

Convergence to Equilibrium in Degenerate Parabolic Equations with Delay

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Abstract

In [10], Busenberg & Huang (1996) showed that small positive equilibria can undergo supercritical Hopf bifurcation in a delay-logistic reaction-diffusion equation with Dirichlet boundary conditions. Consequently, stable spatially inhomogeneous time-periodic solutions exist. Previously in [5] Badii, Diaz & Tesei (1987) considered a similar logistic-type delay-diffusion equation, but differing in two important respects: firstly by the inclusion of nonlinear degenerate diffusion of so-called porous medium type, and secondly by the inclusion of an additional ‘dominating instantaneous negative feedback’ (where terms local in time majorize the delay terms, in some sense). Sufficient conditions were given ensuring convergence of non-negative solutions to a unique positive equilibrium.

A natural question to ask, and one which motivated the present work, is: can one still ensure convergence to equilibrium in delay-logistic diffusion equations in the presence of nonlinear degenerate diffusion, but in the absence of dominating instantaneous negative feedback? The present paper considers

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this question and provides sufficient conditions to answer in the affirmative. In fact the results are much stronger, establishing global convergence for a much wider class of problems which generalize the porous medium diffusion and delay-logistic terms to larger classes of nonlinearities. Furthermore the results obtained are independent of the size of the delay.

Keywords: Delay, degenerate, parabolic, diffusion, logistic, convergence

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^n with boundary of class $C^{2+\alpha}$ for some $\alpha \in (0, 1)$. Define $Q_T = \Omega \times (0, T]$, $S_T = \partial\Omega \times (0, T]$ and $\Gamma_r = \overline{\Omega} \times [-r, 0]$. We consider the following nonlinear degenerate diffusion equation with delay

$$(D) \quad \begin{cases} \partial_t u = \Delta\phi(u) + f(u(x, t))h(u(x, t - r)) & \text{in } Q_T, \\ u = 0 & \text{in } S_T, \\ u = \eta_s \geq 0 & \text{in } \Gamma_r, \end{cases}$$

where $r > 0$ is the delay and $\eta_s(x) := \eta(x, s)$ the initial data. As usual Δ denotes the Laplacian operator and $\partial_t u$ denotes the partial time derivative $\partial u / \partial t$. Throughout we will write Q instead of Q_∞ and we will sometimes abuse notation slightly by writing $u(t)$ instead of $u(\cdot, t)$, for a function $u(x, t)$.

The associated time-independent stationary problem for (D) is given by

$$(DS) \quad \begin{cases} \Delta\phi(u) + f(u)h(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

This paper is concerned with the large-time behaviour of non-negative solutions of (D) and their convergence to solutions of (DS) as $t \rightarrow \infty$. Its novelty lies in the combination of three distinct types of nonlinearity: degenerate

diffusion $\phi(u)$, local source term $f(u)$ and time delay $h(u(x, t - r))$. Such problems are often considered as models in population dynamics where $\phi(u)$ represents movement of individuals to avoid over-crowding, $f(u)$ an intrinsic growth rate and $h(u(x, t - r))$ a delayed response due to gestation periods, resource conversion, incubation periods, etc.

There is a large literature relating to problem (D) when delays are absent ($r = 0$), see [3, 23, 27] for an overview and extensive bibliographies. We will make use of several key results from this literature on degenerate parabolic equations, using it mainly to provide suitable comparison solutions for the solutions of (D). Several authors have considered the non-delay degenerate parabolic case in the presence of periodic forcing terms and established existence and attractivity properties of periodic solutions [14, 26]. There are also many works dealing with the case of linear diffusion and nonlinear delay terms ($r > 0$), see [19, 24, 29] for an overview and references. More recently researchers have considered problems incorporating degenerate diffusion, delay and periodic forcing [28, 30].

In [10] the authors considered the following linear diffusion case $\phi(u) = u$ with logistic delay $h(u) = 1 - u$ and local source term $f(u) = ku$ ($k > 0$):

$$\partial_t u = u_{xx} + ku(x, t)(1 - u(x, t - r)), \quad x \in (0, \pi), \quad t > 0, \quad (1)$$

$$u = 0, \quad x = 0, \pi, \quad t > 0. \quad (2)$$

It is well known that (1-2) possesses a unique positive equilibrium U_k for all $k > 1$, and only the trivial equilibrium $U = 0$ when $k < 1$, with $\|U_k\|_\infty \rightarrow 0$ as $k \rightarrow 1^+$. In the case of no delay ($r = 0$) it is also well known that U_k attracts all non-negative non-trivial solutions for $k > 1$; when $k < 1$ the trivial equilibrium $U = 0$ attracts such solutions. In [10] the authors fixed k slightly

greater than, sufficiently close to, 1 and showed that U_k undergoes Hopf bifurcation as r increases through an infinite sequence of positive values $0 < r_0(k) < r_1(k) < r_2(k), \dots$. In particular they showed that the first bifurcation at $r_0(k)$ is supercritical, giving rise to stable, spatially inhomogeneous, time-periodic solutions of (1-2). Consequently, there exist values of k and of the delay parameter r for which the (small) positive equilibrium U_k is not locally attractive. The present work was motivated in part by asking whether this kind of ‘delay-induced instability’ can occur when linear diffusion is replaced by nonlinear degenerate diffusion, such as $\Delta(u^m)$ for $m > 1$ (the so-called porous medium slow diffusion operator).

No comprehensive literature exists for degenerate parabolic equations including delay terms (and without periodic forcing). To the best of the author’s knowledge the only paper in a similar spirit to the present one is [5]. There the authors considered the equation

$$\partial_t u = \Delta u^m + u \left(a(x) - b(x)u - \int_{-\infty}^t u(x, s)K(x, t - s) ds \right), \quad (3)$$

where b and k are non-negative functions and the positivity set of a in Ω is non-empty. Crucially, and in contrast to the present paper where $b \equiv 0$, it was assumed in [5] that $b > 0$ on $\bar{\Omega}$. Furthermore, in order to guarantee convergence to a unique positive equilibrium, b was assumed [5, Theorem 2.5] to satisfy the stronger condition

$$b(x) \geq \int_0^\infty K(x, s)ds, \quad \forall x \in \Omega. \quad (4)$$

Assumptions such (4) are sometimes referred to as ‘diagonally dominant’ or having ‘negative instantaneous feedback’ in the delay-differential equation

literature and in the theory of competitive population dynamics. Mathematically this property is often used to overcome the absence of a comparison principle in situations where the delay term has a negative response effect, corresponding to $h' < 0$ in our context. See [16, 15, 17, 19].

Such terminology refers to the assumption that the local, instantaneous term bu dominates the non-local, delayed term $K \star u$ (the convolution term in (3)). The work of [5] provides a second motivation for the present paper, namely to obtain sufficient conditions for global convergence of non-negative solutions in the absence of negative instantaneous feedback (i.e. with $b = 0$.)

