Learning, Heterogeneity, and Complexity in the New Keynesian Model*

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Abstract

We present a New Keynesian model in which a fraction n of agents are fully rational, and a fraction $1 - n$ of agents are bounded rational. After deriving a simple reduced form, we demonstrate that the Taylor condition is sufficient for determinacy and stability, both when the proportion of fully rational agents is held fixed, and when it is allowed to vary according to reinforcement learning. However, this result relies on the absence of persistence in the monetary policy rule, and we demonstrate that the Taylor condition is not sufficient for determinacy and stability in the presence of interest rate smoothing. For monetary policy rules that imply indeterminacy, we demonstrate the existence of limit cycles via Hopf bifurcation, and explore a rational route to randomness numerically. Our results support the broader literature on behavioural New Keynesian models, in which the Taylor condition is known to be a useful guide to monetary policy, despite not always being sufficient for determinacy and/or stability.

Keywords: Behavioural New Keynesian model, anticipated utility, learning, heterogeneous expectations.

JEL Codes: E03, E12, E32, E70, E71.

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1 Introduction

The workhorse New Keynesian model is used by central banks, governments, and policy institutions, and defines the contemporary orthodoxy in monetary policy. This states that the central bank should raise interest rates more than one-for-one with any observed increase in inflation, such that the real interest rate increases in response to inflationary shocks, and aggregate demand is subject to central bank control. This principle, translated into the mathematics of DSGE models, ensures that the dynamics of output and inflation are determinate and stable. However, the workhorse model relies on a number of simplifying assumptions. In particular, the basic model assumes that any heterogeneity between households and firms can be ignored, and that all agents are endowed with the ability to form rational expectations. These characteristics of the contemporary orthodoxy have been heavily criticised since the 2008 crisis, encouraging the growth of a literature on behavioural New Keynesian models (Calvert Jump and Levine, 2019).

The behavioural New Keynesian literature builds on the pioneering work of Branch and Evans (2007), Branch and McGough (2004, 2009, 2010), and De Grauwe (2011, 2012a,b), who present models in which a subset of agents form expectations in a bounded rational manner. The size of this subset can be fixed, or can vary according to a learning dynamic. Although the Taylor principle is neither necessary nor sufficient for determinacy and stability in these models (Branch and McGough, 2010, 2016), it remains an important guide to monetary policy. Pecora and Spelta (2017), for example, present a simple model in which the Taylor condition is sufficient for stability, although convergence to the steady state can be slow. This general result, in which the orthodox approach to monetary policy is qualified, but remains correct in its basic logic, is supported by the review of monetary policy under imperfect knowledge in Eusepi and Preston (2018). It is a useful contribution to the current state of knowledge, and largely supports the existing framework.

In this paper, we present a model that supports this general result. It goes beyond the existing behavioural New Keynesian literature by deriving analytical stability conditions in a model with bounded rationality and rational expectations. We make two main contributions to the literature. Our first contribution is the derivation of analytical stability conditions. The existing literature tends to rely on numerical simulation to study the dynamics of behavioural New Keynesian models. While the benefits of numerical simulation are numerous, we are of the opinion that analytical results, arrived at by the use of small models, provide important insights[1]. Our second contribution is the use of the anticipated utility approach of Kreps (1998). The majority of the existing literature on behavioural New Keynesian models employs Euler learning, in which agents' decisions are based on first order conditions to maximisation problems. In contrast to the rational expectations solution, in which model consistent expectations enter the first order conditions, Euler learning uses simple bounded rational predictors alongside knowledge of the form of the rational expectations solution.

[1] This is a standpoint shared by, for example, Turnovsky (2011).

In the anticipated utility approach, henceforth AU, agents follow an optimal decision rule conditional on their beliefs over aggregate states and prices[2]. This takes into account all information available to the agent, and involves forecasts of variables external to them. AU is similar - but not identical - to the internal rationality approach of Adam and Marcet (2011), in which "agents maximize utility under uncertainty, given their constraints and given a consistent set of probability beliefs about payoff-relevant variables that are beyond their control or external". With internal rationality, henceforth IR, beliefs take the form of a well-defined probability measure over a stochastic process - the fully Bayesian plan. Adam and Marcet (2011) and Adam et al. (2017) utilise the IR approach, whereas our paper and a number of the applications cited below adopt AU. Cogley and Sargent (2008) compare AU and IR and encouragingly find that AU can be seen as a good approximation to the fully Bayesian plan[3].

The AU approach was first used in a New Keynesian model in Preston (2005), and a real business cycle model in Eusepi and Preston (2011). Adam and Marcet (2011) apply the IR approach to asset pricing, Spelta et al. (2012) apply AU to a model of house prices, Woodford (2013) apply AU to a New Keynesian framework, and Adam et al. (2017) apply IR to a model of stock market booms. Massaro (2013) constructs a behavioural New Keynesian model in which a fixed subset of agents are AU learners and the remaining subset are fully rational. Of these existing studies, our approach is closest to Massaro (2013). Specifically, we present a New Keynesian model in which a fraction n of agents are fully rational, and a fraction $1 - n$ of agents are AU learners, and use this to demonstrate the following results:

1. The Taylor condition is sufficient for determinacy and stability when n is fixed,
2. The Taylor condition is sufficient for local determinacy and stability when n varies according to reinforcement learning,
3. When monetary policy is such that the dynamics are indeterminate, limit cycles can exist, and may be followed by a rational route to randomness,
4. The Taylor condition is not sufficient for determinacy and stability in the presence of interest rate smoothing.

Thus our results offer qualified support to the existing monetary policy orthodoxy, which is consistent with the message of the behavioural New Keynesian literature.

[2]The anticipated utility approach with infinite time horizons is also referred the *infinite-time horizon* approach. Bounded rationality of this form can be generalized to finite time horizons - see Lustenhouwer and Mavromatis (2017) and Woodford (2019).

[3]See Branch and McGough (2016) and Deak et al. (2017) for further discussion. Sinitskaya and Tesfatsion (2015) introduce forward-looking optimizing agents into an ACE framework. They use a concept that falls within a general definition of AU which they refer to as "constructive rational decision making". Graham (2011) uses the term "individual rationality" to refer to the same general concept.

The remainder of the paper is organised as follows. Section 2 presents the basic New Keynesian framework. Section 3 presents the New Keynesian model with AU households and firms, and demonstrates our first proposition. Section 4 incorporates reinforcement learning, and demonstrates our second and third propositions. Section 5 incorporates interest rate smoothing, and demonstrates our fourth proposition. Section 6 concludes.

2 The New Keynesian model with rational expectations

In this section, we briefly recap the workhorse New Keynesian model with rational expectations. We set up the model in a way the emphasises the link with bounded rationality assuming anticipated utility, which should aid the reader when interpreting the models in sections 3 and 4 below. We first consider the decision problems of households and firms, and then the aggregation and equilibrium conditions.

2.1 Households

Let $C_t(j)$ denote consumption and $H_t(j)$ denote hours worked for the jth household. The within-period utility function is,

$$U_t(j) = \log(C_t(j)) - \frac{H_t(j)^{1+\phi}}{1+\phi},$$

and households choose paths for consumption $C_t(j)$, labour supply $H_t(j)$, and holdings of financial assets $B_t(j)$, to maximise $\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s U_{t+s}(j)$ subject to the flow budget constraint,

$$B_t(j) = R_t B_{t-1}(j) + W_t H_t(j) + \Gamma_t(j) - C_t(j),$$

where W_t denotes the real wage, Γ_t denotes distributed profits, and R_t denotes the ex post real interest rate paid on assets held at the beginning of period t. The first order conditions for the household problem are,

$$\frac{1}{C_t(j)} = \beta \mathbb{E}_t \left[\frac{R_{t+1}}{C_{t+1}(j)} \right], \tag{1}$$

$$H_t(j) = \left(\frac{W_t}{C_t(j)} \right)^{\frac{1}{\phi}}.$$

Usually, one analyses the New Keynesian model by log-linearising the first order conditions, leading to the familiar consumption Euler equation and labour supply function,

$$c_t(j) = \mathbb{E}_t \left[c_{t+1}(j) - r_{t+1} \right],$$

$$h_t(j) = \frac{1}{\phi}(w_t - c_t(j)),$$

where variables in lower case denote log-deviations.

As the labour supply function is static, it does not pose any particular problems when we move from rational expectations to bounded rationality. The consumption Euler equation, on the other hand, involves expectations of future variables, and a useful form of the household's decision rule can be found by solving the household budget constraint forward in time and imposing the Euler and transversality conditions. In symmetric equilibrium with zero net financial assets, this yields a consumption function for the representative household of the form,

$$\mathrm{PV}_t(C_t) = \mathrm{PV}_t\left(\frac{W_t^{1+\frac{1}{\phi}}}{C_t^{\frac{1}{\phi}}}\right) + \mathrm{PV}_t(\Gamma_t),$$

which states that the present value of consumption is equal to the present value of total income, where the present (expected) value of a series $X \equiv \{X_{t+i}\}_{i=0}^{\infty}$ at time t is defined by

$$\mathrm{PV}_t(X_t) \equiv \mathbb{E}_t \sum_{i=0}^{\infty} \frac{X_{t+i}}{R_{t,t+i}} = \frac{X_t}{R_t} + \frac{1}{R_t}\mathrm{PV}_t(X_{t+1}),$$

writing $R_{t,t+i} \equiv R_t R_{t+1} R_{t+2} \cdots R_{t+i}$ as the real interest rate over the interval $[t-1, t+i]$.

Using exogenous point expectations, appendix B and the supplementary appendices D and E demonstrates that the corresponding log-linearised consumption function is given by,

$$\alpha_1 c_t(j) = \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_t) + \alpha_4 \omega_{1,t}, \tag{2}$$

where,

$$\omega_{1,t} = \alpha_5 \mathbb{E}_t w_{t+1} - \alpha_6 \mathbb{E}_t r_{t+1} + \beta \mathbb{E}_t \omega_{1,t+1}, \tag{3}$$

$$\omega_{2,t} = (1-\beta)\gamma_t - r_t + \beta \mathbb{E}_t \omega_{2,t+1}, \tag{4}$$

and,

$$\gamma_t = \frac{1}{1-\alpha} c_t - \frac{\alpha}{1-\alpha}(w_t + h_t),$$

denotes log-linearised dividends. Consumption is therefore a function of the current wage and profit income, expected wage and profit income, and current and expected real interest rates. The parameters and composite parameters are defined in table 1.

