

Behavioural New Keynesian Models*

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Abstract

This paper provides a bird's eye view of the behavioural New Keynesian literature. We discuss three key empirical regularities in macroeconomic data which are not accounted for by the standard New Keynesian model, namely, excess kurtosis, stochastic volatility, and departures from rational expectations. We then present a simple behavioural New Keynesian model that accounts for these empirical regularities in a straightforward manner. We discuss elaborations and extensions of the basic model, and suggest areas for future research.

Keywords: Behavioural macroeconomics, heterogeneous expectations, bounded rationality.

JEL Codes: E70, E71, E30, E32.

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1 Introduction

The New Keynesian three equation model has defined monetary policy orthodoxy since the 1990s. Important contributions to the literature include Goodfriend and King (1997), McCallum and Nelson (1997), Clarida et al. (1999), and Woodford (2003). Large scale New Keynesian models, building on the contributions of Smets and Wouters (2003) and others, became increasingly popular at the turn of the millennium. These models are now used in a variety of central banks and policy institutions, guiding monetary policy and shaping forecasts around the world. Among other things, the New Keynesian orthodoxy is closely associated with inflation targeting and central bank independence.

Around the same time that the New Keynesian three equation model was emerging, economists working in various fields began to explore formal models of bounded rationality and heterogeneous expectations. An important early paper is Evans and Ramey (1992), which embeds costly expectation technologies into a simple macroeconomic model. The authors demonstrate that positive costs of prediction can prevent the emergence of rational expectations equilibria, as the increase in utility associated with accurate prediction may not be worth the costs of accessing those predictions. Their model builds on the earlier theories of consistent and inconsistent learning pioneered in the 1980s.

Evans and Ramey (1992) argue that the optimality of rational expectations is suspect if the costs of their formulation are not taken into account. Essentially, the imposition of rational expectations on macroeconomic models becomes the ultimate free lunch, as agents are endowed with a considerable amount of information at zero cost. Instead, the authors propose that, “any model of expectation formation must begin with an appropriate specification of expectational preferences and technology, if the model is to be regarded as representing rational, optimizing behavior” (ibid., 208). This approach to expectations clearly has the potential to affect theories regarding the conduct of monetary policy, and the authors show in their simple model that long run non-neutrality and hysteresis effects can occur when forecasting is costly.

A considerable advance in the literature on bounded rationality came in Brock and Hommes (1997), which embeds a simple heterogeneous expectations mechanism into a cobweb model of partial equilibrium. As in the standard cobweb model, firms have to forecast the equilibrium price before they set their output level. To do so, they have the choice of using a simple adaptive expectations predictor at zero cost, or perfect foresight at positive cost. The authors argue that firms will choose predictors that result in higher net profits, where the probability of choosing a given predictor is determined by a logit model. This choice is justified by an appeal to the discrete choice model described in Manski and McFadden (1981), which is widely used in microeconomics and econometrics. A similar approach is used in the reinforcement learning literature described in Young (2004).

The insights of Brock and Hommes (1997) were slowly incorporated into the New Keynesian literature in the early 2000s, mainly in the work of William Branch and his co-authors. Branch and McGough (2004) studied the impact of heterogeneous expectations on the existence of sunspot equilibria in rational expectations models, and Branch and Evans (2007) examined discrete choice dynamics of the form considered in Brock and Hommes (1997) in the context of a simple macroeconomic model. Heterogeneous expectations in a New Keynesian framework, albeit without discrete choice, were then examined in considerable detail in Branch and McGough (2009). Heterogeneous expectations and discrete choice were fully incorporated into the New Keynesian three equation model around the time that the USA

and Europe were recovering from the effects of the 2008 financial crisis, in the papers of Branch and McGough (2010) and De Grauwe (2011).

The basic framework set out in Branch and McGough (2010) and De Grauwe (2011) has become known as the behavioural New Keynesian model. This terminology is largely the result of De Grauwe's book length treatment of the subject, *Lectures on Behavioral Macroeconomics* (De Grauwe, 2012). Via the expectational mechanism of Brock and Hommes (1997), the behavioural New Keynesian (BNK) approach incorporates bounded rationality and heterogeneity into the standard New Keynesian (NK) three equation model. This embeds an intuitive form of complexity into the standard approach, where strategy switching generates recurring bouts of instability. Specifically, households and firms have the choice between two (or more) predictors of output and inflation. Around the steady state of a BNK model, both predictors are equally accurate, but one predictor becomes increasingly accurate relative to the other as the economy moves away from the steady state. Any exogenous shock that moves the economy away from the steady state can then lead to agents rapidly switching from one predictor to the other. This creates an endogenous amplification effect which can explain the existence of excess kurtosis and stochastic volatility observed in macroeconomic data.

The purpose of the present paper is to provide the reader with an accessible introduction to the BNK approach, by,

- a. Providing a detailed explanation and empirical evaluation of a simplified version of the BNK model contained in De Grauwe (2012),
- b. Providing a review of the key papers in the BNK literature, complementing existing overviews in Branch and McGough (2018) and Hommes and LeBaron (2018),
- c. Summarising key avenues for future research.

Our approach is therefore to emphasise and explain the connections between existing papers in the BNK literature, and draw together the dispersed empirical results which currently exist. Thus the paper serves a synthetic purpose, amalgamating and comparing existing results, which allows us to identify future research priorities. As argued in Dilaver et al. (2018), we are of the opinion that the BNK model successfully addresses a number of the criticisms levelled at the standard NK model since the crisis, making it an important contender for a new consensus. We hope that the present paper will increase its visibility, and encourage further research in the area.

In section 2 we discuss the main empirical facts that BNK models have sought to explain. We then present the simple BNK model in section 3, which illustrates the basic mechanism described above in a straightforward manner and accounts for the empirical facts discussed in section 2. We then present a bird's eye view of the BNK literature in section 4, including a detailed discussion of the basic model and a survey of the various extensions. Finally, section 5 suggests a number of areas that require further research.

2 Some empirical regularities

The literature preceding the BNK models of Branch and McGough (2010) and De Grauwe (2011) was largely motivated by the apparent falsification of rational expectations in studies

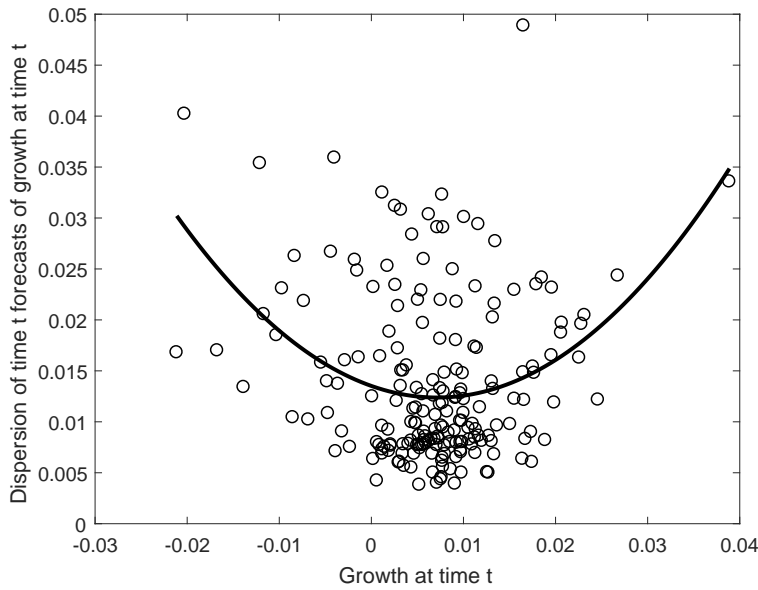
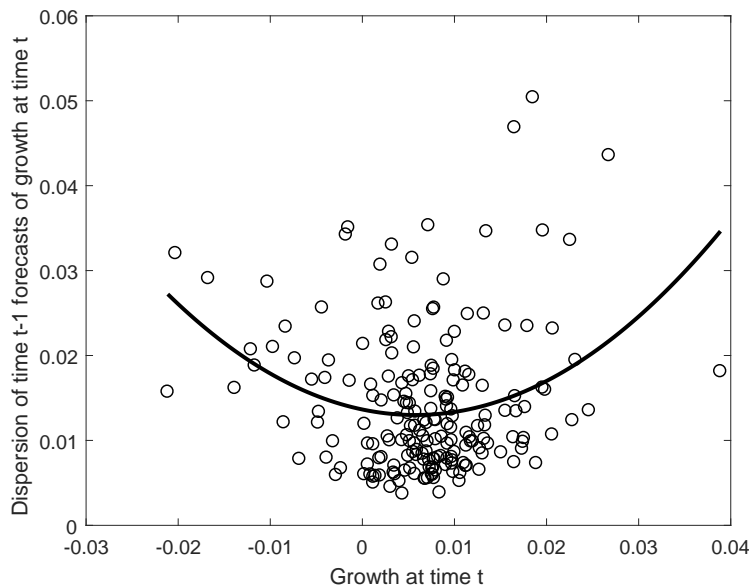
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Figure 1: Scatter plots of the cross-sectional dispersion (75th percentile minus 25th percentile) of real GDP growth from the US Survey of Professional Forecasters, against real GDP growth. Panel A uses forecasts of time t growth made at time t , and panel B uses forecasts of time t growth made at time $t - 1$. Quadratic lines of best fit are included for reference.

of survey data. This is an extremely large literature, with a comprehensive survey of the studies preceding the 2008 financial crisis contained in Pesaran and Weale (2006). The vast majority of these studies, with the notable exception of Keane and Runkle (1990), reject the rational expectations hypothesis when applied to survey data on expectations. Aside from this, casual empiricism indicates the considerable heterogeneity of private sector forecasts of macroeconomic variables, providing *prima facie* justification for the heterogeneous expectations approach of Brock and Hommes (1997).

Figure 1 presents simple scatter diagrams to illustrate the heterogeneity of quarterly GDP growth forecasts in the US Survey of Professional Forecasters. The data are drawn from forecasts of quarter-on-quarter growth made either in the relevant quarter (panel A)

Table 1: Higher moments of US real GDP growth and inflation

Variable	Standard Deviation	Skewness	Kurtosis
Real GDP growth	0.009	-0.022	4.519
Inflation	0.006	1.250	5.176

Notes: The growth rate of a variable x is defined as $(x_t - x_{t-1})/x_{t-1}$. Both series are measured at quarterly frequency, and downloaded from the FRED database on 24/04/2018. The inflation rate is calculated using the implicit GDP deflator.