The remainder of the paper is structured as follows. In Section 2 we define the solution concepts for the problems encountered and establish preliminary existence-uniqueness results. In Section 3 we summarize and extend some known results from the literature concerning sign-indefinite degenerate parabolic equations. Section 4 contains the main results of the paper. Sufficient conditions will be given which ensure global convergence of non-negative solutions of (D) to a positive equilibrium, see Theorem 4.1. The class of problems for which the results are applicable include the logistic-type reaction term described above as a special case. The final Section 5 contains some examples and discussion.

2. Global existence and uniqueness for the delay problem

Let $\mathbb{R}^+ = [0, \infty)$. We begin with the following assumptions:

- (A1) $\phi \in C^1(\mathbb{R}^+)$, $\phi(0) = \phi'(0) = 0$, $\phi' > 0$ on $(0, \infty)$, ϕ^{-1} exists and $\phi^{-1} \in C^\alpha(\mathbb{R}^+)$, there exist $\gamma, \delta > 0$ such that ϕ is convex on $(0, \delta)$ and $u\phi'(u) < \gamma\phi(u)$ on $(0, \infty)$.

(A2) $f \in C^1(\mathbb{R}^+)$ and $f(0) = 0$.

(A3) $\lim_{u \rightarrow \infty} f(u)/\phi(u) = 0$.

From here onwards $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ will denote the function given by

$$g(u) = f(\phi^{-1}(u)). \quad (5)$$

We observe that g has the following properties: $g \in C^1(0, \infty) \cap C^\alpha(\mathbb{R}^+)$, $g(0) = 0$ and $g(u)/u \rightarrow 0$ as $u \rightarrow \infty$ (sublinearity at infinity).

Let us introduce the inhomogeneous, degenerate parabolic problem

$$(\text{II}) \quad \begin{cases} \partial_t u = \Delta \phi(u) + H(x, t)f(u) & \text{in } Q_T, \\ u = 0 & \text{in } S_T, \\ u = \eta_0 \geq 0 & \text{in } \Omega, \end{cases}$$

where $H \in C(\overline{Q})$ and $\eta_0 \in C(\overline{\Omega})$. Such problems appear as a special case of those studied in [2].

Definition 2.1.

(a) (i) A non-negative weak solution u of problem (II) is a function

$u \in C([0, T], L^1(\Omega)) \cap L^\infty(Q_T)$ satisfying

$$\begin{aligned} \int_{\Omega} u(\tau)\zeta(\tau) - \eta_0\zeta(0) \, dx - \int_{Q_\tau} u\partial_t\zeta + \phi(u)\Delta\zeta \, dxdt \\ = \int_{Q_\tau} H(x, t)f(u)\zeta \, dxdt \end{aligned} \quad (6)$$

for all $\tau \in [0, T]$ and $\zeta \in C^2(\overline{Q}_T)$ with $\zeta \geq 0$ and $\zeta = 0$ on S_T . A weak subsolution v (respectively supersolution w) of problem (II) is defined similarly, but with equality replaced by \leq (respectively \geq) in (6).

(ii) A non-negative classical solution u of problem (Π) is a function $u \in C(\overline{Q_T})$ for which the partial derivatives $\partial_t u$ and $\Delta\phi(u)$ exist and are continuous in Q_T and satisfies (Π) pointwise. A classical subsolution v (respectively supersolution w) of problem (Π) is defined similarly but with equality replaced by \leq (respectively \geq) in (Π) .

(b) Non-negative weak and classical solutions of problem (D) are similarly defined, with $H(x, t)$ replaced by $h(u(x, t - r))$.

(c) (i) A non-negative weak solution u of problem (DS) is a function $u \in L^\infty(\Omega)$ satisfying

$$\int_{\Omega} \phi(u)\Delta\zeta + f(u)h(u)\zeta \, dx = 0 \quad (7)$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta \geq 0$ and $\zeta = 0$ on $\partial\Omega$.

(ii) A non-negative classical solution u of problem (DS) is a function $u \in C(\overline{\Omega})$ for which $\Delta\phi(u) \in C^2(\Omega)$ and satisfies (DS) pointwise.

In order to prove existence and uniqueness for the delay problem (D) we will first need the result for the non-delay problem (Π) , which is non-autonomous and inhomogeneous. We prove only that which is sufficient for our analysis of the delay problem.

Lemma 2.1. *Let $H \in C(\overline{Q})$ and $0 \leq \eta \in C(\Gamma_r)$, $\eta_0(x) = 0$ in $\partial\Omega$. If $(A1-A3)$ hold, then for all $T > 0$ there exists a unique, non-negative weak solution u of problem (Π) . Moreover, $u \in C(\overline{Q_T})$.*

Proof. By [2, Theorem 2.1 and Theorem 2.3], there exists a $T > 0$ such that (II) has a unique weak solution on $[0, T)$, and either $T = \infty$ or, $T < \infty$ with $\limsup_{t \rightarrow T^-} \|u(t)\|_\infty = \infty$. We show that $T = \infty$ via a standard comparison argument.

Let e_1 denote the principal eigenfunction of $-\Delta$ on B with homogeneous Dirichlet boundary conditions, where B is any open ball such that $\bar{\Omega} \subset B$. Let $\mu_1 > 0$ denote the corresponding principal eigenvalue and suppose e_1 is normalized such that $\|e_1\|_\infty = 1$. Since $g(u)/u \rightarrow 0$ as $u \rightarrow \infty$ and $e_1 > 0$ on $\bar{\Omega}$, we can choose $c > 0$ sufficiently large such that

$$\|H\|_{L^\infty(Q_T)} g(ce_1(x)) \leq \mu_1 ce_1(x)$$

for all $x \in \bar{\Omega}$. Setting $w(x) = \phi^{-1}(ce_1(x))$, we have

$$\begin{aligned} \partial_t w - \Delta \phi(w) - H(x, t)f(w) &\geq -\Delta(ce_1) - \|H\|_{L^\infty(Q_T)} g(ce_1) \\ &\geq -c(\Delta e_1 + \mu_1 e_1) = 0 \end{aligned}$$

in Q_T . Taking $c > 0$ sufficiently large so that also $ce_1 \geq \eta_0$ in $\bar{\Omega}$ ensures that w is a classical supersolution of (II). It follows by comparison [2, Theorem 2.3] that $u \leq w$ almost everywhere in \bar{Q}_T and so $u \in L^\infty(Q_T)$. Thus T cannot be finite.

The continuity result is well-known [12, Theorem 6.1 and its Corollary].

□

Corollary 2.1. *Let $0 \leq \eta \in C(\Gamma_r)$ and $\eta_0(x) = 0$ for all $x \in \partial\Omega$. If $h \in C(\mathbb{R}^+)$ and (A1-A3) hold, then for any $T > 0$ there exists a unique non-negative weak solution u of (D). Moreover, u is continuous on \bar{Q}_T for all $T > 0$.*

Proof. The method of proof mirrors that used for delay-differential equations [13] or delay reaction-diffusion equations [29].

