Non-existence of local solutions of semilinear heat equations of Osgood type in bounded domains

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Received *****; accepted after revision +++++

Presented by

Abstract

We establish a local non-existence result for the equation $u_t - \Delta u = f(u)$ with Dirichlet boundary conditions on a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and initial data in $L^q(\Omega)$ when the source term f is non-decreasing and $\limsup_{s\to\infty} s^{-\gamma} f(s) = \infty$ for some exponent $\gamma > q(1 + 2/n)$. This allows us to construct a locally Lipschitz f satisfying the Osgood condition $\int_1^\infty 1/f(s)\ \mathrm{d}s = \infty$, which ensures global existence for initial data in $L^\infty(\Omega)$, such that for every q with $1 \le q < \infty$ there is a non-negative initial condition $u_0 \in L^q(\Omega)$ for which the corresponding semilinear problem has no local-in-time solution ('immediate blow-up'). *To cite this article: R. Laister, J.C. Robinson, M. Sierzega, C. R. Acad. Sci. Paris, Ser. I XXX (201*).*

Résumé

Non-existence de solutions locales pour les équations de la chaleur semi-linéaires de type Osgood dans des domaines bornés. Nous établissons un résultat de non-existence locale pour l' équation $u_t-\Delta u = f(u)$ avec des conditions aux limites de Dirichlet sur un domaine borné lisse $\Omega \subset \mathbb{R}^n$ et des données initiales dans $L^q(\Omega)$ lorsque le terme de source f est non-décroissant et $\limsup_{s\to\infty} s^{-\gamma} f(s) = \infty$ pour un exposant $\gamma > q(1 + 2/n)$. Ceci nous permet de construire un f localement Lipschitz qui satisfait la condition de Osgood $\int_1^\infty 1/f(s)\ \mathrm{d}s = \infty$, ce qui garantit l'existence globale pour des données initiales dans $L^\infty(\Omega)$, de telle sorte que pour chaque q tel que $1 \le q < \infty$ il existe une condition initiale non-négative $u_0 \in L^q(\Omega)$ pour laquelle le problème semi-linéaire correspondant n'admet pas de solution locale en temps ('blow-up immédiat'). *Pour citer cet article : R. Laister, J.C. Robinson, M. Sierzega, C. R. Acad. Sci. Paris, Ser. I XXX (201*).*

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1. Introduction

In a previous paper [7] we showed that for locally Lipschitz f with $f > 0$ on $(0, \infty)$, the Osgood condition

$$\int_1^\infty \frac{1}{f(s)}\,\mathrm{d}s = \infty, \tag{1}$$

which ensures global existence of solutions of the scalar ODE $\dot{x} = f(x)$, is not sufficient to guarantee the local existence of solutions of the 'toy PDE'

$$u_t = f(u), \qquad u(x,0) = u_0 \in L^q(\Omega) \tag{2}$$

unless $q = \infty$.

In [5] we considered the Cauchy problem for the semilinear PDE

$$u_t = \Delta u + f(u), \qquad u(0) = u_0, \tag{3}$$

on the whole space $\mathbb{R}^n$ and showed that even with the addition of the Laplacian, for each q with $1 \le q < \infty$ one can find a non-negative, locally Lipschitz f satisfying the Osgood condition (1) such that there are non-negative initial data in $L^q(\mathbb{R}^n)$ for which there is no local-in-time integrable solution of (3).

In this paper we obtain a similar non-existence result for equation (3) when posed with Dirichlet boundary conditions on a smooth bounded domain $\Omega \subset \mathbb{R}^n$. More explicitly, we focus throughout the paper on the following problem:

$$u_t = \Delta u + f(u), \qquad u|_{\partial\Omega} = 0, \qquad u(x,0) = u_0 \in L^q(\Omega). \tag{P}$$

In all that follows we assume that the source term $f : [0,\infty) \to [0,\infty)$ is non-decreasing. We show in Theorem 3.2 that if f satisfies the asymptotic growth condition

$$\limsup_{s\to\infty} s^{-\gamma} f(s) = \infty \tag{4}$$

for some $\gamma > q(1 + 2/n)$ then one can find a non-negative $u_0 \in L^q(\Omega)$ such that there is no local-in-time solution of (P). We then (Theorem 4.1) construct a Lipschitz function f that grows quickly enough such that (4) holds for every $\gamma \ge 0$, but nevertheless still satisfies the Osgood condition (1). This example shows that there are functions f for which (P) has solutions for any u_0 belonging to $L^\infty(\Omega)$, but that there are non-negative $u_0 \in L^q(\Omega)$ for any $1 \le q < \infty$ for which the equation has no local integral solution.

