decides whether ϕ is satisfiable while using space polynomial in $|cl(\phi)|$. Before giving the algorithm for deciding $RBCL$-satisfiability, we need to introduce some notation. As defined above, given a closure $cl(\varphi)$, let $CON(\varphi)$ be the set of all finite nonempty subsets $\{[C_1^{b_1}]\psi_1, \ldots, [C_k^{b_k}]\psi_k\} \subseteq cl(\varphi)$ which match either the first or the second condition of Lemma 2. Moreover, each set $\Gamma = \{[C_1^{b_1}]\psi_1, \ldots, [C_k^{b_k}]\psi_k\} \in CON(\varphi)$ is associated with a formula, denoted as φ_Γ, which is in the form of either $\wedge_{i \leq k}\psi_i$ or $\wedge_{i<k}\psi_i \wedge \neg\psi_k$, depending on whether Γ is for the first or the second condition of Lemma 2, respectively. Then, the algorithm for the satisfiability problem, given a formula φ, is as follows.

1. Non-deterministically select a valuation v for $cl(\varphi)$.

2. For every set $\Gamma \in CON(\varphi)$, recursively check that φ_Γ is satisfiable.

Note that the algorithm requires space polynomial in $cl(\varphi)$ to record the valuation and check that it satisfies the conditions of Lemma 2. Therefore its complexity is NPSPACE (hence PSPACE, since PSPACE=NPSPACE) in $|cl(\varphi)|$. □

If we measure the size of the input to the algorithm (the formula φ) assuming that the resource bounds are written in *unary*, then the algorithm is PSPACE in $|\varphi|$ (since in this case the size of $cl(\varphi)$ is polynomial in $|\varphi|$). However, if the resource bounds are written in binary, then $|cl(\varphi)|$ is exponential in $|\varphi|$ and hence the algorithm requires space exponential in $|\varphi|$ to record the valuation.

5.1 Example

Let us consider a formula $\varphi_0 = [1^1]p \wedge [2^2](\neg p)$. Note that, for convenience, we write $[1^1]\varphi$ instead of $[\{1\}^1]\varphi$. We now follow Lemma 2 to show that φ_0 is satisfiable and how the model is constructed. For simplicity, we assume that the set of agents contains 2 agents and the set of resources contains only time.

Firstly, we compute the set $cl(\varphi_0)$,

$$\begin{array}{rcl} cl(\varphi_0) & = & \{\varphi_0, [1^1]p, [2^2](\neg p), [2^1](\neg p), [2^1][2^1](\neg p), p, \\ & & \neg\varphi_0, \neg[1^1]p, \neg[2^2](\neg p), \neg[2^1](\neg p), \neg[2^1][2^1](\neg p), \neg p\} \end{array}$$

Then, to show that φ_0 is satisfiable, we introduce a valuation v which assigns 1 for formulas φ_0, $[1^1]p$, $[2^2](\neg p)$, $\neg[2^1](\neg p)$, $[2^1][2^1](\neg p)$ and $\neg p$ and assigns 0 to all other formulas in $cl(\varphi_0)$. We need to show for the following tuples of formulas from $cl(\varphi_0)$, that the corresponding formulas determined by Lemma 2, are satisfiable.

- For the tuple $([1^1]p)$ with $v([1^1]p) = 1$, which matches case 1 of Lemma 2, the satisfaction of the sub-formula p is immediate by the following model $M_1 = (\{s_1\}, E_1, V_1)$ in which $E_1(s_1)(C^b) = \{s_1\}$ for all $C \subseteq \{1, 2\}$ and $b \geq 1$ and $V_1(p) = \{s_1\}$.