Table 2: GARCH model results, US real GDP growth and inflation

Variable	ARCH(1)	GARCH(1)
Real GDP growth	0.22 (0.05)	0.78 (0.04)
Inflation	0.27 (0.08)	0.70 (0.05)

Notes: The estimated models are $x_t = \beta_0 + \sum_{s=1}^4 \beta_s x_{t-s} + \epsilon_t$ with $\epsilon_t = \sigma_t z_t$, where z_t is Gaussian white noise and $\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \sigma_{t-1}^2$. Standard errors are given in parentheses under the point estimates.

or one quarter ahead (panel B). The data plotted are the differences between the 75th and 25th percentiles of the forecast distribution. The region of highest density is where GDP growth is close to its long run average and forecast dispersion is low. When GDP growth is away from its long run average, in either direction, forecast dispersion tends to increase. A similar observation was made for inflation expectations by Mankiw et al. (2003), who observed that the interquartile ranges of forecasts in the Michigan Survey, Livingston Survey, and Survey of Professional Forecasters all increase with the distance of year-on-year inflation growth from its mean. The interested reader might compare figure 1 in the present paper with figure 6 in Mankiw et al. (2003). In a related paper, Doern et al. (2012, 1091) suggest that the effects of GDP growth on forecast dispersion should be non-linear, with disagreement rising during recessions as well as booms, although the effects of recessions on forecast dispersion are the main focus of their study.

The first empirical application of the Brock and Hommes (1997) framework is contained in Branch (2004). This paper finds evidence for discrete switching between a small number of forecasting mechanisms in the Michigan Survey of inflation expectations. A notable later paper is Pfajfar and Santoro (2010), who find that only 10% of the forecasts in the Michigan Survey reflect regular information updating, with forecasts to the left of the median being particularly static. Finally, more recent papers examine expectation formation in laboratory experiments, and generally find supportive evidence for heterogeneity. Hommes (2011) is a good example of this approach.

It is therefore reasonable to conclude that US data displays considerable departures from rational expectations, with forecast disagreement rising when GDP growth and inflation are away from their long run averages. As well as departures from rational expectations, De Grauwe (2012) argues that excess kurtosis in GDP growth and inflation is a basic empir-

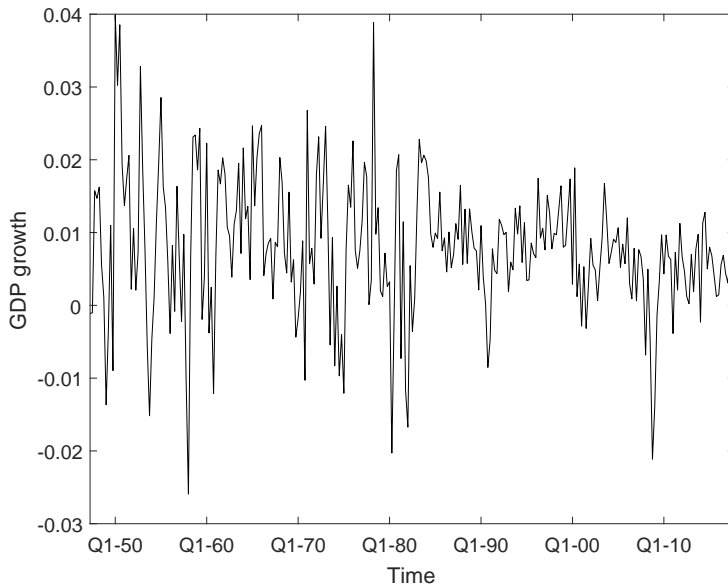
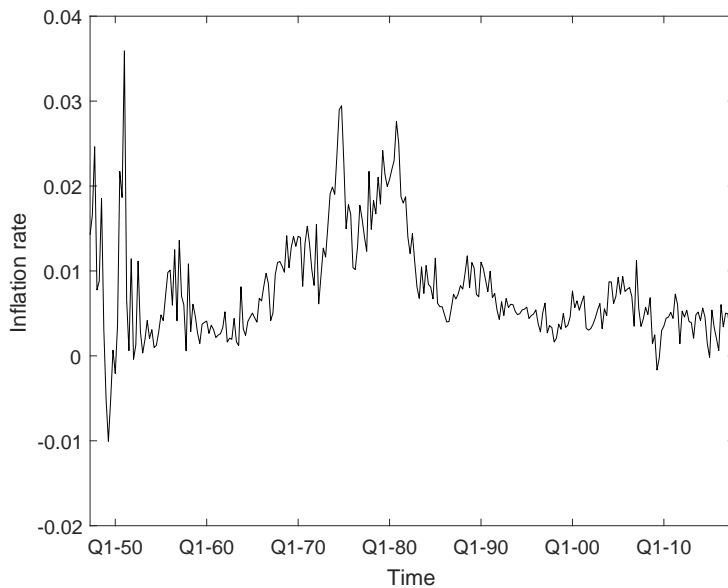
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Figure 2: Time series of US real GDP growth (panel A) and US inflation (panel B), Q2-1947 to Q4-2017.

B

ical fact which should be explained by BNK models. He offers some evidence that a simple BNK model can indeed generate excess kurtosis of a similar degree to that observed in US data. Interestingly, Ascari et al. (2015), building on preceding work by Fagiolo et al. (2008), argue that the standard NK model is incapable of matching the excess kurtosis observed in US time series. This is fairly obvious in the linearised NK model subject to Gaussian shocks, as a linear transformation of Gaussian noise is still Gaussian. However, they demonstrate that a medium scale non-linear NK model is also unable to match the higher moments of US data, even when it is subject to heavy tailed shocks. This is because the standard adjustment frictions in NK models, including habit persistence in consumption, adjustment costs to capital, and so on, all serve to smooth out the effects of shocks. As a result, the observed series become more Gaussian than the underlying shock processes.

The higher moments of US quarterly real GDP growth and the inflation rate are presented

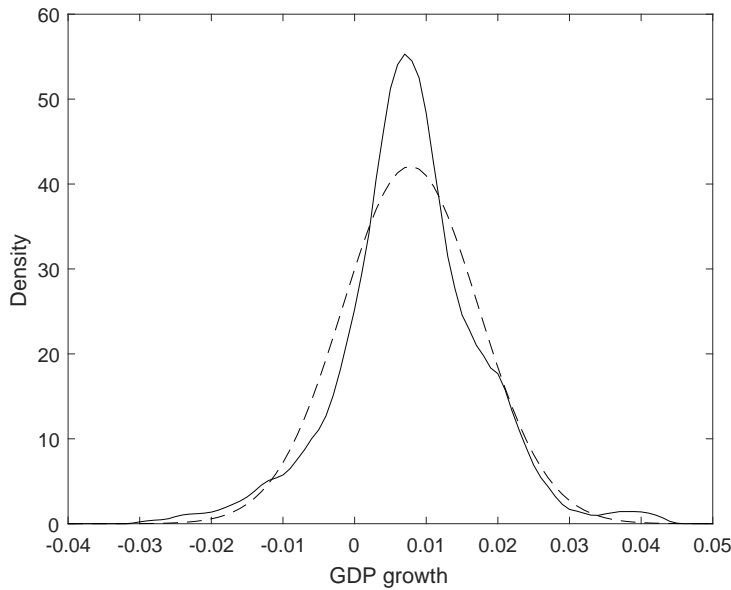
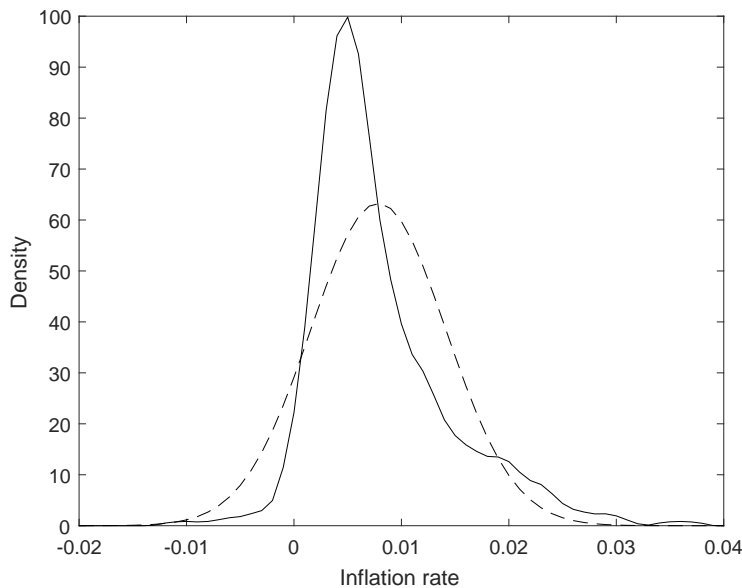
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Figure 3: Kernel density estimates of US real GDP growth (panel A) and US inflation (panel B), Q2-1947 to Q4-2017. Epanechnikov kernel used in both panels; normal density provide for comparison. Jarque-Bera statistics are 27.2 and 129.5 respectively, where the 5% critical value is 5.761.

in table 1, and the sample characteristics of the series are illustrated graphically in the time series in figure 2 and the kernel density plots in figure 3. As discussed in Fagiolo et al. (2008), De Grauwe (2012), and Ascari et al. (2015), both real GDP growth and inflation exhibit excess kurtosis. In addition, while real GDP growth is relatively symmetrical, the inflation rate is positively skewed over the 1947 - 2017 sample period.