Parameter	Definition
α	Elasticity of output with respect to labour input ($\alpha > 0$)
β	Representative household discount rate ($0 < \beta < 1$)
Υ	Fixed cost of rational expectations predictor ($-\infty < \Upsilon < \infty$)
ζ	Elasticity of substitution between consumption goods ($\zeta \geq 0$)
θ_π	Monetary policy rule elasticity of inflation ($\theta_\pi \geq 0$)
θ_y	Monetary policy rule elasticity of output ($\theta_y \geq 0$)
μ	Intensity of choice parameter ($\mu > 0$)
ξ	Calvo probability that firms change price ($0 \leq \xi \leq 1$)
ϕ	Inverse Frisch elasticity of labour supply ($\phi > 0$)
α_1	$\alpha_1 = 1 + \alpha/\phi$
α_2	$\alpha_2 = \alpha(1-\beta)\,(1+1/\phi)$
α_3	$\alpha_3 = 1 - \alpha$
α_4	$\alpha_4 = \alpha\beta$
α_5	$\alpha_5 = (1-\beta)\,(1+1/\phi)$
α_6	$\alpha_6 = 1 + 1/\phi$
δ	$\delta = (1-\xi)(1-\beta\xi)^{-1}$
κ	$\kappa = (1-\xi)(1-\beta\xi)(1+\phi)(\alpha\xi)^{-1}$
ψ	$\psi = (1-\beta\xi)^{-1}$
A	$A = (\theta_\pi\kappa - \theta_\pi\kappa\psi)(\beta\theta_y)^{-1}$
B	$B = (\theta_y + \theta_\pi\kappa\psi)(\beta\theta_y)^{-1}$
C	$C = (\kappa - \delta\beta\theta_y - \kappa\psi)(\beta\theta_y)^{-1}$
D	$D = (\delta\beta\theta_y + \kappa\psi)(\beta\theta_y)^{-1}$

Table 1: Parameters, parameter definitions, and composite parameter definitions.

2.2 Firms

Firms in the retail sector uses a homogeneous wholesale good to produce a basket of differentiated goods for aggregate consumption. Consumers choose the consumption of variety m at a price $P_t(m)$ to maximise a standard CES sub-utility function with elasticity of substitution equal to ζ, which yields the demand functions,

$$C_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} C_t,$$

or,

$$Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} Y_t,$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm\right]^{\frac{1}{1-\zeta}}$ is the aggregate price index, and C_t, Y_t, and P_t are Dixit-Stigliz aggregates (Dixit and Stiglitz, 1977).

For each variety m the retail good is produced costlessly from wholesale production,

$$Y_t(m) = Y_t^W = A_t H_t(m)^\alpha.$$

Following Calvo (1983), there is a probability $1-\xi$ in each period that the price of each variety m is set optimally to $P_t^0(m)$. If the price is not re-optimized, then it is held fixed.[4] For each retail producer m, given its real marginal cost MC_t, the objective is at time t to choose $\{P_t^0(m)\}$ to maximize discounted real profits,

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left[P_t^O(m) - P_{t+k} MC_{t+k}\right],$$

subject to

$$Y_{t+k}(m) = \left(\frac{P_t^O(m)}{P_{t+k}}\right)^{-\zeta} Y_{t+k},$$

where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}}{U_{C,t}}$ is the stochastic discount factor over the interval $[t, t+k]$. The solution to this is standard and given by

$$\frac{P_t^O(m)}{P_t} = \frac{\zeta}{\zeta - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} \left(\Pi_{t,t+k}\right)^{\zeta} Y_{t+k} MC_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} \left(\Pi_{t,t+k}\right)^{\zeta} \left(\Pi_{t,t+k}\right)^{-1} Y_{t+k}}.$$

[4]Thus we can interpret $\frac{1}{1-\xi}$ as the average duration for which prices are left unchanged.

Denoting the numerator and denominator by $\Omega_{3,t}$ and $\Omega_{4,t}$ and introducing a mark-up shock MS_t to MC_t, as detailed in appendix C we can write in recursive form,

$$
\begin{aligned}
\frac{P_t^O(m)}{P_t} &= \frac{\Omega_{3,t}}{\Omega_{4,t}}, \\
\Omega_{3,t} - \xi \mathbb{E}_t[\Lambda_{t,t+1} \Pi_{t+1}^{\zeta} \Omega_{3,t+1}] &= \frac{1}{1-\frac{1}{\zeta}} Y_t MC_t MS_t, \\
\Omega_{4,t} - \xi \mathbb{E}_t[\Lambda_{t,t+1} \Pi_{t+1}^{\zeta-1} \Omega_{4,t+1}] &= Y_t.
\end{aligned}
$$

Using the fact that all resetting firms will choose the same price, by the Law of Large Numbers we can find the evolution of inflation given by,

$$1 = \xi \left(\Pi_{t-1,t}\right)^{\zeta-1} + (1-\xi) \left(\frac{P_t^O}{P_t}\right)^{1-\zeta}.$$

In a zero-net inflation steady state, the linear choice for the optimizing retail firm m found by linearizing these equations about the deterministic steady state is given by,

$$p_t^o(m) - p_t = \omega_{3,t} - \omega_{4,t}, \tag{5}$$

where $p_t^o(m)$ is the optimal price for firm m, and,

$$\omega_{3,t} = \xi\beta \mathbb{E}_{t+1}\left[\zeta \pi_{t+1} + \omega_{3,t+1}\right] + (1-\beta\xi)(y_t + u_{C,t} + mc_t + ms_t),$$

$$\omega_{4,t} = \xi\beta \mathbb{E}_{t+1}\left[(\zeta - 1)\pi_{t+1} + \omega_{4,t+1}\right] + (1-\beta\xi)(y_t + u_{C,t}),$$

where π_t is the aggregate inflation rate, y_t is aggregate output, $u_{C,t}$ is household marginal utility, mc_t is marginal cost, and ms_t is an exogenous supply shock. Finally, for the wholesale sector we have,

$$y_t = \alpha h_t,$$

$$mc_t = w_t - y_t + h_t.$$

Note that labour productivity is assumed to be constant, so the only exogenous driving variable is the shock process ms_t.

2.3 Aggregation and equilibrium

Assuming a unit measure of households and retail firms, aggregation under symmetry entails $c_t(j) = c_t$, $h_t(j) = h_t$, $p_t^o(m) = p_t^o$, and $\xi\pi_t = (1-\xi)(p_t^o - p_t)$. Equilibrium in the output market requires $y_t = c_t$. The model is completed with a Fisher equation,

$$r_t = r_{n,t-1} - \pi_t,$$

where $r_{n,t}$ is the nominal interest rate, and a policy rule of the form,

$$r_{n,t} = \theta_\pi \pi_t + \theta_y y_t. \tag{6}$$

We confine our attention to implementable policy rules, and postpone until section 5 a discussion of rules with persistence.

2.4 Reduced form

Imposing the aggregation and equilibrium conditions, we arrive at the workhorse New Keynesian three equation model,

$$y_t = \mathbb{E}_t y_{t+1} - (r_{n,t} - \mathbb{E}_t \pi_{t+1}), \tag{7}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa(y_t + ms_t), \tag{8}$$

$$r_{n,t} = \theta_\pi \pi_t + \theta_y y_t. \tag{9}$$

Before presenting the determinacy condition, two points about this formulation need to be made. First, there is no lagged output in the demand curve (7), nor lagged inflation in the Phillips curve (8). These can enter through the introduction of external habit in households' utility functions and price indexing, respectively. But we choose to focus on bounded rationality as a persistence mechanism, so both of these features are omitted. Second, even without these persistence terms, the linearisation is only correct about a zero inflation steady state.

To find the determinacy and stability condition for the rational expectations model in (7) - (9), we write the model in state space form, setting $ms_t = 0$ and substituting out $r_{n,t}$ from (7) using (9). We then have,

$$\begin{bmatrix} \mathbb{E}_t y_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \theta_y + \kappa/\beta & \theta_\pi - 1/\beta \\ -\kappa/\beta & 1/\beta \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}. \tag{10}$$

Denote the trace of the system in (10) by τ, and the determinant by Δ. These are,

$$\tau = 1 + \theta_y + \kappa/\beta + 1/\beta,$$

$$\Delta = \frac{1 + \theta_y + \kappa\theta_\pi}{\beta}.$$

For stability, we simply require a stable shock process ms_t. For determinacy, we require that both of the eigenvalues of the system in (10) lie outside the unit circle, as both y_t and π_t are jump variables (Blanchard and Kahn, 1980). Necessary and sufficient conditions are (Woodford, 2003a),

1. $\Delta > 1$,
2. $1 - \tau + \Delta > 0$,
3. $1 + \tau + \Delta > 0$.

As $\beta < 1$ and $\theta_y + \kappa\theta_\pi > 0$, condition 1 is always satisfied, and the binding condition is condition 2. Substituting in the trace and determinant, we arrive at the familiar condition,

$$\theta_\pi + \left(\frac{1-\beta}{\kappa}\right)\theta_y > 1. \tag{11}$$

3 The New Keynesian model with anticipated utility

We now extend the standard New Keynesian model to include both AU and fully rational households and firms. This allows us to demonstrate our first proposition, and forms the basis of the model with reinforcement learning in section 4.