One can see this result as in some sense dual to that of Fila et al. [3] (see also Section 19.3 of [8]), who show that there exists an f such that all positive solutions of $\dot{x} = f(x)$ blow up in finite time while all solutions of (P) are global and belong to $L^\infty(\Omega)$.

2. A lower bound on solutions of the heat equation

Without loss of generality we henceforth assume that Ω contains the origin. For $r > 0$, $B(r)$ will denote the Euclidean ball in $\mathbb{R}^n$ of radius r centred at the origin, and ω_n the volume of the unit ball in $\mathbb{R}^n$.

As an ingredient in the proof of Theorem 3.2, we want to show that the action of the heat semigroup on the characteristic function of a ball

$$\chi_R(x) = \begin{cases} 1 & \text{for } x \in B(R) \\ 0 & \text{for } x \notin B(R) \end{cases}$$

does not have too pronounced an effect for short times.

We denote the solution of the heat equation on Ω at time t with initial data u_0 by $S_\Omega(t)u_0$, i.e. the solution of

$$u_t - \Delta u = 0, \qquad u|_{\partial\Omega} = 0, \qquad u(x,0) = u_0 \in L^q(\Omega).$$

This solution can be given in terms of the Dirichlet heat kernel $K_\Omega(x,y;t)$ by the integral expression

$$[S_\Omega(t)u_0](x) = \int_\Omega K_\Omega(x,y;t)u_0(y)\,\mathrm{d}y.$$

We note for later use that $K_\Omega(x,y;t) = K_\Omega(y,x;t)$ for all $x, y \in \Omega$.

We use the following Gaussian lower bound on the Dirichlet heat kernel, which is obtained by combining various estimates proved by van den Berg in [9] (Theorem 2 and Lemmas 8 and 9). A simplified proof is given in [6].

Theorem 2.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and denote by $K_\Omega(x,y;t)$ the Dirichlet heat kernel on Ω. Suppose that*

$$\epsilon := \inf_{z\in[x,y]} \operatorname{dist}(z,\partial\Omega) > 0, \tag{5}$$

where $[x,y]$ denotes the line segment joining x and y (so in particular $[x,y]$ is contained in the interior of Ω). Then for $0 < t \le \epsilon^2/n^2$

$$K_\Omega(x,y;t) \ge \tfrac{1}{4}\, G_n(x,y;t), \quad \textit{where} \quad G_n(x,y;t) = (4\pi t)^{-n/2}\mathrm{e}^{-|x-y|^2/4t}. \tag{6}$$

We can now bound $S_\Omega(t)\chi_R$ from below.

Lemma 2.2 *There exists an absolute constant $c_n > 1$, which depends only on n, such that for any R for which $B(2R) \subset \Omega$,*

$$S_\Omega(t)\chi_R \ge \frac{1}{c_n}\,\chi_{R/2}, \qquad \textit{for all} \quad 0 < t \le R^2/n^2. \tag{7}$$

Proof. Take $x \in B(R/2)$; then when $y \in B(R)$ certainly $\epsilon \ge R$, so (6) implies that for $0 < t \le R^2/n^2$

$$\begin{aligned}[S_\Omega(t)\chi_R](x) &= \int_{B(R)} K_\Omega(x,y;t)\,\mathrm{d}y \\ &\ge \tfrac{1}{4}(4\pi t)^{-n/2} \int_{B(R)} \mathrm{e}^{-|x-y|^2/4t}\,\mathrm{d}y.\end{aligned}$$

Since $|x| \le R/2$, it follows that $\{w = x - y : \ y \in B(R)\} \supset B(R/2)$ and so

$$\begin{aligned}[S(t)\chi_R](x) &\ge \mathrm{e}^{-\pi^2/4}(4\pi t)^{-n/2} \int_{B(R/2)} \mathrm{e}^{-|w|^2/4t}\,\mathrm{d}w \\ &= \tfrac{1}{4}\pi^{-n/2} \int_{B(R/4\sqrt{t})} \mathrm{e}^{-|z|^2}\,\mathrm{d}z \\ &\ge \tfrac{1}{4}\pi^{-n/2} \int_{B(n/4)} \mathrm{e}^{-|z|^2}\,\mathrm{d}z =: c_n^{-1},\end{aligned}$$

since $t \le R^2/n^2$. $\square$

3. Non-existence of local solutions

In this section we prove the non-existence of local L^q-valued solutions, taking the following definition from [8] as our (essentially minimal) definition of such a solution. Note that any classical or mild solution is a local integral solution in the sense of this definition [8, p. 77–78], and so non-existence of a local L^q-valued integral solution implies the non-existence of classical and mild L^q-valued solutions.