Finally, Branch and Evans (2007) note that stochastic volatility is an empirical regularity in post-war US time series. This is usually explained by changes in monetary policy or shifts in the volatility of exogenous shocks hitting the economy, with arguments surrounding the causes of the Great Moderation being a good example of this. Branch and Evans (2007), however, propose that endogenous switching between heterogeneous predictors in the spirit of Brock and Hommes (1997) is an alternative explanation of stochastic volatility. Indeed, given the basic BNK logic described in section 1, we should not be particularly surprised

if stochastic volatility is the result. Following Branch and Evans (2007), table 2 presents results from AR(4) models with GARCH(1,1) errors fitted to US real GDP growth and inflation. As the results demonstrate, the ARCH and GARCH parameters for both models are significant at the 5% level, which is consistent with the excess kurtosis reported in table 1. We can therefore state with some confidence that stochastic volatility is present in US real GDP growth and inflation.

Thus US survey data display departures from rational expectations, and US macroeconomic data display excess kurtosis and stochastic volatility. These are the basic empirical facts that the BNK literature has sought to explain. Section 3 presents a simple BNK model, which illustrates the basic mechanism described in section 1 in a transparent manner, and accounts for the empirical facts discussed above.

3 A simple BNK model

3.1 The standard NK model

Consider the standard NK model in its three-equation form,

$$y_t = \mathbb{E}_t y_{t+1} - (i_t - \mathbb{E}_t \pi_{t+1}) + u_{1t}, \quad (1)$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa(y_t + u_{2t}), \quad (2)$$

$$i_t = \theta_\pi \pi_t + \theta_y y_t + u_{3t}, \quad (3)$$

where y_t denotes the log output gap, π_t denotes the log inflation gap, i_t denotes the log interest rate gap, and $i_{t-1} - \pi_t$ is the ex post real interest rate. The parameters are β , the representative household's discount factor, κ , the composite Phillips curve parameter, and θ_π and θ_y , the elasticities of the interest rate with respect to inflation and output. The shock processes u_{1t} , u_{2t} , and u_{3t} , are shocks to the effective discount factor or risk premium, marginal costs, and monetary policy, respectively, and are usually autoregressive processes.

By substituting (3) into (1), we arrive at the two dimensional vector system,

$$\begin{bmatrix} 1 + \theta_y & \theta_\pi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t y_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \kappa \end{bmatrix} \begin{bmatrix} u_{1t} - u_{3t} \\ u_{2t} \end{bmatrix}, \quad (4)$$

which in turn yields the reduced form,

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \frac{1}{\kappa\theta_\pi + \theta_y + 1} & \frac{1 - \beta\theta_\pi}{\kappa\theta_\pi + \theta_y + 1} \\ \frac{\kappa}{\kappa\theta_\pi + \theta_y + 1} & \frac{\kappa + \beta(\theta_y + 1)}{\kappa\theta_\pi + \theta_y + 1} \end{bmatrix} \begin{bmatrix} \mathbb{E}_t y_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} \frac{1}{\kappa\theta_\pi + \theta_y + 1} & \frac{-\kappa\theta_\pi}{\kappa\theta_\pi + \theta_y + 1} \\ \frac{\kappa}{\kappa\theta_\pi + \theta_y + 1} & \frac{\kappa(\theta_y + 1)}{\kappa\theta_\pi + \theta_y + 1} \end{bmatrix} \begin{bmatrix} u_{1t} - u_{3t} \\ u_{2t} \end{bmatrix}. \quad (5)$$

The representation of the standard NK model in (4) is useful for illustrating its connection with the standard AD-AS model, while the reduced form in (5) will be useful when we

consider the BNK model in section 3.2. The Blanchard-Kahn condition for saddle-path stability is the familiar,

$$\theta_\pi + \left(\frac{1-\beta}{\kappa}\right)\theta_y > 1, \quad (6)$$

which approximates the Taylor principle for monetary policy. As discussed in section 2, the model is unable to generate excess kurtosis unless the shock processes u_{1t} , u_{2t} , and u_{3t} are themselves heavy-tailed, and elaborations of the standard NK model are often unable to generate excess kurtosis even in the presence of heavy-tailed shocks.

3.2 A BNK model in action

In this section, we present a simple BNK model that captures the basic insight discussed in section 1. BNK models depart from the standard NK model by replacing the rational expectations operator \mathbb{E}_t in (1) - (3) with a bounded rational predictor \hat{E}_t , which can be achieved using the Euler learning approach of Branch and McGough (2010) and De Grauwe (2011). Other approaches, such as the anticipated utility approach of Massaro (2013) and Calvert Jump et al. (2018), or the ‘‘cognitive discounting’’ approach of Gabaix (2016), yield similar results. We return to these alternative approaches in section 4.

Using the Euler learning approach, the reduced form (5) remains the same as in the standard NK case other than the incorporation of bounded rational prediction. This relies on an aggregation result presented in Branch and McGough (2009), in which expectations are both uniform in the steady state and possess certain linearity properties, and the law of iterated expectations holds at both the individual and aggregate level. We then have,

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \frac{1}{\kappa\theta_\pi + \theta_y + 1} & \frac{1-\beta\theta_\pi}{\kappa\theta_\pi + \theta_y + 1} \\ \frac{\kappa}{\kappa\theta_\pi + \theta_y + 1} & \frac{\kappa + \beta(\theta_y + 1)}{\kappa\theta_\pi + \theta_y + 1} \end{bmatrix} \begin{bmatrix} \hat{E}_t y_{t+1} \\ \hat{E}_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} \frac{1}{\kappa\theta_\pi + \theta_y + 1} & \frac{-\kappa\theta_\pi}{\kappa\theta_\pi + \theta_y + 1} \\ \frac{\kappa}{\kappa\theta_\pi + \theta_y + 1} & \frac{\kappa(\theta_y + 1)}{\kappa\theta_\pi + \theta_y + 1} \end{bmatrix} \begin{bmatrix} u_{1t} - u_{3t} \\ u_{2t} \end{bmatrix}, \quad (7)$$

where $\hat{E}_t y_{t+1}$ and $\hat{E}_t \pi_{t+1}$ are average expectations. The model is a version of the familiar textbook AD-AS model considered in, for example, Romer (2006). However, unlike in the textbook AD-AS model or the standard NK model, endogenous shifts in expectations in the BNK model can amplify the effects of exogenous demand and supply shocks, resulting in extreme events, non-Gaussian fluctuations, and stochastic volatility.

In the BNK model, these endogenous shifts in expectations are modelled using the discrete choice approach of Manski and McFadden (1981) or the reinforcement learning approach discussed in Young (2004). To limit the departure from rationality, the reinforcement learning approach proposes that, although adaptation can be slow and there can be a random component of choice, the greater the net benefit from taking an action in the past, the more likely it will be taken in the future. In the BNK literature, the net benefit from using a predictor is usually defined as minus its squared forecasting error plus the cost of obtaining that forecasting rule. Then, if we index the m possible predictors of a variable x_t by $i \in \mathbb{N}$, $i \leq m$, the proportion of agents n_{jt} using predictor j is updated using the multinomial logit model,

$$n_{jt} = \frac{e^{-\mu[(x_{t-1} - \hat{E}_{t-2}^j x_{t-1})^2 + C_j]}}{\sum_{i=1}^m e^{-\mu[(x_{t-1} - \hat{E}_{t-2}^i x_{t-1})^2 + C_i]}} \quad (8)$$

where C_i is the cost of obtaining predictor i and μ is a speed of adjustment parameter. The key features of (8) are that the best performing rule will attract the most followers, and that the proportions $n_{1t}, n_{2t}, \dots, n_{mt}$ sum to one in each period.

We are now in a position to fully specify a simple version of the BNK model. We suppose that there are two possible predictors for output and inflation, an extrapolative predictor and a fundamentalist predictor. Specifically, we propose that,

$$\hat{E}_t^e y_{t+1} = y_{t-1},$$

$$\hat{E}_t^f y_{t+1} = 0,$$

$$\hat{E}_t^e \pi_{t+1} = \pi_{t-1},$$

$$\hat{E}_t^f \pi_{t+1} = 0.$$

These predictors are the same as those used in chapter 1 of De Grauwe (2012), and the model is therefore a simplified version of that model. The extrapolative rule predicts that output and inflation will continue on their current trajectories, which is optimal in a mean squared error sense when agents believe that output and inflation follow random walks. As the model does in fact generate highly persistent output and inflation series, the extrapolative predictors are also relatively accurate.

Households and firms using the fundamentalist predictor believe that output and inflation will return to their long run averages in the next period, which is an extreme version of the return to normality model discussed in Pesaran and Weale (2006). Agents that use this predictor are essentially using a static forecasting mechanism, and incorporate very little relevant information. The existence of this type of agent is supported by the evidence of Pfajfar and Santoro (2010), who find that inflation forecasts to the left of the median forecast are essentially static, entail significant forecast errors, and do not incorporate information from any relevant macroeconomic statistics. Similarly, Carroll (2003) finds that only around a quarter of the participants in the Michigan Survey update their forecasts in every quarter. Clearly, the fundamentalist predictor is only accurate when the economy is close to the steady state.

Denote the proportion of agents using the extrapolative predictor for output by n_t^y , and the proportion of agents using the extrapolative predictor for inflation by n_t^π . Average expectations at any point in time are then given by,

$$\hat{E}_t y_{t+1} = n_t^y y_{t-1}, \quad (9)$$

$$\hat{E}_t \pi_{t+1} = n_t^\pi \pi_{t-1}, \quad (10)$$

and the proportions are updated according to,

Table 3: Higher moments of simulated real GDP growth and inflation

Variable	Standard Deviation	Skewness	Kurtosis
Real GDP growth	0.008	0.00	3.424
Inflation	0.007	0.00	5.457

Notes: The growth rate of real GDP is defined as $y_t - y_{t-1}$, recalling that the model is specified in log deviations from the steady state.