3.1 Households

We distinguish between the consumption of fully rational households, c_t^{RE}, and AU households, c_t^{AU}. The consumption of fully rational households is pinned down by the rational expectations Euler equation as before,

$$c_t^{RE} = \mathbb{E}_t\left[c_{t+1}^{RE} - (r_{n,t} - \pi_{t+1})\right], \tag{12}$$

where we have omitted the household index to reduce notational clutter. With Euler learning (henceforth EL), as in Branch and McGough (2010), the consumption of bounded rational households would be pinned down by the Euler equation,

$$c_t^{EL} = \mathbb{E}_t^*\left[c_{t+1}^{EL} - (r_{n,t} - \pi_{t+1})\right],$$

where $\mathbb{E}_t^*$ denotes a bounded rational expectations operator. Hence households base their consumption decisions on forecasts of the same decision in future periods.

As discussed in the introduction, we replace Euler learning with anticipated utility. Expectation formation discussed in more detail in the next section uses the predictor $\mathbb{E}_t^*[x_{t+j}] = \mathbb{E}_t^*[x_{t+1}]$ for $j \geq 1$.[5] In this case iterating (3) and (4) forward in time and using $\sum_{j=1}^{\infty} \beta^j \mathbb{E}_t^* x_{t+j} = \sum_{j=1}^{\infty} \beta^j \mathbb{E}_t^* x_{t+1} = \frac{\beta}{1-\beta} \mathbb{E}_t^* x_{t+1}$, the consumption function for AU households becomes:

$$\alpha_1 c_t^{AU} = \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_{n,t-1} - \pi_t) + \alpha_4 \omega_{1,t}, \tag{13}$$

with,

$$\omega_{1,t} = \frac{1}{1-\beta}\left[\alpha_5 \mathbb{E}_t^* w_{t+1} + \alpha_6 \mathbb{E}_{h,t}^* \pi_{t+1}\right] - \alpha_6 \left(r_{n,t} + \frac{\beta}{1-\beta} \mathbb{E}_t^* r_{n,t+1}\right),$$

$$\omega_{2,t} = (1-\beta)\gamma_t^{AU} + \beta \mathbb{E}_t^* \gamma_{t+1}^{AU} - \left(r_{n,t-1} + \beta r_{n,t} + \frac{\beta^2}{1-\beta} \mathbb{E}_t^* r_{n,t+1}\right) + \pi_t + \left(\frac{\beta}{1-\beta}\right) \mathbb{E}_{h,t}^* \pi_{t+1},$$

Hence AU households base their consumption decisions on forecasts of the variables exogenous to them - wages, profits, interest rates, and inflation rates.

We now have to differentiate between the profit flows accruing to AU and fully rational households. In the general case, with a fully specified market for the ownership of firms, an individual household's profit earnings would depend on their entire history of strategy choice over fully rational and AU behaviour, leading to a complicated distribution over households. To avoid this - and ensure tractability - Massaro (2013) assumes that profit is distributed equally across households. We take a different approach, and assume that profits accrue to households in proportion to their contribution to overall profit, i.e.,

$$\gamma_t^{RE} = \frac{1}{1-\alpha} c_t^{RE} - \frac{\alpha}{1-\alpha}(w_t + h_t^{RE}), \tag{14}$$

$$\gamma_t^{AU} = \frac{1}{1-\alpha} c_t^{AU} - \frac{\alpha}{1-\alpha}(w_t + h_t^{AU}). \tag{15}$$

This is the major simplifying assumption that allows us to derive straightforward expressions of the model's reduced form, which in turn allows us to derive analytical stability and bifurcation conditions. In fact, it is the only material difference between the microfoundations of our model and that of Massaro (2013). Although the assumption is relatively unusual, it is in a similar spirit to an assumption of equal distribution, and ensures that $\gamma_t = n\gamma_t^{RE} + (1-n)\gamma_t^{AU}$ in each period, where n is the proportion of fully rational agents.

As before, optimal labour supply is an intra-temporal decision, so we have,

[5] This will hold with a general adaptive expectations rule.

$$h_t^{RE} = \frac{1}{\phi}\left(w_t - c_t^{RE}\right), \tag{16}$$

$$h_t^{AU} = \frac{1}{\phi}\left(w_t - c_t^{AU}\right), \tag{17}$$

where h_t^{RE} is the labour supply of fully rational households, and h_t^{AU} is the labour supply of AU households. These labour supply functions provide more insight into the profit distribution rules, as they allow us to re-write (14) and (15) as,

$$\gamma_t^{RE} = \frac{\alpha + \phi}{\phi(1-\alpha)} c_t^{RE} - \frac{\alpha(1+\phi)}{\phi(1-\alpha)} w_t,$$

$$\gamma_t^{AU} = \frac{\alpha + \phi}{\phi(1-\alpha)} c_t^{AU} - \frac{\alpha(1+\phi)}{\phi(1-\alpha)} w_t.$$

Thus RE households receive more dividends relative to AU households when they are consuming more, and vice versa. Note that, as both consumption and labour supply are procyclical, dividends increase with output and hours worked in equilibrium.

3.2 Firms

As for firms with rational expectations, optimal price setting for AU retail firms is given by,

$$(p_t^o - p_t)^{AU} = \beta\xi \mathbb{E}_t^*[\pi_{t+1} + (p_{t+1}^o - p_{t+1})^{AU}] + (1-\beta\xi)(mc_t + ms_t). \tag{18}$$

Solving forwards yields,

$$(p_t^o - p_t)^{AU} = \mathbb{E}_t^* \sum_{i=0}^{\infty} (\beta\xi)^i [\beta\xi\pi_{t+i+1} + (1-\beta\xi)(mc_{t+i} + ms_{t+i})]. \tag{19}$$

Note that AU for retail firms is more straightforward than for households, as the rational expectations solution is already in recursive form and there is no retail firm budget constraint. Note also that with AU we do not impose the aggregation relationship $\xi\pi_t = (1-\xi)(p_t^o - p_t)$ used in the RE solution as this requires firms to know that they are identical.

3.3 Aggregation and equilibrium

Without loss in generality, for reasons given in section 3.5, suppose that the proportion n of fully rational households in the economy is equal to the proportion of fully rational firms. Assuming a unit measure of households, aggregation entails,

$$nc_t^{RE} + (1-n)c_t^{AU} = c_t, \tag{20}$$

$$nh_t^{RE} + (1-n)h_t^{AU} = h_t, \tag{21}$$

$$n(p_t^o - p_t)^{RE} + (1-n)(p_t^o - p_t)^{AU} = p_t^o - p_t, \tag{22}$$

$$\xi\pi_t = (1-\xi)(p_t^o - p_t). \tag{23}$$

The equilibrium conditions, Fisher equation, and the monetary policy rule are exactly the same as in the standard rational expectations model.

3.4 Expectation formation of AU agents

Equations (12) - (23) define the New Keynesian model with AU up to the definition of the bounded rational predictor $\mathbb{E}_t^*$. To close the model, we therefore need to specify the manner in which AU households and firms form their expectations. As discussed in Calvert Jump and Levine (2019), there is a large literature discussing departures from full rationality in expectations formation, with a comprehensive survey of the studies preceding the 2008 financial crisis contained in Pesaran and Weale (2006). A notable post-crisis paper is Pfajfar and Santoro (2010), who find that only 10% of the forecasts in the Michigan Survey reflect regular information updating. A useful simplified predictor in this context is the static predictor in which future values of a variable are forecast as equal to the last observed value of that variable, i.e.,

$$\mathbb{E}_t^*[x_{t+1}] = x_{t-i},$$

for some random variable x, where $i \geq 0$ determines the last observed value. This is the extrapolative predictor used in chapter 1 of De Grauwe (2012b), and is a special case of the bounded rational predictor used in Branch and McGough (2010). It is the optimal predictor when agents believe that x follows a random walk, which is a relatively accurate approximation to most macroeconomics variables (Nelson and Plosser, 1982).

Given the foregoing, we assume that AU households and firms assume that the variables of interest to them follow random walks, and therefore forecast all variables as equal to their last observed values. We assume that variables which are local to the agents, in a geographical sense, are observable within the period, whereas variables that are strictly macroeconomic are only observable with a lag. This categorization regarding information about the current state of the economy follows Nimark (2014). He distinguishes between the local information that agents acquire directly through their interactions in markets and statistics that are collected and summarised, usually by governments, and made available to the wider public[6]. The only exception to this is the nominal interest rate, which we assume

[6]His paper actually focuses on a third category, information provided by the news media, and allows for imperfect information in the form of noisy signals, issues which go beyond the scope of our paper.

is observable within the period given the timing structure of New Keynesian models. Thus AU household expectations are given by,

$$\mathbb{E}_t^* w_{t+1} = w_t, \tag{24}$$

$$\mathbb{E}_t^* \gamma_{t+1} = \gamma_t, \tag{25}$$

$$\mathbb{E}_t^* r_{n,t+1} = r_{n,t}, \tag{26}$$

$$\mathbb{E}_t^* \pi_{t+1} = \pi_{t-1}, \tag{27}$$

and AU firm expectations are given by,

$$\mathbb{E}_t^* mc_{t+1} = mc_t, \tag{28}$$

$$\mathbb{E}_t^* \pi_{t+1} = \pi_{t-1}. \tag{29}$$

AU firms can observe their own marginal costs within the period, but in a similar manner to AU households, can only observe aggregate inflation with a lag. Note that firms observing their real marginal costs within the period, and households observing their real wage and profits within the period, does not imply that firms and households observe the aggregate price level within the period. We assume that they observe their *own* price within the period, and therefore their own real marginal costs, real wages, and dividends, but not the aggregate price level. This is reasonable given the considerable data-gathering costs of observing aggregate macroeconomic variables like inflation, as discussed in Nimark (2014). Note, however, that fully rational agents observe all variables within the period, and that we retain the Taylor rule (9) and assume that the central bank observes current inflation and output, thus having the same information advantage as rational agents.

3.5 Reduced form

Equations (12) - (29) fully describe the New Keynesian model with AU, where the proportion n of fully rational agents is held constant. Deriving the reduced form is relatively straightforward. First, by rearranging the AU household consumption function (13) after substituting in the expectations functions, we find that AU households choose their level of consumption such that,

$$r_{n,t} = \pi_{t-1}, \tag{30}$$

in each period. The derivation of (30) is discussed in some detail in appendix A.