Definition 3.1 *Given* $f \geq 0$ *and* $u_0 \geq 0$ *we say that* u *is a* local integral solution *of* (P) *on* $[0,T)$ *if* $u : \Omega \times [0,T) \to [0,\infty]$ *is measurable, finite almost everywhere, and satisfies*

$$u(t) = S_\Omega(t)u_0 + \int_0^t S_\Omega(t-s)f(u(s))\,\mathrm{d}s \tag{8}$$

almost everywhere in $\Omega \times [0,T)$. *We say that* u *is a* local L^q-valued integral solution *if in addition* $u(t) \in L^q(\Omega)$ *for almost every* $t \in (0,T)$.

We now prove our main result, in which we obtain non-existence of a local L^q-valued integral solution for certain initial data in $L^q(\Omega)$, $1 \leq q < \infty$, under the asymptotic growth condition (9) when f is non-decreasing.

Theorem 3.2 *Let* $q \in [1,\infty)$. *Suppose that* $f : [0,\infty) \to [0,\infty)$ *is non-decreasing. If*

$$\limsup_{s\to\infty} s^{-\gamma} f(s) = \infty \tag{9}$$

for some $\gamma > q(1 + \frac{2}{n})$, *then there exists a non-negative* $u_0 \in L^q(\Omega)$ *such that* (P) *possesses no local* L^q*-valued integral solution.*

Proof. We find a $u_0 \in L^q(\Omega)$ such that $u(t) \notin L^1_{\mathrm{loc}}(\Omega)$ for all sufficiently small $t > 0$ and hence $u(t) \notin L^q(\Omega)$ for all sufficiently small $t > 0$. Note that this is a stronger form of ill-posedness than 'norm inflation' (cf. Bourgain & Pavlović [1]).

Set $\alpha = (n+2)/\gamma < n/q$, so that

$$\limsup_{s\to\infty} s^{-(n+2)/\alpha} f(s) = \infty.$$

Then in particular there exists a sequence $\phi_i \to \infty$ such that

$$f(\phi_i)\phi_i^{-(n+2)/\alpha} \to \infty \qquad \text{as} \quad i \to \infty. \tag{10}$$

Now choose $R > 0$ such that $B(2R) \subset \Omega$ (recall that we assumed that $0 \in \Omega$), and take $u_0 = |x|^{-\alpha}\chi_R(x) \in L^q(\Omega)$. Noting that by comparison $u(t) \geq S_\Omega(t)u_0 \geq 0$, it follows from (8) that for every $t > 0$

$$\int_{B(R)} u(t)\,\mathrm{d}x \geq \int_{B(R)} \int_0^t [S_\Omega(t-s)f(S_\Omega(s)u_0)](x)\,\mathrm{d}s\,\mathrm{d}x.$$

Now choose and fix $t \in (0, R^2/n^2]$. Observe that

$$u_0 \geq \psi\chi_{\psi^{-1/\alpha}}$$

for any $\psi > R^{-\alpha}$. In particular, choosing $\psi = c_n\phi_i$, it follows from Lemma 2.2 and the monotonicity of S_Ω that for all i sufficiently large

$$S_\Omega(s)u_0 \geq \phi_i\chi_{\frac{1}{2}(c_n\phi_i)^{-1/\alpha}}, \qquad 0 \leq s \leq t_i := (c_n\phi_i)^{-2/\alpha}/n^2.$$

Therefore, for any i large enough that $t_i \leq t$ and $c_n\phi_i > R^{-\alpha}$,

$$\begin{aligned}\int_{B(R)} u(t)\,\mathrm{d}x &\geq \int_{B(R)} \int_0^{t_i} S_\Omega(t-s)f(\phi_i\chi_{\frac{1}{2}(c_n\phi_i)^{-1/\alpha}})\,\mathrm{d}s\,\mathrm{d}x \\ &\geq f(\phi_i)\int_0^{t_i}\int_{B(R)} S_\Omega(t-s)\chi_{\frac{1}{2}(c_n\phi_i)^{-1/\alpha}}\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

using Fubini's Theorem and the fact that $f(0) \geq 0$.

Now observe that since $K_\Omega(x,y;t) = K_\Omega(y,x;t)$, for any $t > 0$ and r, R such that $B(R), B(r) \subset \Omega$,

$$\int_{B(R)} [S_\Omega(t)\chi_r](x)\,\mathrm{d}x = \int_{B(R)}\int_{B(r)} K_\Omega(x,y;t)\,\mathrm{d}y\,\mathrm{d}x = \int_{B(r)} [S_\Omega(t)\chi_R](y)\,\mathrm{d}y.$$

Thus

$$\int_{B(R)} u(t)\,\mathrm{d}x \geq f(\phi_i)\int_0^{t_i}\int_{B(\frac{1}{2}(c_n\phi_i)^{-1/\alpha})} S_\Omega(t-s)\chi_R\,\mathrm{d}x\,\mathrm{d}s.$$