- For the tuple $([2^1][2^1](\neg p))$, the satisfaction of the sub-formula $\varphi_1 = [2^1](\neg p)$ is shown by applying Lemma 2 again. We have

$$cl(\varphi_1) \quad = \quad \{\varphi_1, p, \neg\varphi_1, \neg p\}$$

  Consider the valuation v_1 which assigns 1 to only φ_1 and p; we just need to show that for the tuple $([2^1](\neg p))$, the sub-formula $\neg p$ is satisfiable. This is immediate by the existence of a model $M_2 = (\{s_2\}, E_2, V_2)$ in which $E_2(s_2)(C^b) = \{s_2\}$ for all $C \subseteq \{1, 2\}$ and $b \geq 1$ and $V_2(p) = \emptyset$. We construct a model which satisfies φ_1 according to Lemma 2, by using M_2 and a new state s_3. Let us call this model $M_3 = (\{s_2, s_3\}, E_3, V_3)$, in which E_3 and V_3 are as follows (notice that we only provide the description of E_3 for the case when the bound is 1, the cases of the greater bounds are defined inductively on the cases of the smaller ones as shown in the proof of Lemma 2).

$$\begin{array}{rcl}
E_3(s_3)(\emptyset^1) & = & \{S\} \text{ since no formula } [\emptyset^1]\psi \text{ is in } cl(\varphi_1) \\
E_3(s_3)(1^1) & = & \{S\} \\
E_3(s_3)(2^1) & = & \{\{s_2\}, S\} \text{ since } [2^1](\neg p) \text{ is in } cl(\varphi_1) \\
E_3(s_3)(\{1,2\}^1) & = & \{X \mid X \neq \emptyset\} \\
E_3(s_2)(\emptyset^1) & = & \{S\} \\
E_3(s_2)(1^1) & = & \{S\} \\
E_3(s_2)(2^1) & = & \{\{s_2\}, S\} \\
E_3(s_2)(\{1,2\}^1) & = & \{X \mid X \neq \emptyset\} \\
V_3(p) & = & \{s_3\}
\end{array}$$

- For the tuple $([1^1]p, [2^1][2^1](\neg p))$, we will show that $\varphi_2 = p \wedge [2^1](\neg p)$ is satisfiable by applying Lemma 2 again as follows.

  We have

$$cl(\varphi_2) \quad = \quad \{\varphi_2, p, [2^1](\neg p), \neg\varphi_2, \neg p, \neg[2^1](\neg p)\}$$

Then we consider a valuation v_2 which assigns 1 to φ_2, p and $[2^1](\neg p)$, and 0 to the other formulas in $cl(\varphi_2)$. Again, according to the condition of Lemma 2, we need to show that, for the tuple $([2^1](\neg p))$ with $v_2([2^1](\neg p)) = 1$ we have $\neg p$ is satisifiable, which is immediate by the model $M_4 = (\{s_4\}, E_4, V_4)$ in which $E_4(s_4)(C^b) = \{s_4\}$ for all $C \subseteq \{1, 2\}$ and $b \geq 1$ and $V_4(p) = \emptyset$. We construct a model which satisfies φ_2 according to Lemma 2, by using M_4 and a new state s_5. Let us call this model $M_5 = (\{s_4, s_5\}, E_5, V_5)$, in which E_5 and V_5 are as follows:

$$\begin{array}{rcl}
E_5(s_5)(\emptyset^1) & = & \{S\} \text{ since no formula } [\emptyset^p]\psi \text{ is in } cl(\varphi_2) \\
E_5(s_5)(1^1) & = & \{S\} \\
E_5(s_5)(2^1) & = & \{\{s_4\}, S\} \\
E_5(s_5)(\{1,2\}^1) & = & \{X \mid X \neq \emptyset\} \\
E_5(s_4)(\emptyset^1) & = & \{S\} \\
E_5(s_4)(1^1) & = & \{S\} \\
E_5(s_4)(2^1) & = & \{\{s_4\}, S\} \\
E_5(s_4)(\{1,2\}^1) & = & \{X \mid X \neq \emptyset\} \\
V_5(p) & = & \{s_5\}
\end{array}$$

- For the tuple $([2^1](\neg p))$ with $v([2^1](\neg p)) = 0$ which matches case 2 of Lemma 2, the satisfaction of the sub-formula $\neg(\neg p) = p$ is immediate by the model $M_6 = (\{s_6\}, E_6, V_6)$, in which $E_6(s_6)(C^b) = \{s_6\}$ for all $C \subseteq \{1, 2\}$ and $b \geq 1$ and $V_6(p) = \{s_6\}$.