Combining (30) with the monetary policy rule (9), we see that,

$$y_t = -\left(\frac{\theta_\pi}{\theta_y}\right)\pi_t + \left(\frac{1}{\theta_y}\right)\pi_{t-1}, \tag{31}$$

which greatly simplifies the analysis, as we will not need to track output as a separate state variable. In fact, as (31) means that we do not have to separately track the consumption levels of fully rational and AU households in the state space form, it is this result that allows us to derive analytical stability conditions in the sequel. Also note that (31) means that the proportion of fully rational households does not affect the equation of motion for y_t, which allows us to assume that the proportion of fully rational households is equal to the proportion of fully rational firms without loss of generality.

Using the aggregation conditions (22) and (23), and the price setting conditions (18) and (19), we can derive the reduced form New Keynesian Phillips curve with fully rational and AU firms,

$$\pi_t = n(\beta \mathbb{E}_t \pi_{t+1} + \kappa y_t) + (1-n)(\delta\beta\pi_{t-1} + \kappa\psi y_t), \tag{32}$$

where the shocks process ms_t is set equal to zero, $mc_t = y_t(1+\phi)/\alpha$, and the composite parameters κ, δ, and ψ are defined in table 1. Finally, by substituting the equation of motion for output (31) into the New Keynesian Phillips curve (32) and rearranging, we arrive at the reduced form model,

$$\mathbb{E}_t \pi_{t+1} = \left(A + \frac{B}{n}\right)\pi_t - \left(C + \frac{D}{n}\right)\pi_{t-1}, \tag{33}$$

where the composite parameters are defined in table 1.

3.6 State space form and stability

The New Keynesian model with fixed proportions n of fully rational agents and $(1-n)$ of AU agents, has a reduced form (33) described by a second order forward looking difference equation in inflation. Define the auxiliary variable $z_t = \pi_{t-1}$. Then the state space form of our model is given by,

$$\begin{bmatrix} \mathbb{E}_t \pi_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} A + B/n & -(C+D/n) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_t \\ z_t \end{bmatrix}, \tag{34}$$

where π_t is a jump variable and z_t is a pre-determined variable. We are now in a position to demonstrate our first proposition:

Proposition 1: If the monetary policy rule is such that the condition in (11) holds, then the model in (34) is determinate and stable.

Proof of proposition 1: The proof is composed of two lemmas:

Lemma 1: For $n \in (0, 1]$, if the monetary policy rule is such that the condition in (11) holds, then the model in (34) is determinate and stable.

Proof of lemma 1: Determinacy and stability in the model described by (34) with $n \in (0, 1]$ requires one eigenvalue inside the unit circle and one eigenvalue outside the unit circle (Blanchard and Kahn, 1980). Denote the trace of the system in (34) by $\tau = A + B/n$ and the determinant by $\Delta = C + D/n$. As τ and Δ are positive, the necessary and sufficient condition for determinacy and stability in the model described by (34) is $\tau - \Delta > 1$. Now, suppose that $n = 1$ and,

$$\theta_\pi = 1 - \left(\frac{1-\beta}{\kappa}\theta_y\right) + \epsilon, \tag{35}$$

where ϵ is an arbitrarily small but positive constant (i.e. the condition in (11) holds), so the model is determinate and stable. A sufficient condition for stability when $n \in (0, 1)$ is then,

$$\frac{d}{dn}(\tau - \Delta) < 0 \ \forall n \in (0, 1], \tag{36}$$

i.e. $\tau - \Delta$ increases from 1 as n decreases from 1. From (34), we have,

$$\frac{d}{dn}(\tau - \Delta) = (D - B)n^{-2}.$$

Taking advantage of the definitions of B and D in table 1, this yields,

$$\frac{d}{dn}(\tau - \Delta) = \left[\frac{\theta_y(\delta\beta - 1) - \kappa\psi(\theta_\pi - 1)}{\beta\theta_y}\right] n^{-2}, \tag{37}$$

which, by substituting (35) into (37), yields,

$$\frac{d}{dn}(\tau - \Delta) = (D - B)n^{-2} = -\left[\frac{\kappa\psi\epsilon}{\beta\theta_y}\right] n^{-2}. \tag{38}$$

As κ, ψ, β, and θ_y are all positive (see table 1), (38) implies (36). This is illustrated graphically in figure 1, which shows the standard stability plot in the trace and determinant for a second order difference equation when both the trace and determinant are positive (see e.g. Hamilton 1994, chapter 1). ■

Lemma 2: For $n = 0$, if the monetary rule is such that the condition in (11) holds, then the model in (34) is stable.

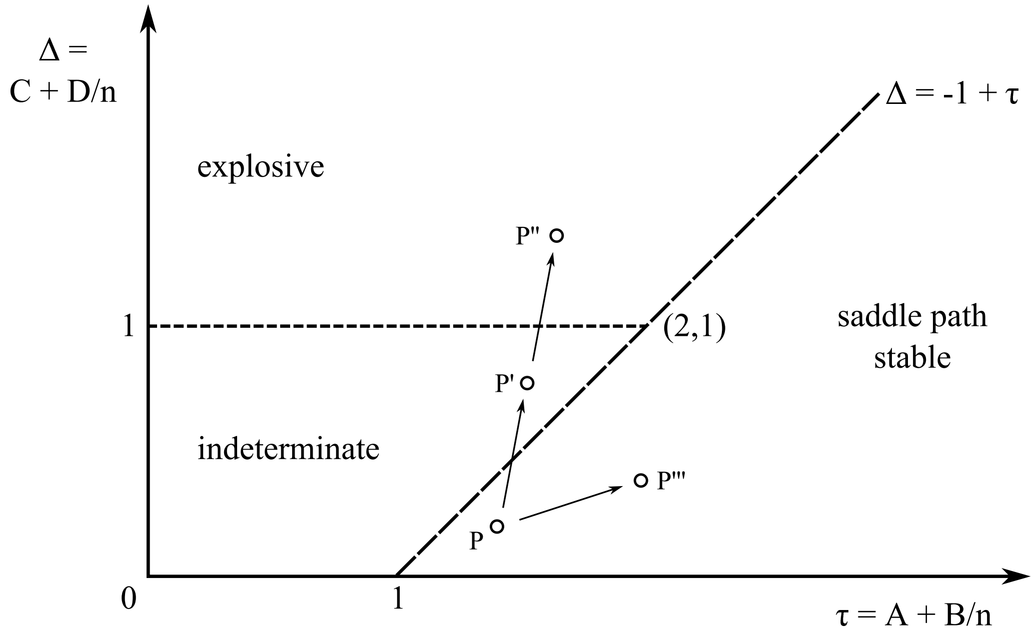

Figure 1: Graphical illustration of proposition 1, showing a stability plot in the trace τ and determinant Δ for a second order difference equation when $\tau > 0$ and $\Delta > 0$. At point P, which lies within the saddle path stable region (i.e. it satisfies $\tau - \Delta > 1$), a decrease in n moves the model to P′ or P″ if $\frac{\partial \tau}{\partial(-n)} < \frac{\partial \Delta}{\partial(-n)}$, or to P''' if $\frac{\partial \tau}{\partial(-n)} > \frac{\partial \Delta}{\partial(-n)}$. Thus a decrease in n moves the model to P′ or P″ if $\frac{\partial \tau}{\partial n} > \frac{\partial \Delta}{\partial n}$, or to P''' if $\frac{\partial \tau}{\partial n} < \frac{\partial \Delta}{\partial n}$.

Proof of lemma 2: When $n = 0$, there are no agents with rational expectations, and therefore determinacy is irrelevant. From (32), the New Keynesian Phillips curve is given by,

$$\pi_t = \delta\beta\pi_{t-1} + \kappa\psi y_t, \tag{39}$$

when all agents are AU. Substituting out y_t using (31), we have the reduced form,

$$\pi_t = \left(\frac{\delta\beta\theta_y + \psi\kappa}{\theta_y + \psi\kappa\theta_\pi}\right)\pi_{t-1}. \tag{40}$$

The model in (40) is stable when the coefficient on π_{t-1} is less than one in absolute value. As the coefficient will be positive given the parameter definitions in table 1, this is the case when,

$$\frac{\delta\beta\theta_y + \psi\kappa}{\theta_y + \psi\kappa\theta_\pi} < 1. \tag{41}$$

Rearranging, and taking advantage of the parameter definitions, we arrive at a stability condition identical to (11). Therefore, the model in (34) with $n = 0$ is stable if the condition in (11) holds, which completes the proof of proposition 1. ■

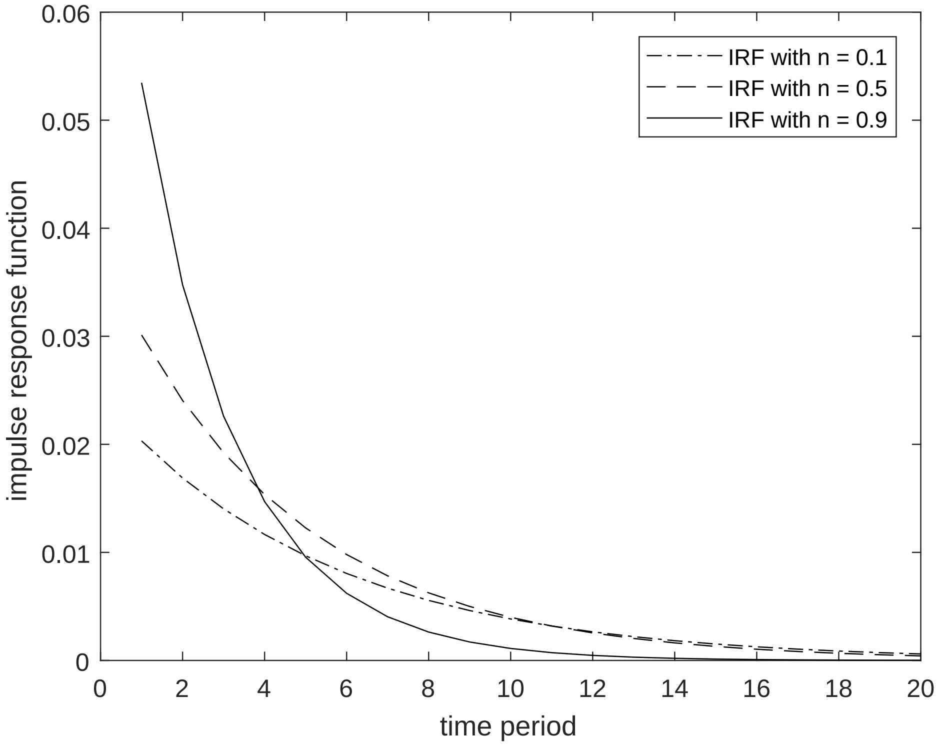

Figure 2: Impulse response functions of inflation to a positive marginal cost shock, for the model with n fixed, for three different values of n. The remaining parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 1.25$, $\theta_y = 0.5$.