Let u_1 be the unique, non-negative and continuous weak solution on $[0, r]$ of the problem

$$\begin{aligned} \partial_t u &= \Delta \phi(u) + H_1(x, t) f(u) && \text{in } Q_r, \\ u &= 0 && \text{in } S_r, \\ u_0 &= \eta_0 && \text{in } \bar{\Omega}, \end{aligned}$$

where $H_1(x, t) = h(\eta(x, t - r))$ is continuous in $\bar{\Omega} \times [0, r]$. Such a solution is guaranteed to exist by Lemma 2.1. Clearly, u_1 is the unique weak solution of (D) on $[0, r]$. Now extend u_1 (but still denoting the extension by u_1) such that $u_1 = \eta$ on Γ_r . Again by Lemma 2.1 there then exists a unique, non-negative and continuous weak solution u_2 on $[0, 2r]$ of the problem

$$\begin{aligned} \partial_t u &= \Delta \phi(u) + H_2(x, t) f(u) && \text{in } Q_{2r}, \\ u &= 0 && \text{in } S_{2r}, \\ u_0 &= \eta_0 && \text{in } \bar{\Omega}, \end{aligned}$$

where $H_2(x, t) = h(u_1(x, t - r))$ is continuous in $\bar{\Omega} \times [0, 2r]$. Clearly u_2 is then the unique weak solution of (D) on $[0, 2r]$. Continuing in this manner inductively we obtain a unique, non-negative and continuous weak solution u_j of (D) on $[0, jr]$ for any positive integer j . For any $T > 0$ we may choose j such that $jr \geq T$ and obtain the unique, non-negative and continuous weak solution u on $[0, T]$ as the restriction $u = u_j|_{[0, T]}$, yielding the required result. \square

3. Some auxiliary degenerate elliptic and parabolic problems

In proving the main convergence results for the delay problem (D) we will construct a contracting sequence of ordered upper and lower solutions of some related equations. We first require some additional assumptions, recalling that the function g is given by equation (5).

(A4) f is strictly increasing on $(0, \infty)$;

(A5) $G(x) := \int_0^x \frac{ds}{g(s)}$ exists for $x > 0$ and $\lim_{x \rightarrow 0} G(x) = 0$;

(A6) g is strictly concave on $(0, \infty)$.

Remark 3.1. *The following properties of g follow easily from (A1-A6):*

(g1) $g \in C^1(0, \infty) \cap C^\alpha(\mathbb{R}^+)$, $g(0) = 0$ and g is strictly increasing;

(g2) $g(s)/s \rightarrow \infty$ as $s \rightarrow 0$;

(g3) $g(s)/s \rightarrow 0$ as $s \rightarrow \infty$.

Following [6], for $a \in C^\alpha(\bar{\Omega})$ we now consider the degenerate parabolic problem

$$(P) \quad \begin{cases} \partial_t z = \Delta \phi(z) + a(x)f(z) & \text{in } Q, \\ z = 0 & \text{in } S_\infty, \\ z = z_0 \geq 0 & \text{in } \bar{\Omega} \end{cases}$$

and its associated stationary problem

$$(PS) \quad \begin{cases} \Delta \phi(z) + a(x)f(z) = 0 & \text{in } \Omega, \\ z = 0 & \text{in } \partial\Omega, \end{cases}$$

where $a(x)$ may change sign in Ω . Problems like (P) and (PS) have been studied by several authors, see [5, 20, 21, 22]. The main interesting feature of (PS) is the existence of non-negative, non-trivial solutions vanishing on open subsets of Ω , so-called ‘dead cores’. It can be shown that there exists a unique maximal solution of (DS) in the class of non-negative functions, but no such minimal solution exists and the situation is much more complicated than the case where $a(x)$ is positive. We describe the main features of the solutions of (P) and (PS) sufficient for our purposes.

The *positivity set* of a is defined by

$$\Omega^+(a) = \{x \in \Omega : a(x) > 0\}$$

and $\Omega_i^+(a)$ denote the connected components of $\Omega^+(a)$, for $i \in M$, where M is an indexing set which is at most countable (due to the regularity of Ω and a). The stationary solution set $S(a)$ is defined to be the set of all non-negative weak solutions u of (PS) satisfying $u > 0$ on $\Omega^+(a)$.

We now observe that if $a > 0$ on $\partial\Omega$, then $\partial\Omega \subset \overline{\bigcup_{i \in M} \Omega_i^+(a)}$ and so

$$\overline{\Omega^+(a)} = \overline{\bigcup_{i \in M} \Omega_i^+(a)} = \overline{\bigcup_{i \in M} \Omega_i^+(a)} \cup \partial\Omega \subset \bigcup_{i \in M} \overline{\Omega_i^+(a)} \cup \partial\Omega.$$

This allows us to apply [6, Theorem 2.1] (in the special case where $I = M$ and $S_M = N_M$, in the notation of that paper) to deduce that (PS) has at most one solution with the property of being positive on $\Omega^+(a)$. Then [6, Theorem 2.2(i)] guarantees the existence of such a solution (again taking $I = M$). We summarize this and related results from [6] as follows:

Lemma 3.1. *Let $a \in C^\alpha(\overline{\Omega})$ and suppose $\Omega^+(a)$ is non-empty. If (A1-A6) hold, then:*

- (i) *Maximality* ([6, Theorems 2.1 and 2.2]): there exists $z^* = z^*(a) \in C^\alpha(\bar{\Omega})$ such that $S(a) = \{z^*(a)\}$. Moreover, $z^*(a)$ is maximal with respect to the set of non-negative solutions of (PS).
- (ii) *Monotonicity* ([6, Lemma 2.4]): if $a_1 \geq a_2$ in $\bar{\Omega}$ then $z^*(a_1) \geq z^*(a_2)$ in $\bar{\Omega}$.
- (iii) *Convergence* ([6, Theorem 3.1] or [22, Theorem 8]): if $z_0 \in L^\infty(\Omega)$ and $z_0 > 0$ on some open subset of $\Omega_i^+(a)$ for every $i \in M$, then the solution z of (P) converges to $z^*(a)$ in $L^p(\Omega)$, for any $p \geq 1$, as $t \rightarrow \infty$. If $n = 1$, or $n \geq 2$ and z is uniformly continuous on $\Omega \times [\varepsilon, \infty)$ for some $\varepsilon > 0$, then convergence to $z^*(a)$ is in $L^\infty(\Omega)$.