Table 4: GARCH model results, simulated real GDP growth and inflation

Variable	ARCH(1)	GARCH(1)
Real GDP growth	0.06	0.40
Inflation	0.01	0.87

Notes: The estimated models are $x_t = \beta_0 + \sum_{s=1}^4 \beta_s x_{t-s} + \epsilon_t$ with $\epsilon_t = \sigma_t z_t$, where z_t is Gaussian white noise and $\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \sigma_{t-1}^2$. Standard errors are not given, as the length of the simulation = $5 \cdot 10^5$.

$$n_t^y = \frac{e^{-\mu[(y_{t-1}-y_{t-3})^2+C]}}{e^{-\mu[(y_{t-1}-y_{t-3})^2+C]} + e^{-\mu[y_{t-1}^2]}}, \quad (11)$$

$$n_t^\pi = \frac{e^{-\mu[(\pi_{t-1}-\pi_{t-3})^2+C]}}{e^{-\mu[(\pi_{t-1}-\pi_{t-3})^2+C]} + e^{-\mu[\pi_{t-1}^2]}}. \quad (12)$$

To reduce the number of parameters in (11) and (12), we have assumed that there is zero cost associated with the fundamentalist predictors, and the cost C of using the extrapolative output predictor is the same as the cost of using the extrapolative inflation predictor. Note that the cost parameter in (11) and (12) is measured in mean squared error units.

The full model is described by (7) and (9) - (12). To simulate the model, we calibrate the parameters shared with the standard NK model in a similar manner to Gali (2008), i.e. $\beta = 0.99$, $\kappa = 0.1$, $\theta_\pi = 1.5$, $\theta_y = 0.125$. To further reduce the number of parameters, we switch off the monetary policy shock, and we calibrate the demand and supply shocks u_{1t} , u_{2t} as AR(1) processes with autoregressive parameter equal to 0.9 and error standard deviations of 0.009 and 0.012, respectively. We set the cost parameter $C = 0$, for the simple reason that the nonlinear parameters in switching models of this form are notoriously difficult to identify. This is discussed in Jang and Sacht (2016) in the context of a behavioural New Keynesian model, and (for example) Terasvirta (1994) in the context of smooth transition autoregressive models. In order to concentrate on a single nonlinear parameter, we therefore set C equal to zero and choose the adjustment parameter μ using a simple moment matching routine described in appendix A; this results in $\mu = 210$.

Higher moments of the simulated output series are presented in table 3. These compare favourably with the empirical moments presented in table 1, although the model generates lower kurtosis for output growth and higher kurtosis for inflation compared to the US data.

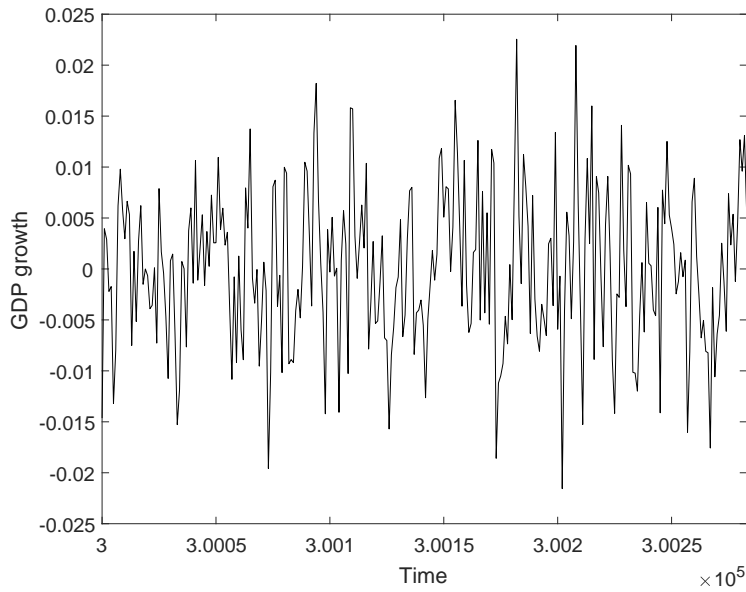
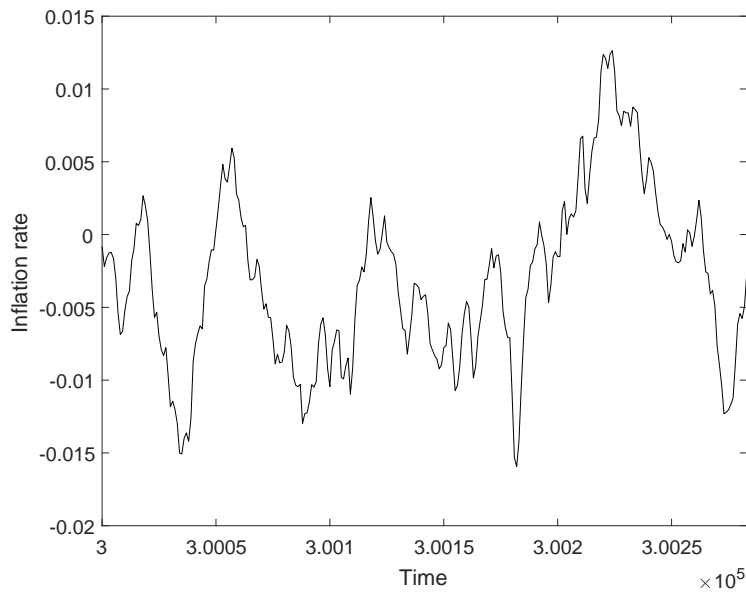
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Figure 4: Time series of simulated real GDP growth (panel A) and inflation (panel B), $t = 300,000$ to $t = 300,283$, using the simple BNK model in section 3.2.

B

In addition, the model cannot capture the positive skewness in US inflation, which is unsurprising as there are no structural asymmetries. Notwithstanding the lack of skew, it is reasonable to conclude that the model can account for the non-normality observed in US output and inflation in a relatively successful manner, given its simplicity. In particular, as illustrated in figure 4, the probabilities of deep recessions and inflationary episodes are non-negligible.

Results from estimating AR(4) models with GARCH(1,1) errors on the simulated data are presented in table 4. The results are qualitatively similar to those of the equivalent models estimated on US data, although the ARCH(1) parameters are somewhat lower than those in table 2, particularly for the simulated inflation series. This should still be considered a success, however, as the GARCH estimates were not used in the moment matching routine used to calibrate the model. Finally, figures 5 and 6 demonstrate that the simulated

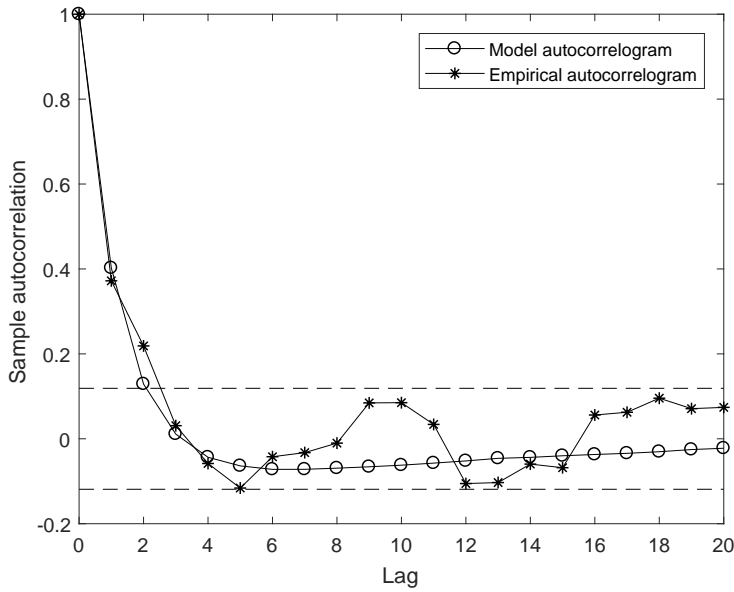
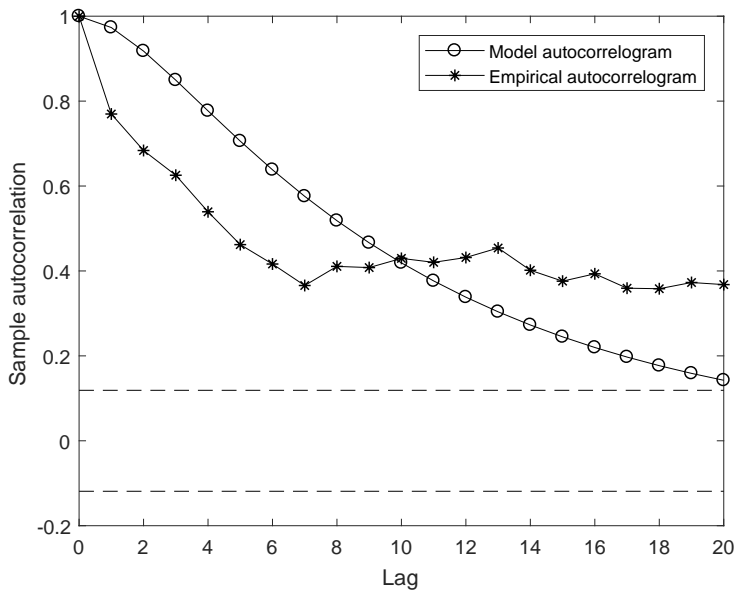
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Figure 5: Auto-correlation functions of simulated and empirical (US) real GDP growth (panel A) and inflation (panel B), using the simple BNK model in section 3.2.

autocorrelations and cross-correlations compare favourably with the US data, with GDP growth negatively correlated with the previous year's inflation rate. This, however, is to be expected, as it is known that the linear NK model is capable of adequately matching the second moments of US output and inflation. Appendix B examines the robustness of these results to alternative interest rate rules and an alternative parameterisation.