Proposition 1 states that the rational expectations determinacy condition is sufficient for determinacy and stability in the model with fixed proportions of fully rational and AU agents. However, the dynamics of the model will vary with n, as the magnitude of the eigenvalues will change as n changes. This is illustrated in figure 2, which plots impulse response functions of inflation in response to an ms shock with $n = 0.1$, $n = 0.5$, and $n = 0.9$. The remaining parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 1.25$, $\theta_y = 0.5$, such that the condition in (11) holds, and the marginal cost shock has no persistence. Although the determinacy and stability properties of the model are unaffected by a reduction in n, given that (11) holds, the response of the model to shocks becomes increasingly persistent as the proportion of fully rational agents decreases. This result is consistent with the results of Pecora and Spelta (2017), who find that convergence to the steady state can be slow in models with heterogeneous expectations, despite the Taylor principle being sufficient for stability.

4 The New Keynesian model with strategy switching

In this section, we extend the analysis to allow n to vary. Following the literature, we assume that n varies according to a reinforcement learning mechanism laid out in section 4.1. We then derive the reduced form in section 4.2, and consider the state space form and local stability conditions in section 4.3. We establish our second and third propositions in this section. First, the rational expectations determinacy condition ensures local determinacy and stability in the model with n variable. Second, if the model starts from a position of indeterminacy, an increase in the fixed cost of being fully rational can lead to the loss of local stability via a Hopf bifurcation. This Hopf bifurcation appears to be super-critical, giving rise to stable limit cycles. As the speed at which agents learn increases, a rational route to randomness appears to follow, which we explore with numerical methods.

4.1 Reinforcement learning and predictor fitness

We extend the model to allow n to vary with the perceived relative forecasting strength of the fully rational and AU predictors. Following Branch and McGough (2010) and the literature described in the introduction, denote the fitness of the rational expectations predictor by v_t^{RE}, and the fitness of the AU predictor by v_t^{AU}. Then the proportion of fully rational agents at any point in time is given by,

$$n_t = \frac{\exp[\mu v_t^{RE}]}{\exp[\mu v_t^{RE}] + \exp[\mu v_t^{AU}]}. \tag{42}$$

The parameter μ in (42) is referred to as the intensity of choice parameter, as a higher μ increases the rate at which agents choose strategies with a high fitness level. In this sense, μ governs the speed of learning.

Denote the perceived mean squared error of the AU predictor by Φ_t, and define it as follows,

$$\Phi_t = (\pi_t - \mathbb{E}^*_{t-1}[\pi_t])^2 = (\pi_t - \pi_{t-2})^2. \tag{43}$$

If - as we will do in the sequel - we consider a deterministic economy, the mean squared error of the fully rational predictor is zero, as rational expectations is equivalent to perfect foresight in this context. Finally, and in accordance with the literature, we define the fitness measures as follows,

$$v_t^{RE} = -\Upsilon, \tag{44}$$

$$v_t^{AU} = -\Phi_t, \tag{45}$$

where Υ is a fixed cost of using the fully rational predictor. The AU predictor is then fit relative to the fully rational predictor when the mean squared error falls below the fixed cost of being fully rational.

4.2 Reduced form

Equations (12) - (29), extended to allow n to vary with equations (42) - (45), fully describe the New Keynesian model with fully rational and AU agents, where the proportion n of fully rational agents varies over time according to the perceived relative fitness of the two strategies. By substituting (43) - (45) into (42), we find that,

$$n_t = \frac{\exp[-\mu\Upsilon]}{\exp[-\mu\Upsilon] + \exp[-\mu\Phi_t]},$$

$$\Rightarrow n_t = \frac{\exp[-\mu\Upsilon]}{\exp[-\mu\Upsilon] + \exp[-\mu(\pi_t - \pi_{t-2})^2]}. \tag{46}$$

Thus, as the perceived mean squared error of the AU predictor falls below the fixed cost, Υ, of being fully rational, agents move towards being AU and n falls. The speed of this process is determined by the intensity parameter μ. Note that (46) implies,

$$n_t^{-1} = 1 + \exp[-\mu((\pi_t - \pi_{t-2})^2 - \Upsilon)]. \tag{47}$$

As we have changed nothing in the original model other than allowing n to vary, the original reduced form (33) becomes,

$$\mathbb{E}_t\pi_{t+1} = \left(A + \frac{B}{n_t}\right)\pi_t - \left(C + \frac{D}{n_t}\right)\pi_{t-1}, \tag{48}$$

with A, B, C, and D defined as before. Finally, substituting (47) into (48), we arrive at the reduced form New Keynesian model with n variable,

$$\mathbb{E}_t\pi_{t+1} = \left[A + B\left(1 + e^{-\mu((\pi_t - \pi_{t-2})^2 - \Upsilon)}\right)\right]\pi_t - \left[C + D\left(1 + e^{-\mu((\pi_t - \pi_{t-2})^2 - \Upsilon)}\right)\right]\pi_{t-1}. \tag{49}$$

The reduced form (49) is a highly non-linear third order difference equation. The state space form, which we turn to next, simplifies the expression somewhat and allows analytical stability conditions to be derived.

4.3 State space form and stability

As before, define the auxiliary variable $z_t = \pi_{t-1}$, and define a second auxiliary variable $zz_t = z_{t-1} = \pi_{t-2}$. Then the state space form of the model in (49) is given by,

$$\begin{bmatrix} \mathbb{E}_t\pi_{t+1} \\ z_{t+1} \\ zz_{t+1} \end{bmatrix} = \begin{bmatrix} A + B\left(1 + e^{-\mu((\pi_t - zz_t)^2 - \Upsilon)}\right) & -\left[C + D\left(1 + e^{-\mu((\pi_t - zz_t)^2 - \Upsilon)}\right)\right] & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_t \\ z_t \\ zz_t \end{bmatrix},$$

where π_t is a jump variable and z_t and zz_t are pre-determined variables. In the steady state, $\pi_t = z_t = zz_t = 0$, and $n_t = (1 + e^{\mu\Upsilon})^{-1}$. Therefore, the Jacobian matrix J evaluated at the steady state is as follows:

$$J|_{\pi_t = z_t = zz_t = 0} = \begin{bmatrix} A + B\left(1 + e^{\mu\Upsilon}\right) & -\left[C + D\left(1 + e^{\mu\Upsilon}\right)\right] & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{50}$$

For local determinacy and stability we require two eigenvalues of the Jacobian matrix (50) inside the unit circle, and one eigenvalue outside. Local indeterminacy occurs when all eigenvalues of the Jacobian matrix (50) are inside the unit circle. If a pair of eigenvalues are complex conjugates, as they pass through the unit circle a Hopf (or Neimark-Sacker) bifurcation occurs (see e.g. Hommes (2013), chapter 3). Proposition 2 considers the case of local determinacy and stability, and proposition 3 considers the case of local indeterminacy and Hopf bifurcation.

Proposition 2: If the monetary policy rule is such that the condition in (11) holds, then the model in (49) is locally determinate and stable.

Proof of proposition 2: At the steady state, $n_t = (1 + e^{\mu\Upsilon})^{-1}$. As $\mu \in [0, \infty)$ and $\Upsilon \in (-\infty, \infty)$, $n_t \in (0, 1)$ at the steady state. The proof then follows directly from lemma 1: as the model with fixed n is stable and determinate when the condition in (11) holds, the model with variable n is locally stable and determinate when the condition in (11) holds. ■

Proposition 3: Local indeterminacy and stability in the model described by (49) requires all eigenvalues inside the unit circle. In this case, an increase in Υ can lead to a loss of local stability via a Hopf bifurcation.

Proof of proposition 3: Consider the mapping $x_{t+1} = F(x_t, \varphi)$, $x_t \in \mathbb{R}^n$, and $\varphi \in \mathbb{R}$ is a parameter. Following Iooss et al. (1981) and Gabisch and Lorenz (1987), we have the following theorem:

> **Hopf:** Let the mapping $x_{t+1} = F(x_t, \varphi)$, $x_t \in \mathbb{R}^n$, $\varphi \in \mathbb{R}$, have a fixed point at the origin. If there is a φ_0 such that the Jacobian matrix evaluated at the origin has a pair of complex conjugate eigenvalues $\lambda_{1,2}$ which lie on the unit circle, while the remainder of its spectrum lies at a non-zero distance from the unit circle, and the Hopf transversality condition holds, i.e.
>
> $$\frac{d(\text{mod}\lambda(\varphi))}{d\varphi} > 0,$$

then if $\lambda^n(\varphi_0) \neq \pm 1$ for $n = 1, 2, 3, 4$, there is an invariant closed curve bifurcating from $\varphi = \varphi_0$. So, as a parameter φ is varied, a stable fixed point loses stability as a pair of complex conjugate eigenvalues crosses the unit circle[7].