Remark 3.2. *The conditions on the initial data z_0 in Lemma 3.1 (iii) are clearly satisfied if z_0 is continuous and $z_0 > 0$ in Ω or, more generally, if $z(x, T) > 0$ in Ω for some $T > 0$ since problem (P) is autonomous.*

The proof of Lemma 3.1 can be found in the references given, but for exposition purposes we briefly outline the ideas here. Setting $v = \phi(z)$, (PS) is equivalent to

$$\Delta v + a(x)g(v) = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{in } \partial\Omega. \quad (8)$$

As in the proof of Lemma 2.1, (g3) guarantees the existence of supersolutions $\bar{v}(x)$ of (8) with arbitrarily large minimum over $\bar{\Omega}$. Similarly, (g2) ensures that arbitrarily small (weak) subsolutions $\underline{v}(x)$ of (8) exist such that $\underline{v} \not\equiv 0$ on $\Omega_i^+(a)$, for every $i \in M$, again by using a principal eigenfunctions of $-\Delta$ on a small ball contained within the interior of each $\Omega_i^+(a)$. Each subsolution

(respectively supersolution) then generates a monotonically increasing (respectively decreasing) solution $\underline{z}(t; \underline{z}_0)$ of (P) in time, (respectively $\bar{z}(t; \bar{z}_0)$) where $\underline{z}_0(x) = \phi^{-1}(\underline{v}(x))$ (respectively $\bar{z}_0(x) = \phi^{-1}(\bar{v}(x))$). See [11, 22] for example. The limiting functions $\underline{z}^*(x)$ and $\bar{z}^*(x)$ are then weak solutions of (PS) satisfying $\bar{z}^* \geq \underline{z}^* > 0$ on $\Omega^+(a)$ and the interval $[\underline{z}^*, \bar{z}^*]$ attracts (in $L^p(\Omega)$) all solutions of (P) having initial data in $[\underline{z}^*, \bar{z}^*]$. The assumptions (A5-A6) ensure uniqueness via a maximum principle argument (see [6, Theorem 2.1]), so that $\underline{z}^* = \bar{z}^* = z^*(a)$, say. For any initial data z_0 satisfying (iii), \underline{z} and \bar{z} can then be chosen such that $z_0 \in [\underline{z}, \bar{z}]$, so that $z(t; z_0) \rightarrow [\underline{z}^*, \bar{z}^*] = \{z^*(a)\}$ in $L^p(\Omega)$ as $t \rightarrow \infty$.

Corollary 3.1. *Let $z^* : C^\alpha(\bar{\Omega}) \rightarrow C^\alpha(\bar{\Omega})$, $a \mapsto z^*(a)$ be defined as in Lemma 3.1.*

(a) *Suppose $a^{(m)} \in C^\alpha(\bar{\Omega})$ is a monotonically increasing sequence such that $a^{(m)} \rightarrow a \in C^\alpha(\bar{\Omega})$ pointwise as $m \rightarrow \infty$. If $\Omega^+(a^{(m)}) \neq \emptyset$ for all $m \geq 1$, then $z^*(a^{(m)}) \rightarrow z^*(a)$ pointwise as $m \rightarrow \infty$.*

(b) *Suppose $b^{(m)} \in C^\alpha(\bar{\Omega})$ is a monotonically decreasing sequence such that $b^{(m)} \rightarrow b \in C^\alpha(\bar{\Omega})$ pointwise as $m \rightarrow \infty$. If $\Omega^+(b) \neq \emptyset$, then $z^*(b^{(m)}) \rightarrow z^*(b)$ pointwise as $m \rightarrow \infty$.*

Proof. (a). By Lemma 3.1 $z^*(a^{(m)})$ is an increasing sequence with $z^*(a^{(m)}) \leq z^*(a)$. Hence $(z^*(a^{(m)}))(x)$ converges pointwise for $x \in \bar{\Omega}$ as $m \rightarrow \infty$, with limit $v(x) \leq (z^*(a))(x)$ say. Hence $\phi(z^*(a_m)(x)) \rightarrow \phi(v(x))$ pointwise for $x \in \bar{\Omega}$ as $m \rightarrow \infty$. Lebesgue's dominated convergence theorem then allows one to pass to the limit as $m \rightarrow \infty$ in the definition of weak solution for

(PS), so that $v \in L^\infty(\Omega)$ satisfies

$$\Delta\phi(v) + a(x)f(v) = 0 \text{ in } \Omega, \quad v = 0 \text{ in } \partial\Omega.$$

By elliptic regularity it then follows that $v \in C^\alpha(\bar{\Omega})$ and $\phi(v) \in C^{2+\alpha}(\bar{\Omega})$.

It remains to show that $v = z^*(a)$. But for any $x \in \Omega^+(a)$ (i.e. $a(x) > 0$) there exists j such that $a^{(j)}(x) > 0$, and so $x \in \Omega^+(a^{(j)})$ and $0 < (z^*(a^{(j)}))(x) \leq v(x)$. Since $x \in \Omega^+(a)$ was arbitrary we have $v > 0$ in $\Omega^+(a)$ and so $v = z^*(a)$, by definition of z^* . This proves part (a).

Part (b) follows in an identical way, until one has to show $v = z^*(b)$. But for any $x \in \Omega^+(b)$ (i.e. $b(x) > 0$), $(z^*(b^{(m)}))(x) \geq (z^*(b))(x) > 0$ and so (letting $m \rightarrow \infty$), $v(x) \geq (z^*(b))(x) > 0$. Since $x \in \Omega^+(b)$ was arbitrary we have $v > 0$ in $\Omega^+(b)$ and so $v = z^*(b)$. \square

In the special case where a is a positive constant, $a(x) \equiv k > 0$ say, problem (PS) becomes

$$(PS_k) \quad \begin{cases} \Delta\phi(z) + kf(z) = 0 & \text{in } \Omega, \\ z = 0 & \text{in } \partial\Omega. \end{cases}$$

In what follows we denote the unique positive solution of (PS_k) by ψ_k (i.e. $\psi_k = z^*(k)$) and write Ω_k^+ for $\Omega^+(h \circ \psi_k)$, i.e. $\Omega_k^+ = \{x \in \Omega : h(\psi_k(x)) > 0\}$.

4. Main results: global convergence for the delay problem

The methods of this section are similar in spirit to those in [5, 11, 21, 22]. The main difference however is that, unlike the non-delay cases considered in these references, solutions of (D) are not generally monotonic in time regardless of any specific choice of initial data (such as sub- and supersolutions of the associated elliptic problem).

For $v, w \in C(\overline{\Omega})$ satisfying $v \leq w$ in $\overline{\Omega}$ we define the interval subset $[v, w]$ by

$$[v, w] := \{u \in C(\overline{\Omega}) : v(x) \leq u(x) \leq w(x) \forall x \in \overline{\Omega}\}.$$

We will say that $u(t) \rightarrow [v, w]$ in $C(\overline{\Omega})$ as $t \rightarrow \infty$ if

$$v(x) \leq \liminf_{t \rightarrow \infty} u(x, t) \leq \limsup_{t \rightarrow \infty} u(x, t) \leq w(x)$$

uniformly for $x \in \overline{\Omega}$.

We now make the following assumption concerning the delay function h :

(A7) $h \in C^\alpha(\mathbb{R}^+) \cap C^1(0, \infty)$, $h' < 0$ on $(0, \infty)$ and $k := h(0) > 0$.

It follows from (A7) and the boundary conditions on ψ_k that $\Omega_k^+ \neq \emptyset$.