3.3 The BNK mechanism

To understand how the model generates excess kurtosis and stochastic volatility, consider a trajectory near the steady state. Households and firms using the extrapolative predictors believe that output and inflation will remain close to the steady state, and households and

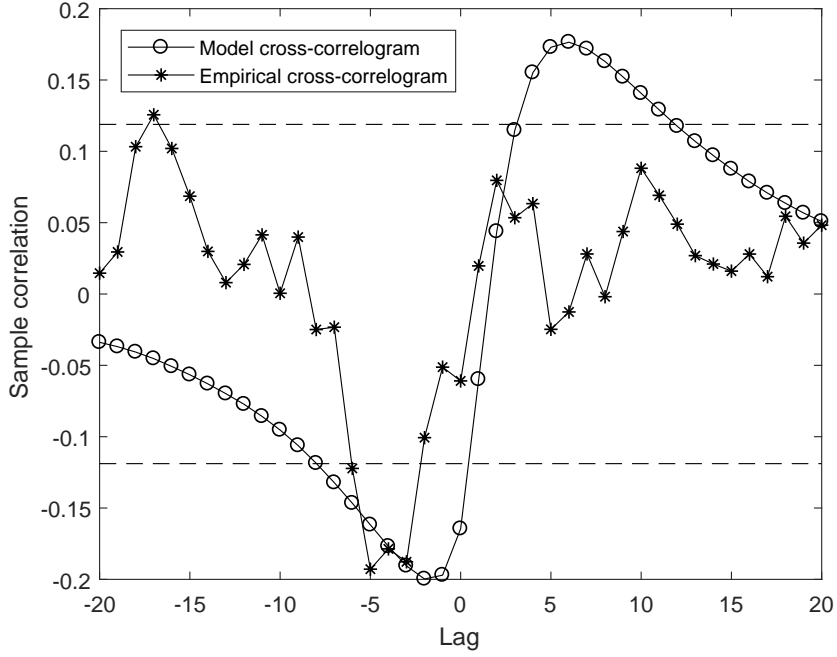


Figure 6: Cross-correlogram of simulated and empirical GDP growth and lagged inflation, using the simple BNK model in section 3.2. Precisely, the graph is of $\text{corr}[\Delta y_t, \pi_{t+\text{Lag}}]$.

firms using the fundamentalist predictors believe the same. The two sets of predictors will therefore have similar forecasting errors, and given a non-negative cost to using the extrapolative predictors, most agents will be using the fundamentalist predictors. The variance of output and inflation will be relatively low as a result, with the covariance matrix when all agents are using the fundamentalist predictors given by,

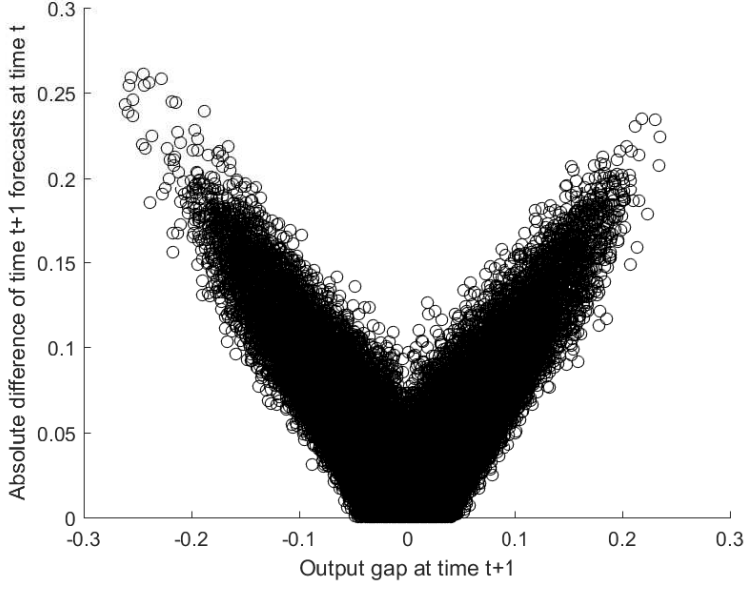
$$\text{Var} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} \Big|_{n_t^y = n_t^\pi = 0} = \tag{13}$$

$$\begin{bmatrix} \frac{1}{\kappa\theta_\pi + \theta_y + 1} & \frac{-\kappa\theta_\pi}{\kappa\theta_\pi + \theta_y + 1} \\ \frac{\kappa}{\kappa\theta_\pi + \theta_y + 1} & \frac{\kappa(\theta_y + 1)}{\kappa\theta_\pi + \theta_y + 1} \end{bmatrix} \text{Var} \begin{bmatrix} u_{1t} - u_{3t} \\ u_{2t} \end{bmatrix} \begin{bmatrix} \frac{1}{\kappa\theta_\pi + \theta_y + 1} & \frac{-\kappa\theta_\pi}{\kappa\theta_\pi + \theta_y + 1} \\ \frac{\kappa}{\kappa\theta_\pi + \theta_y + 1} & \frac{\kappa(\theta_y + 1)}{\kappa\theta_\pi + \theta_y + 1} \end{bmatrix}^T.$$

Now, consider a demand or supply shock that moves the economy away from the steady state. The mean squared forecasting error of the fundamentalist predictors will quickly become greater than the mean squared error of the extrapolative predictors, and households will switch to using the latter following the reinforcement learning processes in (11) and (12). Thus, following an impulse that pushes the economy away from the steady state, households and firms switch to using the extrapolative predictors. In comparison to (13), the covariance matrix of output and inflation when all agents are using the extrapolative predictors is given by,

$$\text{vec} \left(\text{Var} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} \Big|_{n_t^y = n_t^\pi = 1} \right) = (I_{4 \times 4} - (A \otimes A))^{-1} \text{vec} \left(\text{Var} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} \Big|_{n_t^y = n_t^\pi = 0} \right), \tag{14}$$

A



B

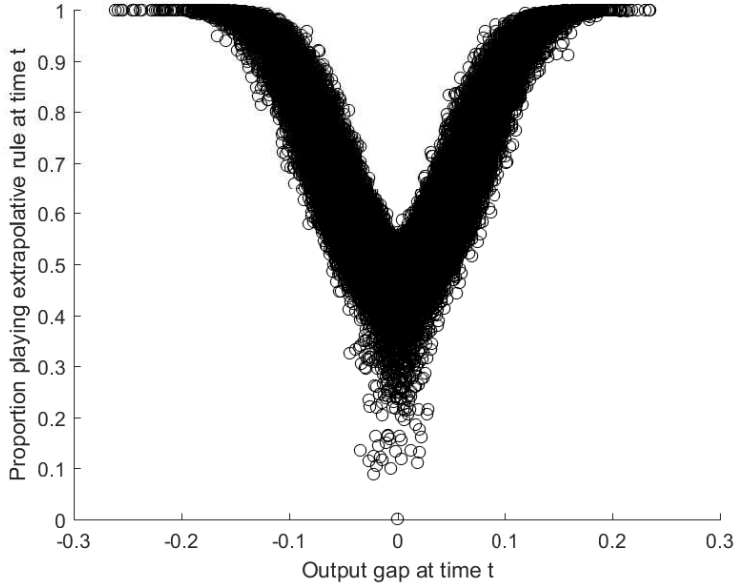


Figure 7: Panel A: Scatter plot of $|\hat{E}_t^e y_{t+1} - \hat{E}_t^f y_{t+1}|$ against time $t + 1$ output. Panel B: Scatter plot of n_t^y against time t output, both using the simple BNK model in section 3.2.

see Hamilton (1994, 265), with,

$$A = \begin{bmatrix} \frac{1}{\kappa\theta_\pi + \theta_y + 1} & \frac{1 - \beta\theta_\pi}{\kappa\theta_\pi + \theta_y + 1} \\ \frac{\kappa}{\kappa\theta_\pi + \theta_y + 1} & \frac{\kappa + \beta(\theta_y + 1)}{\kappa\theta_\pi + \theta_y + 1} \end{bmatrix}. \quad (15)$$

As the eigenvalues of A are inside the unit circle, the eigenvalues λ of $A \otimes A$ are also inside the unit circle. The eigenvalues of $I_{4 \times 4} - (A \otimes A)$ are then $1 - \lambda$, and the eigenvalues of $(I_{4 \times 4} - (A \otimes A))^{-1}$ are given by $(1 - \lambda)^{-1}$. As a result, (14) generally results in greater variances of output and inflation than (13), and volatility increases as agents start to use the extrapolative predictors.

Now consider a trajectory away from the steady state. Households and firms using the

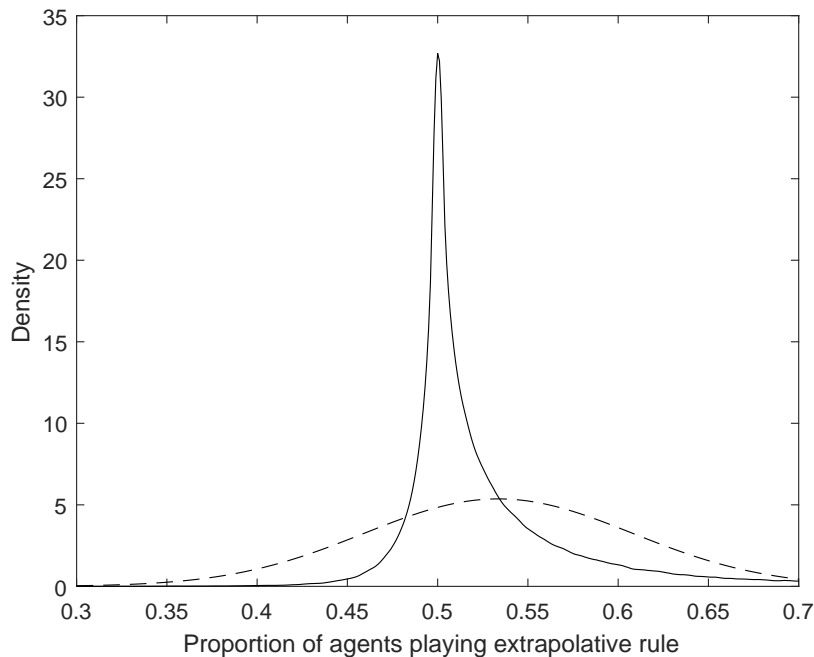


Figure 8: Kernel density estimate of n_t^y , i.e. the proportion of agents playing the extrapolative forecasting rule for GDP at time t . Epanechnikov kernel used; normal density provided for comparison.

extrapolative predictors believe that output and inflation will remain away from the steady state, so most agents will be using these predictors in the absence of large shocks. However, any shock that pushes the economy towards the steady state will lead to households and firms switching towards the fundamentalist predictors, which stabilises the model given (13) and (14). Thus households and firms switch to the destabilising extrapolative predictors when the economy is pushed away from the steady state, and switch to the stabilising fundamentalist predictors when the economy is near the steady state. Given the foregoing, the effects of exogenous demand and supply shocks are amplified by non-linear behavioural changes, which appears to be a sufficient condition for excess kurtosis and stochastic volatility in the New Keynesian model¹.