Denote the trace of the Jacobian in (50) by $\tau = A + B(1 + e^{\mu\Upsilon})$. By inspection, the matrix is non-invertible, so the determinant $\Delta = 0$, and at least one eigenvalue is equal to zero. In fact, the eigenvalues of (50) are given by,

$$\lambda_{1,2} = \tau/2 \pm \sqrt{\tau^2/4 - \Delta_0},\ \lambda_3 = 0,$$

where $\Delta_0 = C + D(1 + e^{\mu\Upsilon})$ is the pseudo-determinant of (50), i.e. the product of the non-zero eigenvalues. When $\Delta_0 > \tau^2/4$ so the non-zero eigenvalues are complex conjugate, let $\lambda_{1,2} = \beta_1 \pm \beta_2 i$, where $\beta_1 = \tau/2$ and $\beta_2 = \sqrt{\Delta_0 - \tau^2/4}$. The modulus of the complex conjugate eigenvalues is then:

$$\mathrm{mod}(\lambda_{1,2}) = \sqrt{\beta_1^2 + \beta_2^2},$$

from which it follows that $\mathrm{mod}(\lambda_{1,2}) = \sqrt{\Delta_0}$. As the remaining eigenvalue $\lambda_3 = 0$, we require Δ_0 to equal unity for a Hopf bifurcation to occur.

Now, as $\Delta_0 = C + D(1 + e^{\mu\Upsilon})$, $\mathrm{mod}(\lambda_{1,2}) = 1$ when,

$$C + D(1 + e^{\mu\Upsilon}) = 1. \tag{51}$$

Taking advantage of the parameter definitions in table 1 and re-arranging, this condition reduces to,

$$\mu\Upsilon = \ln\left[\frac{\beta\theta_y - \kappa}{\delta\beta\theta_y + \kappa\psi}\right]. \tag{52}$$

As the right hand side of (52) is finite, as $\Upsilon \to \infty$, Δ_0 will pass through unity from below if it starts from a parameterisation in which $\Delta_0 < 1$. Precisely, we have,

$$\frac{d(\mathrm{mod}\lambda_{2,3}(\Upsilon))}{d\Upsilon} = \frac{d\sqrt{\Delta_0}}{d\Upsilon} > 0.$$

Therefore, if the non-zero eigenvalues are complex conjugate as Δ_0 passes through unity, the model undergoes a Hopf bifurcation. This is illustrated graphically in figure 3, which presents the same stability plot as in figure 1, as the model in (49) linearised is effectively a second order difference equation, but with the region of complex conjugate eigenvalues highlighted. ■

[7]This wording largely follows Iooss et al. (1981), although it has been altered slightly to fit with the notation of the present paper. Gabisch and Lorenz (1987: 161) considers the case of $x_{t+1} = F(x_t, \varphi)$, $x_t \in \mathbb{R}^2$.

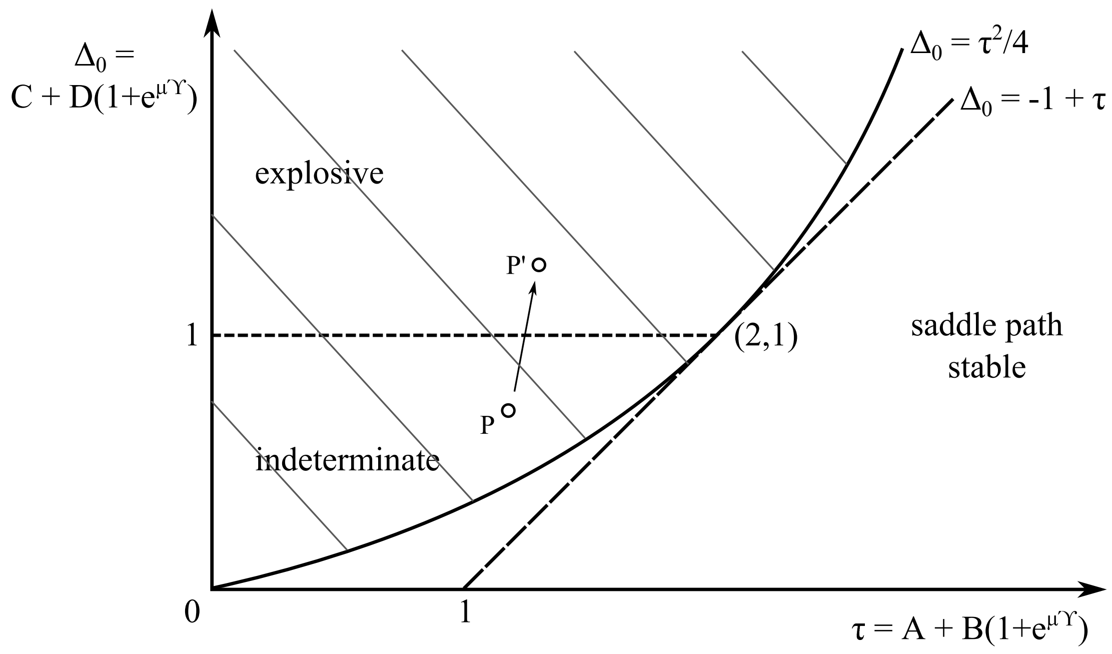

Figure 3: Graphical illustration of proposition 4, showing a stability plot in the trace τ and pseudo-determinant Δ_0 for the model in (49). Note this looks exactly the same as the stability plot in figure 1, as the linearised model is effectively a second order difference equation in π_t and z_t, but we have now shaded the region of complex conjugate eigenvalues with grey lines. As the model moves from points P to P′, as Υ is increased, a Hopf bifurcation takes place.

4.4 Rational route to randomness

Proposition 3 demonstrates that an increase in the fixed cost of being fully rational can lead to the loss of local stability via a Hopf bifurcation if the model starts from a position of local indeterminacy. The existence of limit cycles therefore depends on the monetary policy rule and Υ. Figure 4 presents a plot of a single simulated trajectory of the model in (49), numerically demonstrating the existence of a stable limit cycle in the inflation rate. The underlying parameterisation is the same parameterisation used in the rest of the paper, and is a fairly standard prior for the basic New Keynesian model.

The existence of a Hopf bifurcation and stable limit cycles indicate the possibility of a rational route to randomness. Following Brock and Hommes (1997), this is a bifurcation route to instability, cycles, and chaos as the intensity of choice parameter μ increases. Mathematically, this route to chaos is associated with the emergence of a homoclinic loop, as the equilibrium becomes a saddle-focus with one stable and two unstable eigenvalues after the Hopf bifurcation, associated with a one dimensional stable manifold and a two dimensional unstable manifold, respectively. In fact, proposition 5.5.2 in Hommes (2013) would lead us to expect the existence of a homoclinic loop in the model considered here.

Retaining the same underlying parameterisation, and setting $\Upsilon = 0.1$, figures 5 and 6 plot several trajectories as μ increases. As is evident from the plots, the stable limit cycle quickly

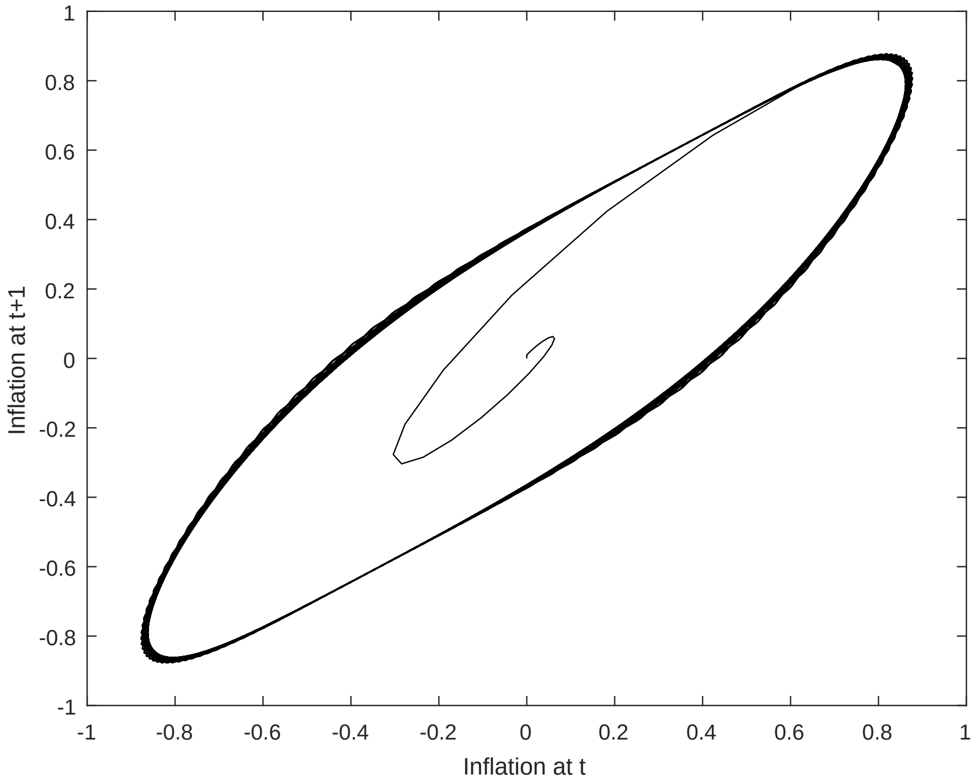

Figure 4: Phase plot of inflation with n variable, illustrating a stable limit cycle. The parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.5$, $\theta_y = 4$, $\mu = 1$, $\Upsilon = 0$.

loses its smoothness as μ increases, and then varies between periodic attractors and strange attractors. This evolution is not dissimilar to the evolution in the Henon-like map discussed in Gonchenko et al. (2014), in which simple Shilnikov scenarios in three dimensional maps are discussed in some detail. Finally, figure 7 plots a bifurcation diagram as μ is increased, and the simulated largest Lyapunov exponents for the model over the same range of μ. Both panels in figure 7 are plotted using the software *E&F Chaos* - see Diks et al. (2008).

The bifurcation diagram is constructed by simulating the model for T periods, k times for k different values of μ equally spaced between 1 and 3. For each of the k values of μ, this yields T different simulated values of inflation which are plotted on the vertical axis (although a long burn-in period for each simulation ensures that the simulated values of inflation constitute the fixed point(s) for the system). The Lyapunov exponents are simulated, and measure the average rate of separation of a trajectory before and after a small perturbation. As a positive Lyapunov exponent is an important indicator of chaos, we can state with some confidence that the model in (49) displays a rational route to randomness.