Proposition 4.1. *Suppose (A1-A7) hold and define the sequences v_n and w_n inductively by: $v_1 = 0$, $w_1 = \psi_k$, $v_{n+1} = z^*(h(w_n))$ and $w_{n+1} = z^*(h(v_{n+1}))$. Then the sequences v_n and w_n are well-defined and have the following properties:*

- (i) $v_n, w_n \in C^\alpha(\overline{\Omega})$ for all $n \geq 1$ and $v_2 > 0$ in Ω_k^+ ;
- (ii) $v_n \leq w_n$ and $[v_{n+1}, w_{n+1}] \subset [v_n, w_n]$ for all $n \geq 1$;
- (iii) v_n, w_n converge pointwise to limits $v^*, w^* \in C^\alpha(\overline{\Omega})$, respectively, as $n \rightarrow \infty$ (with $v^* \leq w^*$) and v^*, w^* satisfy

$$\begin{aligned} \Delta \phi(v^*) + h(w^*)f(v^*) &= 0 & \text{in } \Omega, \\ \Delta \phi(w^*) + h(v^*)f(w^*) &= 0 & \text{in } \Omega, \\ v^* = w^* &= 0 & \text{in } \partial\Omega. \end{aligned}$$

Proof. One observes that since $\Omega^+(h(w_1)) = \Omega^+(h(\psi_k)) = \Omega_k^+$ it follows that $v_2 = z^*(h(w_1)) = z^*(h(\psi_k)) > 0$ in Ω_k^+ .

Now, v_{n+1} is defined to be the maximal solution of

$$\Delta\phi(v) + h(w_n)f(v) = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{in } \partial\Omega \quad (9)$$

and w_{n+1} is defined to be the maximal solution of

$$\Delta\phi(w) + h(v_{n+1})f(w) = 0 \quad \text{in } \Omega, \quad w = 0 \quad \text{in } \partial\Omega. \quad (10)$$

By the boundary conditions, $h(w_n(x)) = h(v_n(x)) = h(0) = k > 0$ for any $x \in \partial\Omega$ and $n \geq 1$. Hence $\Omega^+(h(v_n))$ and $\Omega^+(h(w_n))$ are non-empty and so $z^*(h(w_n))$ and $z^*(h(v_{n+1}))$ are well defined. Elliptic regularity then yields part (i).

We show first by induction that $v_n \leq w_n$ for all $n \geq 1$. The case $n = 1$ is trivially true. Suppose that $v_n \leq w_n$. Then

$$0 = \Delta\phi(v_{n+1}) + h(w_n)f(v_{n+1}) \leq \Delta\phi(v_{n+1}) + h(v_n)f(v_{n+1}),$$

so that v_{n+1} is a subsolution of the problem satisfied by w_n , namely,

$$\Delta\phi(w_n) + h(v_n)f(w_n) = 0 \quad \text{in } \Omega, \quad w_n = 0 \quad \text{in } \partial\Omega.$$

Hence $v_{n+1} \leq w_n$. But then

$$0 = \Delta\phi(v_{n+1}) + h(w_n)f(v_{n+1}) \leq \Delta\phi(v_{n+1}) + h(v_{n+1})f(v_{n+1}),$$

so that v_{n+1} is a subsolution of (10) and hence $v_{n+1} \leq w_{n+1}$ as required.

Next we show by induction that $[v_{n+1}, w_{n+1}] \subset [v_n, w_n]$ for all $n \geq 1$. For $n = 1$ we trivially have $0 = v_1 \leq v_2$ and

$$0 = \Delta\phi(w_2) + h(v_2)f(w_2) \leq \Delta\phi(w_2) + kf(w_2),$$

so that w_2 is a subsolution of problem (PS_k) , and hence $w_2 \leq \psi_k = w_1$.

Suppose now that $v_n \leq v_{n+1}$ and $w_{n+1} \leq w_n$. Then

$$0 = \Delta\phi(v_{n+1}) + h(w_n)f(v_{n+1}) \leq \Delta\phi(v_{n+1}) + h(w_{n+1})f(v_{n+1}),$$

so that v_{n+1} is a subsolution of the problem

$$\Delta\phi(v) + h(w_{n+1})f(v) = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{in } \partial\Omega$$

and hence $v_{n+1} \leq v_{n+2}$. Similarly,

$$0 = \Delta\phi(w_{n+2}) + h(v_{n+2})f(w_{n+2}) \leq \Delta\phi(w_{n+2}) + h(v_{n+1})f(w_{n+2}),$$

so that w_{n+2} is a subsolution of the problem

$$\Delta\phi(w) + h(v_{n+1})f(w) = 0 \quad \text{in } \Omega, \quad w = 0 \quad \text{in } \partial\Omega$$

and hence $w_{n+2} \leq w_{n+1}$, as required. This completes part (ii) of the proof.

Since v_n and w_n are bounded monotonic sequences, the pointwise limits $v_n(x) \rightarrow v^*(x)$ and $w_n(x) \rightarrow w^*(x)$ exist for all $x \in \bar{\Omega}$ as $n \rightarrow \infty$, and $v^* \leq w^*$. By Lebesgue's monotone convergence theorem we may pass to the limit as $n \rightarrow \infty$ in the definition of weak solution, to obtain $v^*, w^* \in L^\infty(\Omega)$ as weak solutions of

$$\Delta\phi(v^*) + h(w^*)f(v^*) = 0 \quad \text{in } \Omega, \tag{11}$$

$$\Delta\phi(w^*) + h(v^*)f(w^*) = 0 \quad \text{in } \Omega, \tag{12}$$

$$v^* = w^* = 0 \quad \text{in } \partial\Omega. \tag{13}$$

Elliptic regularity then yields part (iii).

□

Remark 4.1. Assumption (A1) and [18, Theorem 2.5] ensure that solutions of (P) are in fact precompact in $C(\overline{\Omega})$. In this case the L^p -convergence referred to in Lemma 3.1 (iii) can be strengthened to convergence in $C(\overline{\Omega})$.

Proposition 4.2. Let $0 \leq \eta \in C(\Gamma_r)$ and $\eta_0(x) = 0$ for all $x \in \partial\Omega$. Suppose that the solution u of (D) satisfies $u(x, t) > 0$ in $\Omega \times [T, \infty)$, for some $T > r$ and let v^* and w^* be as in Proposition 4.1. If (A1-A7) hold, then $u(t) \rightarrow [v^*, w^*]$ in $C(\overline{\Omega})$ as $t \rightarrow \infty$.

Proof. We first prove that $u(t) \rightarrow [v_n, w_n]$ in $C(\overline{\Omega})$ as $t \rightarrow \infty$ for all $n \geq 1$, where v_n, w_n are as in from Proposition 4.1.

By classical regularity theory for uniformly parabolic equations, the solution u of (D) is a classical solution for $t > T$, and so from (A7) we have $\partial_t u - \Delta\phi(u) = h(u(x, t-r))f(u) \leq kf(u)$ for $t > T$. Consequently, u is a classical subsolution of Problem (P) with $a(x) \equiv k$ and $z(x, T) = u(x, T)$. Hence, by comparison, $u \leq \bar{z}_1$ where \bar{z}_1 is the solution of

$$\begin{aligned} \partial_t z &= \Delta\phi(z) + kf(z) && \text{in } \Omega \times (T, \infty), \\ z &= 0 && \text{in } \partial\Omega \times (T, \infty), \\ z(x, T) &= u(x, T) && \text{in } \overline{\Omega}. \end{aligned}$$

Since $\Omega^+(k) = \Omega$ and $z(x, T) > 0$ in Ω , it follows from Lemma 3.1 (iii) and Remark 4.1 that $\bar{z}_1(t) \rightarrow \psi_k$ in $C(\overline{\Omega})$ as $t \rightarrow \infty$. Hence $u(t) \rightarrow [0, w_1]$ in $C(\overline{\Omega})$ as $t \rightarrow \infty$.