Therefore by Lemma 2 φ_0 is satisfiable. We construct the model satisfying φ_0, according to Lemma 2, by aggregating models $M_1, \ldots, M_6$ into $M = (S, E, V)$ where $S = \{s_0, s_1, \ldots, s_6\}$, $V(p) = \{s_1, s_3, s_5, s_6\}$ and E at s_0 and s_1 is as follows (E at other states is defined in a similar way):

$$\begin{array}{rcl}
E(s_0)(\emptyset^1) & = & \{S\} \\
E(s_0)(1^1) & = & \{X \mid \{s_1, s_3, s_5, s_6\} \subseteq X\} \\
E(s_0)(2^1) & = & \{X \mid \{s_2, s_3, s_4, s_5\} \subseteq X\} \\
E(s_0)(\{1,2\}^1) & = & \{X \mid X \neq \emptyset\} \\
E(s_1)(\emptyset^1) & = & \{S\} \\
E(s_1)(1^1) & = & \{X \mid \{s_1, s_3, s_5, s_6\} \subseteq X\} \\
E(s_1)(2^1) & = & \{S\} \text{ since } M_1, s_1 \not\models [2^1][2^1](\neg p) \\
E(s_1)(\{1,2\}^1) & = & \{X \mid X \neq \emptyset\}
\end{array}$$

6 Model-checking $RBCL$

In this section we describe a model-checking algorithm for $RBCL$. Our is similar to the model-checking algorithm for ATL in [7] and runs in time polynomial in the size of the formula and the structure (if we treat the number of resources as a constant, and measure the size of the input formula assuming that resource bounds are written in unary). The main differences between the ATL model-checking algorithm given by Alur *et al*. and that presented here are that we take the costs of strategies into account, and, instead of working with a straightforward set of subformulas $Sub(\phi)$ of a given formula ϕ, we work with an extended set of subformulas $Sub^+(\phi)$. $Sub^+(\phi)$ includes $Sub(\phi)$, and in addition:

- if $[C^b]\psi \in Sub(\phi)$, then $[C^{b'}]\psi \in Sub^+(\phi)$ for all $b' < b$

Note that $|\{b' \mid b' \leq b\}| = b^r$ where $r = |Res|$. Therefore $|Sub^+(\phi)|$ is polynomial in $|\phi|$ if resource bounds are written in unary, and exponential if resource bounds are written in binary.

Theorem 6. *Given a multi-step resource-bounded action model $M = (A, R, \Sigma, S, T, o, Res, V)$ and a formula ϕ, there is an algorithm which returns the set of states $[\phi]_M$ satisfying ϕ: $[\phi]_M = \{s \mid M, s \models \phi\}$, which runs in time $O(|Sub^+(\phi)|^2 \times |M|)$. In other words, the algorithm is polynomial in the size of the model, and is polynomial in the size of the formula if the resource bounds are written in unary. If the resource bounds are written in binary, the algorithm is exponential in $|\phi|$.*

Proof. Consider the following model-checking algorithm:

for every ϕ' in $Sub^+(\phi)$:

case $\phi' == p$: $[\phi']_M = \{s \mid s \in V(s)\}$

case $\phi' == \neg\psi$: $[\phi']_M = S \setminus [\psi]_M$

case $\phi' == \psi_1 \wedge \psi_2$: $[\phi']_M = [\psi_1]_M \cap [\psi_2]_M$

case $\phi' == [C^b]\psi$ with $t(b) = 1$: $[\phi']_M = Pre(C, [\psi]_M, b)$

case $\phi' == [C^b]\psi$ with $t(b) > 1$:
    $\rho := [false]$;
    foreach $b' < b$ **do**
        $\rho = \rho \cup [[C^{b'}]\psi]_M \cup Pre(C, [[C^{b'}]\psi]_M, b \ominus b')$
    od;
    $[\phi']_S := \rho$

where Pre is a function which, given a coalition C, a set $\rho \subseteq S$ and a bound b, returns all states s in which C has a joint action σ_C with cost $Res(\sigma_C) \leq b$ such that $o(s, \sigma_C) \subseteq \rho$. $\ominus$ is the inverse of $\otimes$, in particular if $c \leq b \ominus b'$ then $b' \otimes c \leq b$.