Since $\frac{1}{2}(c_n\phi_i)^{-1/\alpha} < R/2$ and $t - s \leq t \leq R^2/n^2$ we can use Lemma 2.2 once more to obtain

$$\begin{aligned}\int_{B(R)} u(t)\,\mathrm{d}x &\geq f(\phi_i)\int_0^{t_i}\int_{B(\frac{1}{2}(c_n\phi_i)^{-1/\alpha})} \frac{1}{c_n}\chi_{R/2}\,\mathrm{d}x\,\mathrm{d}s \\ &= \frac{\omega_n}{c_n} f(\phi_i)\,t_i\left[\frac{1}{2}(c_n\phi_i)^{-1/\alpha}\right]^n \\ &= [\omega_n 2^{-n} c_n^{-1-(n+2)/\alpha}/n^2]\, f(\phi_i)\phi_i^{-(n+2)/\alpha} \to \infty \quad \text{as} \quad i \to \infty\end{aligned}$$

due to (10). □

We note that if $f(s) \geq cs$ for some $c > 0$ then arguing as in [5, Theorem 4.1] there can in fact be no local integral solution of (P) whatsoever.

For the canonical Fujita equation

$$u_t = \Delta u + u^p, \tag{11}$$

our argument shows the non-existence of local solutions when $p > q(1+\frac{2}{n})$. The sharp result in this case is known to be $p > 1 + \frac{2q}{n}$ [11,12] with equality allowed if $q = 1$ [2].

The existence of a finite limit in (9) implies that $f(s) \leq c(1+s^\gamma)$, and hence by comparison with (11) is sufficient for the local existence of solutions provided that $\gamma < 1 + \frac{2q}{n}$ [10]. We currently, therefore, have an indeterminate range of γ,

$$1 + \frac{2q}{n} \leq \gamma \leq q(1+\frac{2}{n})$$

for which we do not know whether (9) characterises the existence or non-existence of local solutions.

4. A very 'bad' Osgood f

To finish, using a variant of the construction in [5], we provide an example of an f that satisfies the Osgood condition (1) but for which

$$\limsup_{s\to\infty} s^{-\gamma} f(s) = \infty, \qquad \text{for every} \quad \gamma \geq 0. \tag{12}$$

Theorem 4.1 *There exists a locally Lipschitz function $f : [0,\infty) \to [0,\infty)$ such that $f(0) = 0$, f is non-decreasing, and f satisfies the Osgood condition*

$$\int_1^\infty \frac{1}{f(s)}\,\mathrm{d}s = \infty,$$

but nevertheless (12) holds. Consequently, for this f, for any $1 \leq q < \infty$ there exists a non-negative $u_0 \in L^q(\Omega)$ such that (P) *has no local L^q-valued integral solution.*

Proof. Fix $\phi_0 = 1$ and define inductively the sequence ϕ_i via

$$\phi_{i+1} = \mathrm{e}^{\phi_i}.$$

Clearly, $\phi_i \to \infty$ as $i \to \infty$. Now define $f : [0,\infty) \to [0,\infty)$ by

$$f(s)=\begin{cases}(\mathrm{e}-1)s, & s\in J_0:=[0,1],\\ \phi_i-\phi_{i-1}, & s\in I_i:=[\phi_{i-1},\phi_i/2],\ i\geq 1,\\ \ell_i(s), & s\in J_i:=(\phi_i/2,\phi_i),\ i\geq 1,\end{cases} \tag{13}$$

where ℓ_i interpolates linearly between the values of f at $\phi_i/2$ and ϕ_i. By construction $f(0)=0$, f is Lipschitz and non-decreasing, and f is Osgood since

$$\int_1^\infty \frac{1}{f(s)}\,\mathrm{d}s \geq \sum_{i=1}^\infty \int_{I_i} \frac{1}{f(s)}\,\mathrm{d}s = \sum_{i=1}^\infty \frac{\phi_i/2-\phi_{i-1}}{\phi_i-\phi_{i-1}} = +\infty.$$

However, $f(\phi_i)=\mathrm{e}^{\phi_i}-\phi_i$, and so for any $\gamma\geq 0$

$$\lim_{i\to\infty} \phi_i^{-\gamma} f(\phi_i) \to \infty \qquad \text{as} \quad i\to\infty,$$

which shows that (12) holds. □

This example shows that there exist semilinear heat equations that are globally well-posed in $L^\infty(\Omega)$, yet ill-posed in every $L^q(\Omega)$ for $1\leq q<\infty$.

Acknowledgements The authors acknowledge support under the following grants: EPSRC Leadership Fellowship EP/G007470/1 (JCR); EPSRC Doctoral Prize EP/P50578X/1 (MS). We would all like to thank Claude Baesens for the French translation of the title and abstract.

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