We now proceed by induction. We have just shown that $u(t) \rightarrow [v_1, w_1]$ in $C(\overline{\Omega})$ as $t \rightarrow \infty$, where $v_1 = 0$ and $w_1 = \psi_k$. If $u(t) \rightarrow [v_n, w_n]$ in $C(\overline{\Omega})$ as $t \rightarrow \infty$ then, for any $m \in \mathbb{N}$, there exists $\tau_n > T + r$ such that

$u(x, t - r) \leq w_n(x) + 1/m$ for all $x \in \Omega$ and $t > \tau_n$. Hence for $t > \tau_n$ we have $\partial_t u - \Delta \phi(u) \geq h(w_n(x) + 1/m)f(u)$ and so u is a supersolution of (P) with $a(x) = a_n^{(m)}(x) := h(w_n(x) + 1/m)$ and $z(x, \tau_n) = u(x, \tau_n) > 0$ in Ω . Hence, $u \geq \underline{z}_n^{(m)}$, where $\underline{z}_n^{(m)}$ is the solution of

$$\begin{aligned} \partial_t z &= \Delta \phi(z) + a_n^{(m)}(x)f(z) && \text{in } \Omega \times (\tau_n, \infty), \\ z &= 0 && \text{in } \partial\Omega \times [\tau_n, \infty), \\ z(x, \tau_n) &= u(x, \tau_n) > 0 && \text{in } \Omega. \end{aligned}$$

By Lemma 3.1 (iii) and Remark 4.1, $\bar{z}_n^{(m)}(t) \rightarrow z^*(a_n^{(m)})$ in $C(\bar{\Omega})$ as $t \rightarrow \infty$. Hence

$$\liminf_{t \rightarrow \infty} u(x, t) \geq z^*(a_n^{(m)})$$

uniformly in x for all $m \geq 1$. Clearly, $a_n^{(m)}$ is monotonically increasing in m and $a_n^{(m)} \rightarrow a_n \in C^\alpha(\bar{\Omega})$ as $m \rightarrow \infty$, where $a_n(x) := h(w_n(x))$. For any $x \in \partial\Omega$, $a_n^{(m)}(x) = h(1/m) > 0$ for m sufficiently large, and so $\Omega^+(a_n^{(m)})$ is non-empty for such m . By Corollary 3.1(a), $\lim_{m \rightarrow \infty} z^*(a_n^{(m)}(x)) = z^*(a_n(x)) = z^*(h(w_n)) = v_{n+1}$. Thus,

$$\liminf_{t \rightarrow \infty} u(x, t) \geq v_{n+1}$$

uniformly in x .

Similarly it now follows that, for any $m \in \mathbb{N}$, there then exists $t_n > \tau_n$ such that $u(x, t - r) \geq [v_{n+1}(x) - 1/m]_+ := \max\{v_{n+1}(x) - 1/m, 0\}$ for all $x \in \bar{\Omega}$ and $t > t_n$. Hence for $t > t_n$ we have $\partial_t u - \Delta \phi(u) \leq h(v_{n+1}(x) - 1/m)f(u)$ and so u is a subsolution of (P) with $a(x) = b_n^{(m)}(x) := h([v_{n+1}(x) - 1/m]_+)$

and $z(x, t_n) = u(x, t_n) > 0$. Hence, $u \leq \bar{z}_n^{(m)}$, where $\bar{z}_n^{(m)}$ is the solution of

$$\begin{aligned} \partial_t z &= \Delta \phi(z) + b_n^{(m)}(x) f(z) && \text{in } \Omega \times (t_n, \infty), \\ z &= 0 && \text{in } \partial\Omega \times [t_n, \infty), \\ z(x, t_n) &= u(x, t_n) > 0 && \text{in } \Omega. \end{aligned}$$

Again by Lemma 3.1 (iii) and Remark 4.1, $\bar{z}_n^{(m)}(t) \rightarrow z^*(b_n^{(m)})$ in $C(\bar{\Omega})$ as $t \rightarrow \infty$. Hence

$$\limsup_{t \rightarrow \infty} u(x, t) \leq z^*(b_n^{(m)})$$

uniformly in x for all $m \geq 1$. Clearly, $b_n^{(m)}$ is monotonically decreasing in m and $b_n^{(m)} \rightarrow b_n \in C^\alpha(\bar{\Omega})$ as $m \rightarrow \infty$, where $b_n(x) := h(v_{n+1}(x))$. For any $x \in \partial\Omega$, $b_n(x) = h(0) = k > 0$ and so $\Omega^+(b_n)$ is non-empty. By Corollary 3.1(b), $\lim_{m \rightarrow \infty} z^*(b_n^{(m)}(x)) = z^*(b_n(x)) = z^*(h(v_{n+1})) = w_{n+1}$ and thus

$$\limsup_{t \rightarrow \infty} u(x, t) \leq w_{n+1}$$

uniformly in x . Hence $u(t) \rightarrow [v_{n+1}, w_{n+1}]$ in $C(\bar{\Omega})$ as $t \rightarrow \infty$ and so $u(t) \rightarrow [v_n, w_n]$ in $C(\bar{\Omega})$ as $t \rightarrow \infty$ for all $n \geq 1$, by induction.

Finally, since v_n and w_n are bounded sequences in $C^\alpha(\bar{\Omega})$, there exist subsequences v_{n_j} and w_{n_j} such that $v_{n_j} \rightarrow v^*$ and $w_{n_j} \rightarrow w^*$ in $C(\bar{\Omega})$ as $j \rightarrow \infty$. Hence $u(t) \rightarrow [v^*, w^*]$ in $C(\bar{\Omega})$ as $t \rightarrow \infty$.

□

Remark 4.2. *Eventual positivity of solutions of (D) is by no means obvious in the general case and we do not pursue this interesting question here. See*

[8] for results in this direction in the non-delay case. However, under the assumptions of Theorem 4.1 below, such positivity is guaranteed. Positivity can also be ensured when the initial condition η_0 is assumed positive in Ω . The reason is as follows. By Lipschitz continuity of f and the boundedness of u , u is a supersolution of the problem

$$\begin{aligned} \partial_t z &= \Delta \phi(z) - Mz \quad \text{in } Q_T, \\ z &= 0 \quad \text{in } S_T, \\ z(x, 0) &= \eta_0(x) > 0 \quad \text{in } \Omega \end{aligned} \tag{14}$$

for M sufficiently large. By (A1) (in particular, using $u\phi'(u) < \gamma\phi(u)$) one can then show that $\phi^{-1}(\epsilon \exp(-ct)e_1(x))$ is a subsolution of (14) on any compact subset of Ω , for $\epsilon > 0$ sufficiently small and $c > 0$ sufficiently large. It follows that $u \geq z > 0$ in Q_T for all $T > 0$.