The first three cases are straightforward. For the $[C^b]$ cases, consider any state s_1 from which ψ can be enforced by C within the bound b. Either there exists a one step strategy σ in $s_1$to do this (and s_1 is in $[[C^b]\psi]_M$, $t(b) = 1$), or σ is multi-step. In the latter case, where $t(b) > 1$, the resource bounds $b' < b$ are enumerated in increasing order of $t(b)$, and in an order consistent with $<$ on vectors: if $b_1 < b_2$, then b_1 is before b_2 in the enumeration. Note that on each iteration we only need to include in $[[C^b]\psi]_M$ those states from which there is a single step strategy with cost $c \leq b \ominus b'$ to enforce $[C^{b'}]\psi$, $b' < b$. Consider the cost c of the first step of a multi-step strategy σ. By executing the first step of σ, C enforces a set of states in which they have a strategy of cost $b' \leq b \ominus c$ to enforce ψ, in other words a set of states satisfying $[C^{b'}]\psi$. So s_1 is in $Pre(C, [[C^{b'}]\psi]_M, b \ominus b')$ for some $[C^{b'}]\psi$, $b' \leq b \ominus c$, and all such sets of states are eventually enumerated by the algorithm.

In the the $\phi' == [C^b]\psi$ case, the loop is executed $|Sub^+(\phi)|$ times. The function Pre can be computed in time linear in M. This gives us complexity $O(|Sub^+(\phi)|^2 \times |M|)$. If r is treated as a constant factor and resource bounds are written in unary, we get complexity polynomial in $|\phi|$ and $|M|$.[2] □

$Pre(C, \rho, b)$ can be implemented symbolically, in a way similar to the computation of $Pre(C, \rho)$ in ATL (see, for example, [8]). One way to encode the costs of actions would be to add to each agent i's state a set of boolean variables B_i representing the agent's 'endowment' of resources. This endowment is decremented in the successor state by the cost of the action the agent performs in order to reach that state. The fact that the action executed by the agent costs at most b can then be expressed as a boolean expression $\delta_b(B_i, B_i')$, where B_i' are the values of the variables B_i in the successor state, and δ_b is a suitable boolean arithmetic expression.

7 Conclusions and further work

Alternating-Time Temporal Logic and Coalition Logic [13, 10, 14, 7, 17] allow the expression of many interesting properties of coalitions and strategies. However, there is no natural way of expressing resource requirements in these logics. Logics such as RTL* [9] which introduce resource bounds into temporal logic, allow only the analysis of single agent systems. In this paper, we have proposed a complete

[2] We also treat the number of agents as a constant factor: the complexity of ATL model-checking without this assumption was shown to be exponential in [11]. Other assumptions implicit in the formulation of the problem, e.g., that the set of states is given explicitly, are discussed in [15].

and sound logic $RBCL$ in which we can express the costs of (multi-step) strategies, and hence coalitional ability under resource bounds in multi-shot games. $RBCL$ is related to both Coalition Logic and ATL. The $[C^b]$ operators in $RBCL$ (without resource bounds) correspond to the $[C^*]$ operator in Extended Coalition Logic [13] (which stands for a finite iteration of $[C]$ modalities), and to the $\langle\langle C \rangle\rangle F$ operator of ATL [7]. $RBCL$ is sufficiently expressive to formalise, e.g., the decision problems for Coalitional Resource Games discussed in [18], and the properties of resource-bounded communicating reasoners investigated in [2]. We showed how to verify properties expressed in $RBCL$ and give a decision procedure for the satisfiability problem of $RBCL$ and a model-checking algorithm.

However there are some properties which cannot be expressed in $RBCL$. For example, we cannot express properties such as 'coalition C has a strategy to maintain the property ϕ with resources b', or 'C can maintain ϕ until ψ becomes true provided C has resources b'. Such properties (without resource bounds) can be expressed in Alternating-Time Temporal Logic (ATL), and in future work we plan to investigate extending ATL with costs of actions and hence of strategies. Other possible directions for future work include optimising the model-checking algorithm for $RBCL$ by exploiting resource bounds or by using bounded model-checking for some properties.

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