Figure 7 illustrates this mechanism in action. Panel A demonstrates that the difference between the extrapolative and fundamentalist forecasts of output are low when the economy is near the steady state, and progressively increases as the economy moves away from the steady state. This can be compared to the scatter diagrams of empirical forecast dispersion presented in figure 1. Panel B demonstrates that the proportion of agents playing the extrapolative predictor for output is low when the economy is near the steady state, and increases as the economy moves away from the steady state. This results in a highly volatile series for n_t^y , and the attractor in panel B of figure 7 can be usefully compared to the chaotic attractors in figures 1 and 5 of Brock and Hommes (1997). In fact, as illustrated in figure 8, the distribution of n_t^y is highly non-normal with positive skew and excess kurtosis, and a long run average of approximately 0.54 under the baseline parameterisation.

The basic empirical facts that the BNK literature has set out to explain are departures from rational expectations, excess kurtosis, and stochastic volatility. A simple BNK model

¹Interestingly, this particular BNK model is locally stable if the corresponding rational expectations model is stable, i.e. if (6) holds. This is discussed in appendix C in more detail, but is not a general result.

appears to successfully account for these facts, reinforcing the evidence presented in Branch and Evans (2007) and De Grauwe (2012). We therefore consider the BNK framework to be a promising way to incorporate a number of the criticisms of mainstream economics into the standard NK model. To encourage further work on BNK models, section 4 presents a bird’s eye view of the literature, including elaborations and extensions of the basic model, and suggests key areas for future research.

4 An overview of the literature

Having discussed the intuition behind BNK models, and established that a simple BNK model accounts for the excess kurtosis, stochastic volatility, and departures from rational expectations observed in US data, we now provide an overview of the BNK literature. In section 4.1 we discuss the basic model in its various forms, then in section 4.2 we discuss various extensions and elaborations.

4.1 The basic model

The first model to add endogenous switching between heterogeneous predictors to the standard NK model is contained in Branch and McGough (2010). This builds on previous work contained in Branch and McGough (2009), which incorporated heterogeneous predictors into the standard NK model, but with fixed proportions of agents using each predictor. In the latter paper, the authors note that Mankiw et al. (2003) document disagreements in survey data on expectations, and Branch (2004) finds that survey data on expectations are consistent with a dynamic discrete choice model of a form similar to (8). As discussed in section 1, the models in Branch and McGough (2009, 2010) evolved from a number of previous studies conducted by William Branch and his co-authors on heterogeneous expectations.

The authors draw attention to the difference between their approach and other forms of simple heterogeneity in NK models, including the rational-myopic distinction of Gali et al. (2004) and the learning literature following Evans and Honkapohja (2001)². In the latter approach, agents have access to a predictor which is consistent with the minimum state variable form of the rational expectations equilibrium, but have to estimate the parameters in real time. In the approach of Branch and McGough (2010), agents have access to an incorrectly specified predictor and a rational expectations predictor, but the latter has a fixed cost of access. The standard NK part of the model is almost identical to the model considered in section 3.1, but instead of (3) the central bank sets the nominal interest rate according to,

$$i_t = \theta_\pi \hat{E}_t \pi_{t+1} + \theta_y \hat{E}_t y_{t+1} + u_{3t}. \quad (16)$$

The interest rule in (16) is an expectations-based rule, and combined with (1) - (2) yields the reduced form,

²It is also somewhat different to the later approaches of Kurz et al. (2013), Gabaix (2016), and Woodford (2018).

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 - \theta_y & 1 - \theta_\pi \\ \kappa(1 - \theta_y) & \beta + \kappa(1 - \theta_\pi) \end{bmatrix} \begin{bmatrix} \hat{E}_t y_{t+1} \\ \hat{E}_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \kappa & \kappa \end{bmatrix} \begin{bmatrix} u_{1t} - u_{3t} \\ u_{2t} \end{bmatrix}. \quad (17)$$

However, unlike in the simple BNK model considered in section 3.2, $\hat{E}_t y_{t+1}$ and $\hat{E}_t \pi_{t+1}$ are not entirely backwards looking in Branch and McGough (2010). Instead, we have,

$$\hat{E}_t y_{t+1} = n_t^y \mathbb{E}_t y_{t+1} + (1 - n_t^y) \phi^2 y_{t-1}, \quad (18)$$

$$\hat{E}_t \pi_{t+1} = n_t^\pi \mathbb{E}_t \pi_{t+1} + (1 - n_t^\pi) \phi^2 \pi_{t-1}, \quad (19)$$

i.e. n_t^y agents forecast output using the rational expectations predictor, and $1 - n_t^y$ forecast output using a simple backwards looking predictor, with a similar process in place for inflation. As pointed out in Branch and McGough (2010), bounded rational agents dampen past data in forming expectations when $\phi < 1$, and believe future output and inflation will diverge when $\phi > 1$. The case we use in section 3.2, $\phi = 1$, is the baseline case used in Brock and Hommes (1997). Ceteris paribus, the BNK model is more unstable over a greater part of the parameter space when $\phi > 1$, and a smaller part of the parameter space when $\phi < 1$.

As in (8), households and firms in Branch and McGough (2010) choose their predictor each period according to a reinforcement learning rule, with the rational expectations predictor having a fixed cost associated with it. Under certain parameterisations the steady state conditional on $n_t^y = 1$ is a sink, and the steady state conditional on $n_t^y = 0$ is a source. Around the steady state, however, it is optimal for households and firms to use the backwards looking predictor to save on the cost of accessing the rational expectations predictor, hence n_t^y falls. Away from the steady state, the backwards looking predictor becomes so inaccurate that it pays to use the rational expectations predictor despite its cost, hence n_t^y rises. Thus the steady state can become locally unstable but globally stable via a Hopf bifurcation, which can rapidly lead to chaotic dynamics as a bifurcation parameter is varied.

The dynamics of the model in Branch and McGough (2010) follow the basic logic described in section 1, although in the deterministic model this logic gives rise to limit cycles and chaotic dynamics rather than stochastic volatility. A similar model, apparently developed independently, is presented in De Grauwe (2011). This uses the standard NK framework in (1) - (3), although structural lags in output and inflation are allowed for. The major difference between De Grauwe (2011) and Branch and McGough (2010) is the absence of rational expectations in the former. Instead, $\hat{E}_t y_{t+1}$ and $\hat{E}_t \pi_{t+1}$ are given by,

$$\hat{E}_t y_{t+1} = n_t^y g_t + (1 - n_t^y)(-g_t), \quad (20)$$

$$\hat{E}_t \pi_{t+1} = n_t^\pi \pi^T + (1 - n_t^\pi) \pi_{t-1}, \quad (21)$$

where the difference between g_t and $-g_t$ is an increasing function of the standard deviation of output, and π^T is the central bank's inflation target. Households and firms switch between optimistic and pessimistic output forecasts in this model, and inflation forecasts that imply varying degrees of belief in the official target of the central bank. Again, the dynamics of the model follow the basic logic described in the introduction, although the paper is mainly

concerned with the implications of endogenous waves of animal spirits for the conduct of monetary policy.

The structural simplicity of the models discussed in Branch and McGough (2010), De Grauwe (2011), and section 3.2 of the present paper, is made possible by the use of Euler equation learning. An important alternative approach is the use of anticipated utility, or infinite horizon learning. This was introduced in Kreps (1998), used in a standard New Keynesian framework in Preston (2005), and used in a heterogeneous expectations framework in Massaro (2013). In this model, households solve their utility maximisation problem conditional on point expectations by taking into account their intertemporal budget constraint and associated transversality condition, as well as the consumption Euler equation. Specifically, households in Massaro (2013) maximise their intertemporal utility function,

$$\hat{E}_{it} \sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{C_{is}^{1-\sigma}}{1-\sigma} - \chi \frac{H_{is}^{1+\gamma}}{1+\gamma} \right), \quad (22)$$

where households are indexed by i and H denotes labour supply, subject to the flow budget constraint,

$$C_{it} + B_{it} \leq W_t H_{it} + I_{t-1} \Pi_t^{-1} B_{it-1} + D_t, \quad (23)$$

where B_t denotes bond holdings, I_t denotes the gross nominal interest rate, and D_t denotes dividends received by firms. Using lower case variables to denote log deviations from the steady state, Massaro (2013) demonstrates that (22) and (23) in conjunction with the associated transversality condition lead to a consumption rule of the form,

$$c_{it} = \hat{E}_{it} \sum_{s=t}^{\infty} \beta^{s-t} (\zeta_w w_s + \zeta_d d_s) - \beta \hat{E}_{it} \sum_{s=t}^{\infty} \beta^{s-t} (i_s - \pi_{s+1}), \quad (24)$$

where ζ_w and ζ_d are composite parameters, and we have imposed $b_{it} = 0$ and $\sigma = 1$ for simplicity. Consumption is thus increasing in expected future income flows, and decreasing in expected future real interest rates.

By aggregating across households, and imposing various equilibrium conditions, one can show that (24) implies an aggregate consumption function of the form,

$$c_t = (1 - \beta) \hat{E}_t \sum_{s=t}^{\infty} \beta^{s-t} y_s - \beta \hat{E}_t \sum_{s=t}^{\infty} \beta^{s-t} (i_s - \pi_{s+1}). \quad (25)$$

Massaro (2013) considers a general expectations structure, but if we make the restrictive assumptions that $\hat{E}_t y_s = y_t$ for all $s > t$, $\hat{E}_t \pi_s = \pi_{t-1}$ for all $s > t$, and that $\hat{E}_t i_s = i_t$ for all $s > t$, then (25) reduces to,

$$c_t = y_t - \left(\frac{\beta}{1 - \beta} \right) (i_t - \pi_{t-1}). \quad (26)$$

This is an example of an anticipated utility consumption function, where agents maximise utility conditional on their current beliefs. This approach is similar to, but distinct from, the internal rationality approach in which agents, “maximize utility under uncertainty, given

their constraints and given a consistent set of probability beliefs about payoff-relevant variables that are beyond their control or external” (Adam and Marcet, 2011). The approach of Adam and Marcet (2011) requires a fully Bayesian plan for beliefs, as opposed to the anticipated utility approach, in which households do not consider the possibility that their beliefs might change in the future. The latter is obviously more straightforward than the former, although Cogley and Sargent (2008) demonstrate that the anticipated utility approach can be seen as a good approximation to the fully Bayesian approach.