We are now in a position to establish the main result of this paper, that of global convergence of solutions of (D) to a unique positive equilibrium of (DS). This requires all of the assumptions (A1-A7), which we now summarize in a more convenient, but equivalent form.

(H_ϕ) $\phi \in C^1(\mathbb{R}^+)$, $\phi(0) = \phi'(0) = 0$, $\phi' > 0$ on $(0, \infty)$, ϕ^{-1} exists and $\phi^{-1} \in C^\alpha(\mathbb{R}^+)$, there exist $\gamma, \delta > 0$ such that ϕ is convex on $(0, \delta)$ and $u\phi'(u) < \gamma\phi(u)$ on $(0, \infty)$;

(H_f) $f \in C^1(\mathbb{R}^+)$, $f(0) = 0$ and f is strictly increasing.

(H_h) $h \in C^\alpha(\mathbb{R}^+) \cap C^1(0, \infty)$, $k := h(0) > 0$ and $h' < 0$ on $(0, \infty)$.

(H_g) g is strictly concave on $(0, \infty)$, $\lim_{s \rightarrow \infty} g(s)/s = 0$, $G(x) := \int_0^x \frac{ds}{g(s)}$ exists and $\lim_{x \rightarrow 0} G(x) = 0$.

(H_η) $0 \leq \eta \in C(\Gamma_r)$, $\eta_0(x) = 0$ in $\partial\Omega$ and $\eta_0 \not\equiv 0$.

Theorem 4.1. *Let (H_ϕ) , (H_f) , (H_h) , (H_g) and (H_η) hold and let ψ_k denote the unique positive solution of problem (PS_k) . Suppose also that*

(A8) there exists $\beta > 0$ such that $\frac{\phi(u)h(u)}{f(u)}$ is non-decreasing on $(0, \beta)$.

If $\|\psi_k\|_\infty < \beta$, then

- (i) there exists a time $T_0 = T_0(\eta) \geq 0$ such that the solution u of the delay problem (D) satisfies $u(x, t) > 0$ in $\Omega \times [T_0, \infty)$;*
- (ii) there exists a unique non-negative, non-trivial solution φ of problem (DS) and $\varphi > 0$ in Ω ;*
- (iii) the solution u of the delay problem (D) converges in $C(\bar{\Omega})$ to φ as $t \rightarrow \infty$.*

Proof. We first claim that $h(u) > 0$ on $(0, \beta)$. For if not, then by (H_h) there exists $q \in (0, \beta)$ such that $h(q) = 0$ and $h'(q) < 0$. But then

$$\begin{aligned} (\phi h/f)'(q) &= [f(q)h'(q)\phi(q) + f(q)h(q)\phi'(q) - f'(q)h(q)\phi(q)]/f^2(q) \\ &= h'(q)\phi(q)/f(q) < 0, \end{aligned}$$

contradicting the assumption that $\phi(u)h(u)/f(u)$ is non-decreasing on $(0, \beta)$.

We already know that $\limsup_{t \rightarrow \infty} u(x, t) \leq \psi_k(x)$, uniformly in x . Hence, since $\psi_k < \beta$ in $\bar{\Omega}$, there exists T_1 such that $u(x, t) < \beta$ in $\bar{\Omega} \times [T_1, \infty)$. For $t \geq T_1 + r$, $h(u(x, t - r)) > 0$ and so u is a supersolution of the generalized

porous medium-type equation

$$\partial_t z = \Delta \phi(z) \quad \text{in } \Omega \times (T_1 + r, \infty), \quad (15)$$

$$z = 0 \quad \text{in } \partial\Omega \times (T_1 + r, \infty), \quad (16)$$

$$z(x, T_1 + r) = u(x, T_1 + r) \quad \text{in } \bar{\Omega}. \quad (17)$$

From (H_ϕ) , and in particular the convexity of ϕ near zero, the well-known positivity result [9, Theorem 5] ensures the existence of a time $T_0 \geq T_1 + r$ such that the solution z of (15-17) satisfies $z > 0$ in $\Omega \times [T_0, \infty)$. This yields the required positivity result (i). In particular this now allows us to make use of Proposition 4.2.

Since $\psi_k < \beta$ we also have that $\Omega_k^+ = \Omega$ since $h(\psi_k) > 0$ in Ω . Hence $v^* \geq v_2 > 0$ in Ω by Proposition 4.1 (i). Multiplying (11) by $\phi(w^*)$, (12) by $\phi(v^*)$ and integrating, we obtain

$$0 = \int_{\Omega} f(v^*)f(w^*) \left[\frac{\phi(w^*)h(w^*)}{f(w^*)} - \frac{\phi(v^*)h(v^*)}{f(v^*)} \right] dx. \quad (18)$$

Since $0 < v^* \leq w^* \leq \psi_k < \beta$ in Ω , it now follows from the monotonicity of $\phi h/f$ on $(0, \beta)$ that $v^* = w^*$. Consequently the solution u of (D) converges in $C(\bar{\Omega})$ to v^* as $t \rightarrow \infty$, where v^* is a positive solution of problem (DS). It remains only to show that v^* is the unique, non-negative, non-trivial solution of (DS).

Setting $z = \phi(u)$, problem (DS) is equivalent to

$$\Delta z + F(z) = 0 \quad \text{in } \Omega, \quad z = 0 \quad \text{in } \partial\Omega \quad (19)$$

where $F(z) := h(\phi^{-1}(z))g(z)$. By (H_h) and (H_g) , $\limsup_{z \rightarrow \infty} F(z)/z \leq 0$ and $\lim_{z \rightarrow 0} F(z)/z = +\infty$. It follows from standard results for sublinear elliptic equations [1, 7] that there exists a maximal positive solution \hat{z} of (19). Equivalently, there exists a maximal positive solution $\varphi := \phi^{-1}(\hat{z}) \geq v^* > 0$ of

(DS). By (H_h) , φ is a subsolution of (PS_k) and so $\varphi \leq \psi_k < \beta$ in Ω . Now, for any non-negative non-trivial solution U of (DS) we have $U \leq \varphi$ and so $h(U) \geq h(\varphi) > 0$. Thus $U > 0$ in Ω by the maximum principle. In the obvious way, from (DS) we also have

$$0 = \int_{\Omega} \phi(U)\phi(\varphi) \left[\frac{f(\varphi)h(\varphi)}{\phi(\varphi)} - \frac{f(U)h(U)}{\phi(U)} \right] dx. \quad (20)$$

From (H_h) and (A8) we now have that

$$\left[\frac{f(u)h(u)}{\phi(u)} \right]' = \frac{2h'(u)f(u)}{\phi(u)} - \frac{f(u)^2}{\phi(u)^2} \left[\frac{\phi(u)h(u)}{f(u)} \right]' < 0$$

on $(0, \beta)$ and so fh/ϕ is strictly decreasing on $(0, \beta)$. Hence, by (20), $U = \varphi$ and so φ is the unique, non-negative, non-trivial solution of (DS), proving part (ii). In particular it follows that $v^* = \varphi$, proving part (iii).