Unsurprisingly, as proposition 5.5.2 in Hommes (2013) leads us to expect the existence of a homoclinic loop, there exist parameterisations in which near-homoclinic trajectories are particularly apparent in numerical simulation. Figure 8 presents an example of this, and

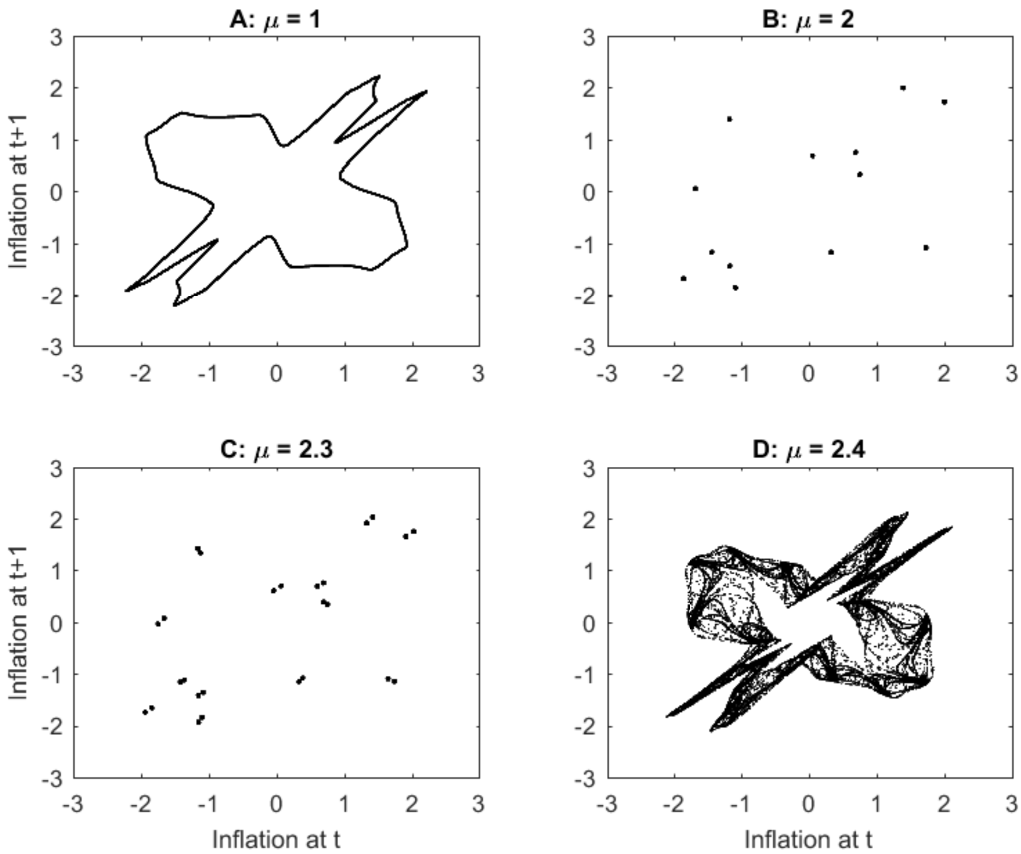

Figure 5: Simulated trajectories for various values of μ, illustrating the *rational route to randomness.* The remaining parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$, $\Upsilon = 0.1$.

plots the phase diagram in two dimensions and three dimensions. The plotted trajectory starts very close to the steady state, and spirals away from it across the unstable manifold. Throughout this process the proportion of AU agents fluctuates with the fluctuations in inflation. As the trajectory gets further from the steady state, it becomes increasingly difficult to forecast, leading to agents shifting away from the AU predictor towards the rational expectations predictor for longer periods of time. At this point the model stabilises, and re-approaches the steady state down the stable manifold. The corresponding time series of inflation and n, the proportion of rational firms, are plotted in figure 9, which illustrates this dynamic from a different perspective. This dynamic is common to models of this form, in which agents shift between destabilising bounded rational predictors and stabilising fully rational predictors, following Brock and Hommes (1997).

4.5 Robustness to the timing assumptions

In the models above, we assume that the proportion of rational and bounded rational agents changes based on inflation information within the period, i.e.,

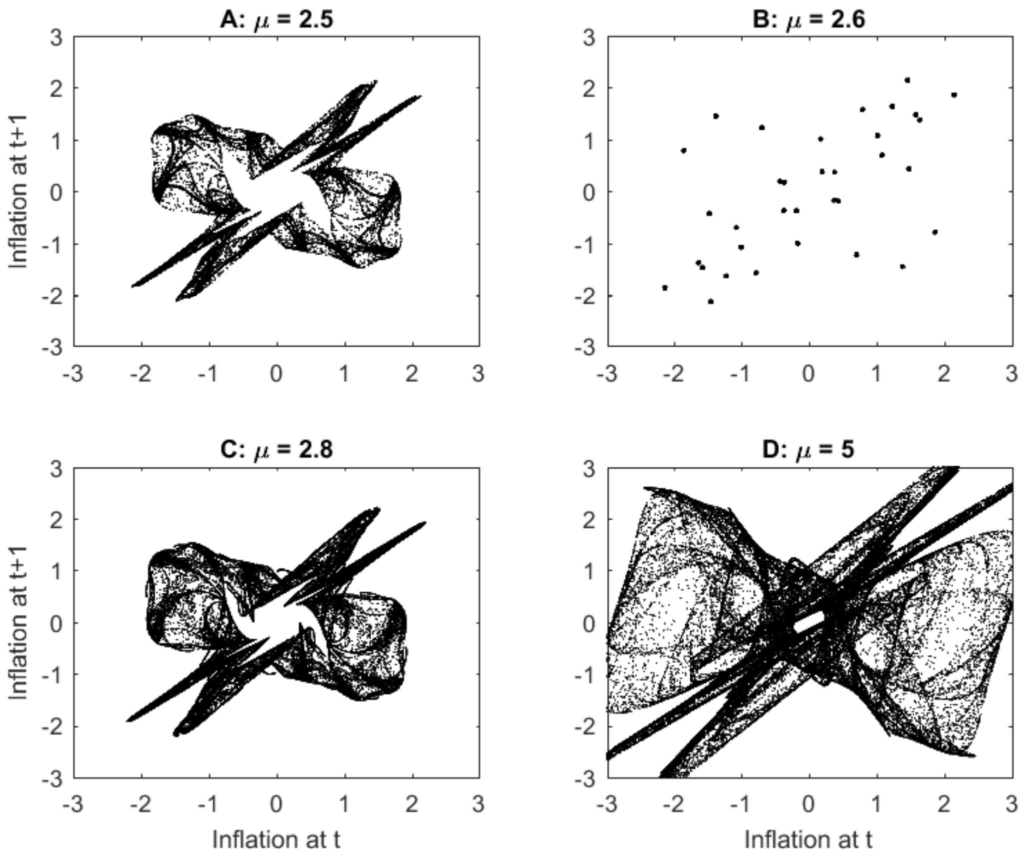

Figure 6: Simulated trajectories for various values of μ, illustrating the *rational route to randomness*. The remaining parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$, $\Upsilon = 0.1$.

$$n_t^{-1} = 1 + \exp[-\mu((\pi_t - \pi_{t-2})^2 - \Upsilon)], \tag{53}$$

as in (47). It might be argued, however, that this is an awkward assumption given that only RE agents observe inflation within the period when they form their expectations. Fortunately this timing assumption is not crucial, as the strategy switching dynamic only enters the model in a nonlinear fashion. To see this, consider the case in which strategy switching only uses information available at time $t-1$. In this case, a reasonable alternative measure of the mean squared error of the bounded rational predictor is,

$$\Phi_t = (\mathbb{E}_t^*[\pi_t] - \mathbb{E}_{t-1}^*[\pi_t])^2 = (\pi_{t-1} - \pi_{t-2})^2, \tag{54}$$

in which case the reduced form model becomes,

$$\mathbb{E}_t\pi_{t+1} = \left[A + B\left(1 + e^{-\mu((\pi_{t-1}-\pi_{t-2})^2-\Upsilon)}\right)\right]\pi_t - \left[C + D\left(1 + e^{-\mu((\pi_{t-1}-\pi_{t-2})^2-\Upsilon)}\right)\right]\pi_{t-1}.$$

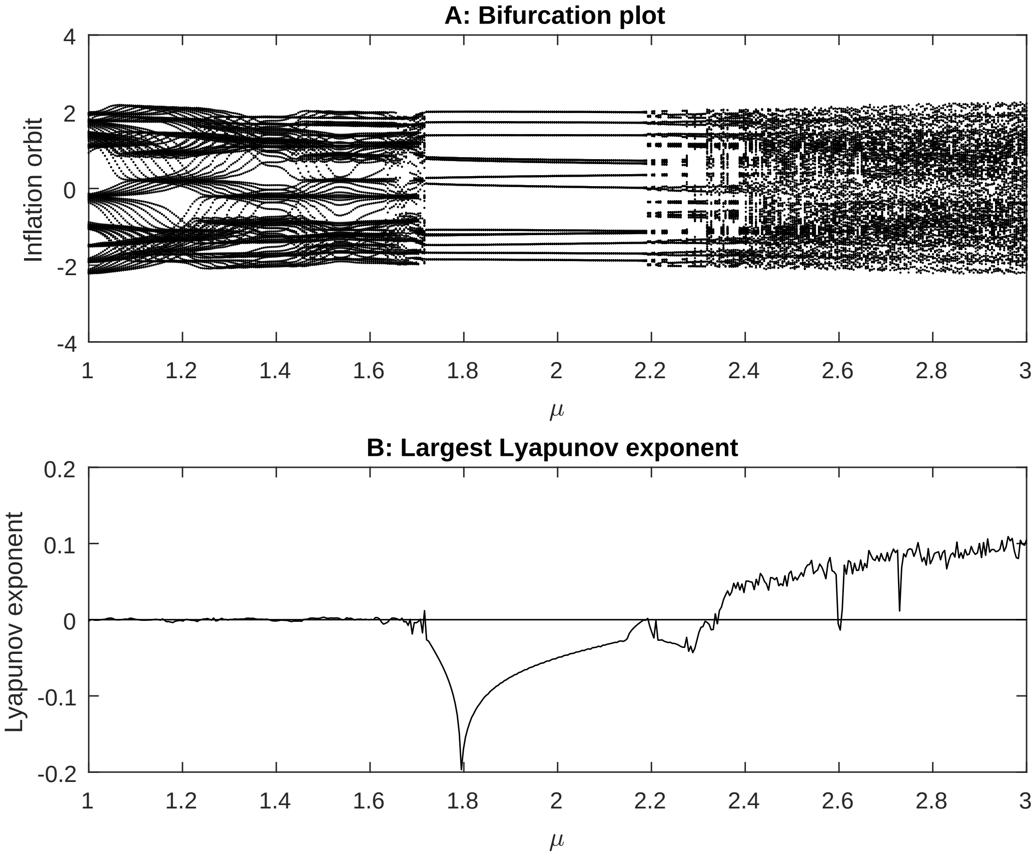

Figure 7: Panel A: Bifurcation plot of the orbit of inflation against μ. Panel B: Largest Lyapunov exponent against μ. The parameter values are $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$, $\Upsilon = 0.1$.