Calvert Jump et al. (2018) introduce anticipated utility into the basic BNK model by allowing agents to switch between anticipated utility and rational expectations in essentially the same manner as Branch and McGough (2010). They demonstrate that the NK Phillips curve with n_t rational firms and $1 - n_t$ anticipated utility firms is given by,

$$\pi_t = n_t(\beta\mathbb{E}_t\pi_{t+1} + \kappa y_t) + (1 - n_t)(\delta\beta\pi_{t-1} + \kappa\psi y_t), \quad (27)$$

when $\hat{E}_t\pi_{t+1} = \pi_{t-1}$, where δ and ψ are composite parameters, but otherwise the notation is identical to that in section 3. Indeed, when $n_t = 1$ the Phillips curve in (27) is identical to the standard NK Phillips curve in (2). Either by making use of the consumption function in (26), or assuming that profit income is split across households according to their relative importance in the economy, Calvert Jump et al. (2018) then derive the equilibrium condition $i_t = \pi_{t-1}$ as a special case. This substantially simplifies the model, and the reduced form becomes,

$$\mathbb{E}_t\pi_{t+1} = \left[\zeta_1 + \zeta_2 \left(1 + e^{-\mu((\pi_t - \pi_{t-2})^2 - C)} \right) \right] \pi_t - \left[\zeta_3 + \zeta_4 \left(1 + e^{-\mu((\pi_t - \pi_{t-2})^2 - C)} \right) \right] \pi_{t-1},$$

where $\zeta_1 - \zeta_4$ are composite parameters, but otherwise the notation is as before. While the basic mechanism in the anticipated utility BNK model is essentially identical to that of Branch and McGough (2010), the simplicity of the reduced form allows analytical stability results to be derived.

The basic BNK model adds heterogeneous predictors and discrete choice dynamics to the standard NK model. This has been done using Euler learning alongside rational expectations, as in Branch and McGough (2010), anticipated utility alongside rational expectations, as in Calvert Jump et al. (2018), and Euler learning alongside a set of misspecified predictors, as in De Grauwe (2011). Section 4.2 surveys some of the ways in which this basic model has been extended.

4.2 Extensions to the basic model

The main use of the BNK model, and the various extensions to it, has been the examination of monetary policy conduct. De Grauwe (2012), for example, points out that strict inflation targeting in the presence of demand shocks, while being optimal in a standard NK model, is not necessarily desirable in a BNK model. This is because output stabilisation in the latter can, to some extent, eliminate extreme swings between optimists and pessimists, which in turn can reduce excess kurtosis in the output gap. Di Bartolomeo et al. (2016) use the linear-quadratic approach to compute optimal monetary policy in the Branch and McGough (2009) model, and find that central banks aiming to reduce consumption variability should set monetary policy to reduce the cross-sectional variability of expectations.

In general, the Taylor principle is neither necessary nor sufficient for determinacy and stability in BNK models. As originally discussed in Branch and McGough (2010), small amounts of heterogeneity can ensure determinacy in models which would otherwise be indeterminate under homogeneous rational expectations, and cause indeterminacy in models which would otherwise be determinate under homogeneous rational expectations. This is discussed in more detail in Branch and McGough (2018). In addition, even when the Taylor principle is desirable, the rationale for aggressive inflation targeting is quite different in BNK models compared to the standard NK model. This is discussed in Anufriev et al. (2013), who argue that central banks can force households and firms to converge on stabilising expectations by responding aggressively to inflation.

Although the Taylor principle is neither necessary nor sufficient in the general case, there are notable examples of NK models with heterogeneous expectations in which it remains an important guide to monetary policy. Pecora and Spelta (2017), for example, present a model in which the Taylor condition is sufficient for stability, although convergence to the steady state can be very slow. The Taylor condition is also sufficient for stability in the Calvert Jump et al. (2018) model when the proportion of rational agents is fixed, and is sufficient for local stability in the case of strategy switching. As the main difference between the Branch and McGough (2010) and Calvert Jump et al. (2018) models - aside from the use of anticipated utility in the latter - is the use of an expectations-based monetary policy rule in the former, this suggests that policy rules that react to expectations are undesirable in BNK models. This is consistent with the conclusions of Eusepi and Preston (2018) in their review of monetary policy under imperfect knowledge, i.e.,

[I]mperfect knowledge and learning can limit the set of policies available to central banks, rendering aggregate demand management and inflation control in general more difficult than under rational expectations Importantly, the dependence of monetary policy on expectations must be of a very specific kind. For example, simple rules that respond to a measure of inflation expectations, while often desirable in a rational expectations analysis, can lead to macroeconomic instability. (Eusepi and Preston, 2018, 4-5).

While the results in Branch and McGough (2010) are robust to the use of a monetary policy rule that reacts to optimal forecasts, to the best of our knowledge robustness to the standard policy rule in (3) has not been investigated. This would be a useful exercise, given the conclusions of Eusepi and Preston (2018). The intuition behind the undesirability of expectations-based policy rules in BNK models is relatively straightforward given that, at the most basic level, the nominal interest rate has to vary such that the real interest rate is pro-cyclical in New Keynesian models. In BNK models, inflation expectations may not correspond particularly closely to actual inflation, so any policy rule that reacts strongly to expectations will not necessarily induce stabilising real interest rate dynamics.

An interesting extension to BNK investigations of monetary policy is contained in Demary (2017), who demonstrates that an extended version of the De Grauwe (2012) model replicates empirical term structure data. Among other things, he demonstrates that long term inflation expectations determine the level of the yield curve, while monetary policy determines its slope. As discussed by Anufriev et al. (2013), the central bank can force private expectations away from extrapolative inflation expectations by reacting aggressively to the inflation rate, but this action can result in a volatile yield curve in the model discussed in Demary (2017). This has obvious implications for fiscal policy, and there may be interesting implications for the interaction of monetary policy and public debt dynamics. Although

the policy interaction problem is yet to be examined in a BNK framework, Gasteiger (2017) discusses the issue in a Neoclassical model with heterogeneous expectations.

Aside from the various analyses of aggregate demand management, the basic BNK model has also been extended to analyse credit cycles and house price cycles. The former have been examined in De Grauwe and Macchiarelli (2015) and De Grauwe and Gerba (2018). These papers follow the basic model set out in De Grauwe (2011), but add a banking sector which charges a mark-up over the central bank interest rate. Following the financial accelerator approach of Bernanke et al. (1999), this mark-up is a decreasing function of firm sector equity, which in turn is pro-cyclical. Then, for example, a negative monetary policy shock reduces output and inflation in the normal manner, which reduces firm sector equity, which in turn increases the spread charged by private banks over the central bank rate. This basic financial accelerator mechanism is amplified if private agents converge on a pessimistic predictor as a consequence of the initial shock.

A similar mechanism is presented in Bofinger et al. (2013), which incorporates the BNK mechanism into a fairly standard house price model. They specify patient and impatient households, where impatient households face a collateral constraint that ties their borrowing to the expected present value of their future housing stock. As in the standard financial accelerator mechanism, an expansionary shock increases activity, which increases house prices, which loosens the collateral constraint on impatient households' borrowing. Thus the effects of the initial shock are amplified as impatient households borrow more, increasing house prices further. Following the same logic as in De Grauwe and Macchiarelli (2015) and De Grauwe and Gerba (2018), the boom can then become self sustaining if the initial increase in house prices causes households to converge on an optimistic predictor. As non-house price inflation does not immediately increase following an increase in house prices in this model, the authors demonstrate that there is a meaningful role for a house price augmented Taylor rule in economies prone to house price booms. As in De Grauwe (2012), this extended monetary policy rule can dampen extreme swings between optimism and pessimism, thus stabilising output and inflation dynamics as a by-product of stabilising house prices.

Adding bounded rationality via the BNK logic to standard financial accelerator models captures the “new era story” emphasised by Akerlof and Shiller (2009) in their theories of animal spirits and irrational exuberance, or the “speculative mania” and “euphoria” of Kindleberger (1978) and Minsky (1982). The policy implications are intuitive, and if one's aim is to account for the mechanisms driving events like the 2008 financial crisis, some departure from the strict rational expectations hypothesis is arguably necessary. Consider Alan Greenspan's discussion of the reasons that economists failed to predict the 2008 crisis:

What went wrong? Why was virtually every economist and policy-maker of note so blind to the coming calamity? How did so many experts, including me, fail to see it approaching? I have come to see that an important part of the answers to those questions is a very old idea: “animal spirits,” the term Keynes famously coined in 1936 to refer to “a spontaneous urge to action rather than inaction.” (Greenspan, 2013, 89).

However, as Greenspan also points out, departures from strict rationality are more predictable than economists have traditionally understood, and such behaviour, “can be measured and should be made an integral part of economic forecasting and economic policy-making” (ibid.). This is a useful summary of the BNK research programme.

5 Where next?

The BNK research programme, along with similar approaches in the broader macroeconomic research community, continues to expand. Recent papers have examined optimal linear-quadratic policy in the framework of Massaro (2013), and the impact of heterogeneous expectations on fiscal consolidations in a New Keynesian setting (Beqiraj et al., 2017; Hommes et al., 2018). Aside from further work on fiscal policy, which is relatively neglected compared to monetary policy in the BNK literature, an important area for further work is the role of bounded rationality and heterogeneity in the open economy setting. This would appear to be a particularly important application of the logic described in the present paper, given the well recorded evidence of non-linearities and stochastic volatility in exchange rate dynamics (Moosa, 2000). However, at the time of writing, there are only two working papers examining open economy issues from a BNK perspective (Jang, 2015; Bertasiute et al., 2018), and a single published article (De Grauwe and Ji, 2017).

Alongside the foregoing, future work should focus on estimation. Without a considerably improved empirical basis for BNK models - aside from the descriptive statistics discussed in this paper and elsewhere - it is unclear why central banks, or any other policy institutions, should replace the standard NK framework in their forecasting and policy analyses. In particular, it would be useful to know whether or not the incorporation of reinforcement learning and heterogeneous expectations into the type of large NK model used in central banks would have the same effects as in the small NK model considered in the present paper. As heavy-tailed shocks are filtered out by the transmission mechanisms of large scale NK models (Ascari et al., 2015), it is not impossible that a similar effect could operate on the BNK mechanism.