□

Remark 4.3. *The requirement that $\|\psi_k\|_{\infty} < \beta$ in Theorem 4.1 can obviously be weakened to $\|w^*\|_{\infty} < \beta$, where w^* is as in Proposition 4.1. The choice of ψ_k in the statement of the Theorem 4.1 has been made for the sake of explicitness and applicability, since it is more easily obtained via solution of (PS_k) only.*

5. Some special cases and discussion

In this final section we consider the power law nonlinearities $\phi(u) = u^m$ and $f(u) = u^p$ where $m > p \geq 1$ so that (D) becomes

$$\partial_t u = \Delta u^m + u^p h(u(x, t - r)),$$

with Dirichlet boundary conditions and initial data η satisfying (H_η) . Hypotheses (H_ϕ) , (H_f) and (H_g) are easily seen to be true. It is also known (see [4]) that $\psi_k = k^{1/(m-p)}\psi$, where ψ is the unique positive solution of

$$\Delta\psi^m + \psi^p = 0 \quad \text{in } \Omega, \quad \psi = 0 \quad \text{in } \partial\Omega.$$

For suitable h , the hypotheses of Theorem 4.1 will then hold for all k satisfying

$$0 < k < \left(\frac{\beta}{\|\psi\|_\infty} \right)^{m-p}. \quad (21)$$

In the case of a one-dimensional domain $\Omega = (0, \pi)$ one easily obtains by quadrature that

$$\|\psi\|_\infty = \left(\frac{(m+p)\pi^2}{8mI^2} \right)^{\frac{1}{m-p}}, \quad I := \int_0^{\pi/2} \sin^{\frac{m+p}{m-p}} \theta \, d\theta.$$

We now consider some specific forms for h satisfying (H_h) .

Example 5.1 (A generalized logistic). Consider the degenerate parabolic delay problem

$$\partial_t u = \Delta u^m + ku^p (1 - u^q(x, t - r))$$

with appropriate boundary and initial conditions, $k > 0$ and $q > 0$. Here, $h(s) = k(1 - s^q)$ clearly satisfies (H_h) . Now, $\phi(u)h(u)/f(u) = ku^{m-p}(1 - u^q)$ is non-decreasing for all $u \in (0, \beta)$, where $\beta = [(m-p)/(m-p+q)]^{1/q}$. From (21) the hypotheses of Theorem 4.1 will be satisfied provided that

$$k < k_0(m, p, q) := \left(\frac{m-p}{(m-p+q)\|\psi\|_\infty^q} \right)^{(m-p)/q}. \quad (22)$$

Hence for $k < k_0$ solutions of (D) converge to the unique positive solution of

$$\Delta\varphi^m + k\varphi^p(1 - \varphi^q) = 0 \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{in } \partial\Omega$$

as $t \rightarrow \infty$.

Example 5.2 (The logistic case). For $p = q = 1$ we have the logistic term considered in [10], but here with nonlinear degenerate diffusion $\phi(u) = u^m$ for $m > 1$. Hence we have global convergence to φ as $t \rightarrow \infty$ if

$$k < k_0(m, 1, 1) = \left(\frac{1 - \frac{1}{m}}{\|\psi\|_\infty} \right)^{(m-1)}.$$

where φ is the unique positive solution of

$$\Delta \varphi^m + k\varphi(1 - \varphi) = 0 \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{in } \partial\Omega. \quad (23)$$

When $\Omega = (0, \pi)$,

$$k_0(m, 1, 1) = \left(\frac{8mI^2}{(m+1)\pi^2} \right) \left(1 - \frac{1}{m} \right)^{m-1}. \quad (24)$$

One observes that $k_0(m, 1, 1) \rightarrow 0$ as $m \rightarrow 1^+$ and $k_0(m, 1, 1) \rightarrow 8/(e\pi^2)$ as $m \rightarrow \infty$.

Example 5.2 shows that the destabilization caused by the Hopf bifurcation (as r increases) of small positive solutions in [10] for the case $m = 1$, does not occur in the degenerate diffusion case $m > 1$. The reason is as follows. For $m > 1$, it is known (see [11]) that the unique positive equilibrium of (23), $\varphi = \varphi_k$, is increasing in k and $\|\varphi_k\|_\infty \rightarrow 0$ as $k \rightarrow 0^+$. Thus for $m > 1$, small positive solutions corresponds to small k . But since k_0 is independent of r , all non-trivial solutions converge to the equilibrium φ_k as $t \rightarrow \infty$, for all $k \in (0, k_0)$, regardless of the size of the delay r .

The question of convergence to equilibrium for values of k larger than k_0 in the logistic problem remains open. The restriction on k in Theorem 4.1 is imposed in order to guarantee uniqueness of solutions (v^*, w^*) of (11-13) satisfying $0 < v^* \leq w^* \leq \psi_k$. The author has been unable to determine

whether such uniqueness holds for all $k > 0$ and is unaware of any results in the literature that guarantee uniqueness for such k . The only other results the author is aware of appear in [25, Proposition 5.4, Corollary 5.6], and are equivalent to requiring that

$$k < \frac{\left(1 - \frac{1}{m}\right)^{m-1}}{\|e_0\|_\infty} \quad (25)$$

where e_0 is the unique positive solution of $-\Delta e_0 = 1$ in Ω with zero Dirichlet boundary conditions. For the case where $\Omega = (0, \pi)$ the bound in (25) becomes $k < 8\left(1 - \frac{1}{m}\right)^{m-1}/\pi^2$; compare with (24). However, one would still need to check whether the condition in (25) is sufficient to guarantee that the attracting interval $[v^*, w^*]$ has $v^* > 0$ in Ω , as opposed to v^* merely being non-negative and non-trivial (called *semi-coexistence states* in [25]).

Example 5.3. In this example we demonstrate that condition (A8) may hold with $\beta = \infty$ and therefore can be sufficient for global convergence of solutions of (D) for all $k > 0$. We consider the problem

$$\partial_t u = \Delta u^m + \frac{ku^p}{c + u^q(x, t - r)}$$

for $c, q > 0$ with Dirichlet boundary conditions. Here, $h(s) = k/(c + s^q)$ for $s \geq 0$. If $m \geq p + q$ then $\phi(u)h(u)/f(u) = ku^{m-p}/(c + u^q)$ is increasing on $(0, \infty)$ and so we may take $\beta = \infty$ in Theorem 4.1 to conclude that φ is globally attractive for all $k > 0$, where φ is the unique positive solution of

$$\Delta \varphi^m + \frac{k\varphi^p}{c + \varphi^q} = 0 \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{in } \partial\Omega.$$

If $m \in (p, p + q)$ however, then $\phi(u)h(u)/f(u)$ is only increasing on $(0, \beta)$, where

$$\beta = \left(\frac{c(m - p)}{p + q - m} \right)^{1/q}.$$

Thus, in the latter case, we can only ensure global attractivity of φ for non-negative solutions of (D) if

$$k < k_0(m, p, q, c) := \left(\frac{c(m-p)}{(p+1-m)\|\psi\|_\infty^q} \right)^{(m-p)/q}.$$

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