Estimation of BNK models could be pursued using likelihood based methods, Bayesian or otherwise, which employ filtering methods to approximate the evolution of unobserved state variables, or moment based methods, which match simulated artificial moments with corresponding empirical moments³. Deak et al. (2015) provides an application of the first method to a log-linearised BNK model with exogenous proportions of agents with and without rational expectations, and Jang and Sacht (2016) provide an application of the second method to a complete BNK model. Franke et al. (2015) compares the two approaches through the lens of the workhorse New Keynesian model with habit in consumption and price indexing. Cornea-Madeira et al. (2017) estimate a New Keynesian Phillips curve with heterogeneous expectations and reinforcement learning, but aside from these papers there are very few attempts to estimate BNK models.

Since BNK models are highly non-linear, the advantages and disadvantages of maximum likelihood versus moment matching should be considered in relation to their ability to estimate non-linear models, rather than their log-linearised counterparts. For the method of moments, non-linearities do not pose a particular problem, but this is not the case for likelihood methods. In addition, moment matching is considered to perform relatively well when models are misspecified. However, this estimation method is subject to some implementation problems, including the relatively arbitrary choice of relevant moments, the high computational cost if moments have to be simulated, and the often unknown sampling distributions of empirical moments in short samples. The use of the sample coefficient of kurtosis as a population kurtosis estimator, for example, is known to perform very poorly

³See Fernandez-Villaverde et al. (2016) for a comprehensive survey of many of the issues we raise here. We thank Rodolfo Arioli for considerable help with this sub-section.

in short samples (Bai and Ng, 2005).

In comparison to moment matching, maximum likelihood estimation requires filtering techniques to approximate the likelihood function when unobserved state variables are present. There are two approaches to this when the shape of the density is unknown - the local (Gaussian) approach, or the global approach. An important example of the latter is the particle filter, as in Fernandez-Villaverde et al. (2015), where it is used to estimate a DSGE model with stochastic volatility. While global methods can produce unbiased estimates of the likelihood function, they are computationally expensive and suffer from the curse of dimensionality as model size increases. Local methods, on the other hand, enable linear filtering techniques to be applied to medium and large scale models, and can often result in acceptable precision compared to particle filters (Arasaratnam and Haykin, 2009; Meyer-Gohde, 2014; Kollmann, 2017).

The various advantages and difficulties of the moment-based and likelihood-based approaches, combined with the small number of papers that estimate BNK models, make it very difficult to suggest a specific avenue for future empirical work. However, work by Del Negro and Schorfheide (2008) and Christiano et al. (2011) on Bayesian maximum likelihood with “endogenous” priors combines the benefits of likelihood and moment-based estimation, which may be a promising agenda to pursue. Alternatively, the approach of penalised indirect inference recently proposed by Blasques and Duplinskiy (2018) allows some of the benefits of Bayesian inference to be incorporated into a frequentist moment matching estimator, and might also be a profitable avenue for future empirical work. Applying either of these estimation strategies, and others that we have not considered, is an important next step in the evolution of the BNK research programme.

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Appendices

A Moment matching routine

The standard NK part of simple BNK model in section 3.2 is calibrated in a similar manner to Gali (2008), i.e. $\beta = 0.99$, $\kappa = 0.1$, $\theta_\pi = 1.5$, $\theta_y = 0.125$. We further calibrate the shock processes u_{1t} and u_{2t} as AR(1) processes with autoregressive parameter equal to 0.9 and error standard deviations of 0.009 and 0.012, respectively, and set $C = 0$ to reduce the number of non-linear parameters. We are then left with the choice for μ . This is calibrated using a simulated minimum distance routine. Denote the 4D vector of the simulated standard deviations of output growth and inflation and the simulated coefficients of kurtosis of output growth and inflation by z_{mod} , i.e.,

$$z_{mod} = [\text{std}(\Delta y_t), \text{std}(\pi_t), \text{kurt}(\Delta y_t), \text{kurt}(\pi_t)]^\top. \quad (\text{A.1})$$

Denote the 4D vector of the empirical standard deviations of output growth and inflation and the empirical coefficients of kurtosis of output growth and inflation by z_{emp} , i.e.,

$$z_{emp} = [\text{std}(\Delta \tilde{y}_t), \text{std}(\tilde{\pi}_t), \text{kurt}(\Delta \tilde{y}_t), \text{kurt}(\tilde{\pi}_t)]^\top, \quad (\text{A.2})$$

where $\Delta \tilde{y}_t$ is the growth rate of real output, and $\tilde{\pi}_t$ is the growth rate of the GDP deflator, using US data downloaded from the FRED database on 24/04/2018. Note that the linear NK model is derived with π_t as the log inflation gap, but as the US inflation rate can be negative we conduct the empirical exercises under the assumption that π_t is the inflation rate. Given (A.1) and (A.2), we choose μ to minimise the Euclidean norm of $z_{mod} - z_{emp}$, i.e.,

$$\mu = \text{argmin} \ ||z_{mod}(\mu) - z_{emp}||, \quad (\text{A.3})$$

where the Euclidean norm of some vector z with n elements is given by $\|z\| = (\sum_{i=1}^n |z_i|^2)^{0.5}$. We evaluate (A.3) using the `fminbnd` routine in Matlab, which is a minimisation algorithm based on golden section search and parabolic interpolation. We limit the parameter space to $\mu \in (0, 1000)$, which results in a minimum at $\mu \approx 210$.

B Alternative parameterisations

In this appendix we present descriptive statistics based on two alternative monetary policy rules and one alternative parameterisation. The alternative monetary policy rules are,

$$i_t = \theta_\pi \pi_{t-1} + \theta_y y_{t-1}, \quad (\text{B.4})$$

$$i_t = \theta_\pi \hat{E}_t \pi_{t+1} + \theta_y \hat{E}_t y_{t+1}, \quad (\text{B.5})$$

where (B.4) is the “lagged data” Taylor rule in Bullard and Mitra (2002), and (B.5) is the “forward expectations” Taylor rule in the same paper, also used in Branch and McGough (2010). The simulations with these alternative Taylor rules use the baseline parameterisation. In addition we simulate an alternative parameterisation of the baseline model in which $\kappa = 0.3$, as in Clarida et al. (2000) and Bullard and Mitra (2002). The results are presented in tables B.1 and B.2 below. They are qualitatively similar to the results reported in the main body of the paper, i.e. they display excess kurtosis and conditional heteroskedasticity.

Table B.1: Higher moments of simulated real GDP growth and inflation

Model	Variable	Standard Deviation	Skewness	Kurtosis
“lagged data” rule	Real GDP growth	0.010	0.00	3.875
	Inflation	0.007	0.00	7.767
“forward exp.” rule	Real GDP growth	0.012	0.00	5.789
	Inflation	0.015	0.00	20.811
alternative κ	Real GDP growth	0.007	0.00	3.050
	Inflation	0.013	0.00	3.570

Table B.2: GARCH model results, simulated real GDP growth and inflation

Model	Variable	ARCH(1)	GARCH(1)
“lagged data” rule	Real GDP growth	0.09	0.21
	Inflation	0.03	0.87
“forward exp.” rule	Real GDP growth	0.13	0.38
	Inflation	0.06	0.87
alternative κ	Real GDP growth	0.01	0.15
	Inflation	0.05	0.9

C Stability properties of the simple BNK model

As is well known, a purely forward looking linear model of the form,

$$\mathbb{E}_t z_{t+1} = \mathbf{B}z_t + \mathbf{C}u_t, \quad (\text{C.6})$$

where z_t is some vector of endogenous variables and u_t is some vector of stationary exogenous variables, requires all eigenvalues of \mathbf{B} outside the unit circle for determinacy. It is therefore the case that the equivalent statement of the model,

$$z_t = \mathbf{A}\mathbb{E}_t z_{t+1} + \mathbf{D}u_t, \quad (\text{C.7})$$

requires all eigenvalues of \mathbf{A} inside the unit circle for determinacy, as $\mathbf{A} = \mathbf{B}^{-1}$. Now, we note that the simple behavioural New Keynesian model with Euler learning outlined in section 3.2 simply replaces \mathbb{E}_t with \hat{E}_t in (C.7), i.e.,

$$z_t = \mathbf{A}\hat{E}_t z_{t+1} + \mathbf{D}u_t, \quad (\text{C.8})$$

where \hat{E}_t is some bounded rational predictor. For stability analysis we can ignore the stochastic part of the model, so (C.7) and (C.8) become,

$$z_t = \mathbf{A}z_{t+1}, \quad (\text{C.9})$$

$$z_t = \mathbf{A}\hat{E}_t z_{t+1}. \quad (\text{C.10})$$

Consider the specification of bounded rational expectations in section 3.2 conditional on some value of $n \in [0, 1]$, i.e. $\hat{E}_t z_{t+1} = n z_{t+1}$, where n is the proportion of extrapolative agents. (C.10) therefore becomes,

$$z_t = \mathbf{A}n z_{t+1}, \quad (\text{C.11})$$

or,

$$z_t = \mathcal{A}z_{t+1}, \quad (\text{C.12})$$

where $\mathcal{A} = \mathbf{A}n$. If λ are the eigenvalues of \mathcal{A} and φ are the eigenvalues of \mathbf{A} , then $\lambda = n\varphi$. Thus if φ are all inside the unit circle then λ are all inside the unit circle, as $n \in [0, 1]$.

From the foregoing, it follows that determinacy of the forward looking rational expectations model in (C.7) is sufficient for stability of the behavioural New Keynesian model in (C.8) for an arbitrary value of n . In addition, it follows that determinacy of the model in (C.7) is sufficient for local stability of the behavioural New Keynesian model when n is some function of z , as long as the gradient of that function is zero at the steady state as in (11) and (12). The result will generally cease to hold when different predictor functions are used by bounded rational agents, different policy rules are used, or backwards looking variables are added to the model, as in the literature on more complex models reviewed in section 4.