Clearly, the Jacobian matrix for this model is exactly the same as the Jacobian for the model in the main body of the paper, i.e.,

$$J|_{\pi_t = z_t = zz_t = 0} = \begin{bmatrix} A + B\left(1 + e^{\mu\Upsilon}\right) & -\left[C + D\left(1 + e^{\mu\Upsilon}\right)\right] & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

as altering the timing assumptions has no effect on the linear part of the model[8]. As such, propositions 2 and 3 are unaffected.

[8]Note that this would also be the case if we assumed a mean squared error of the form $\Phi_t = (\pi_{t-1} - \mathbb{E}^*_{t-2}[\pi_{t-1}])^2 = (\pi_{t-1} - \pi_{t-3})^2$, for example.

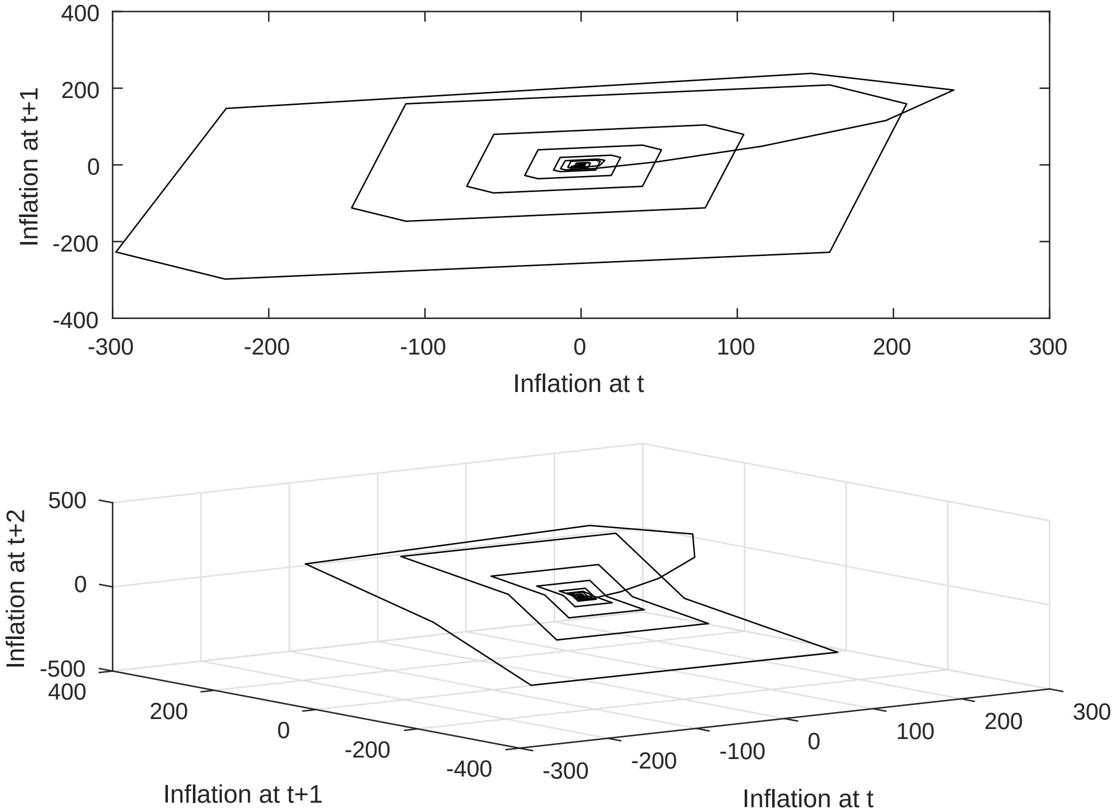

Figure 8: Trajectories in two and three dimensions, respectively, of the first 103 iterations of the model in which $\mu = 1$ and $\Upsilon = 0$. The remaining parameterisation is $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$.

4.6 Local instability and global indeterminacy

The limit cycles and chaotic dynamics explored above exist in the locally explosive part of the parameter space. As the global dynamics are bounded, however, inflation does not diverge to $\pm\infty$ and as such the transversality condition is satisfied. Interestingly, as there are an infinite number of trajectories that converge on a stable limit cycle (as in figure 4), and an infinite number of chaotic trajectories (as in figures 5 and 6), these dynamics are examples of a type of global indeterminacy also analysed by Benhabib et al. (2001, 2002), Airaudo and Zanna (2012), and others. In this situation one can imagine a one-off sunspot pinning down the state vector at $t = 0$, which then evolves along a perfect foresight equilibrium trajectory to a unique limit cycle, or continues to evolve chaotically without repeating itself. This is, in fact, exactly how the simulations in this section are computed.

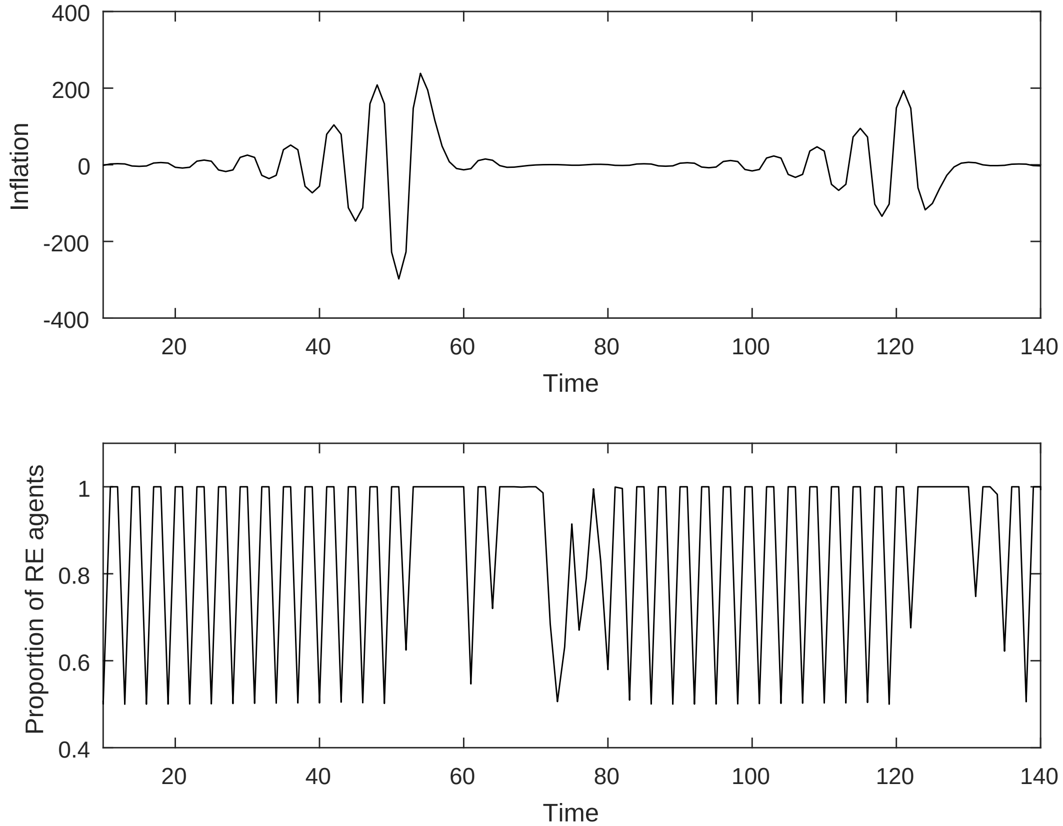

Figure 9: Trajectories, respectively, of the first iterations 10 to 140 of the model in which $\mu = 1$ and $\Upsilon = 0$. The remaining parameterisation is $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_\pi = 0.3$, $\theta_y = 1$.

5 Monetary policy rules with persistence

In sections 3 and 4 we demonstrate three propositions. First, in the model with fixed proportions of fully rational and AU agents, we demonstrate that the condition in (11) is sufficient for determinacy and stability. Second, in the model with variable proportions of fully rational and AU agents, we demonstrate that the condition in (11) is sufficient for local determinacy and stability. Third, in the model with variable proportions of fully rational and AU agents, we demonstrate that an increase in the cost of being fully rational can lead to a Hopf bifurcation if the model starts out from a position of indeterminacy.

These results rely on a lack of persistence in the policy rule. In this final section, we relax this assumption to check the robustness of the results in sections 3 and 4. Specifically, we generalise the monetary policy rule to the standard rule with persistence,

$$r_{n,t} = \rho_r r_{n,t-1} + (1 - \rho_r)(\theta_\pi \pi_t + \theta_y y_t), \tag{55}$$

where $\rho_r \in (0, 1]$.

For the case of pure rational expectations, with n fixed and equal to 1, the policy space for the rule in (55) is given by,

$$\theta_\pi + \frac{1-\beta}{\kappa}\theta_y > 1 - \rho_r, \tag{56}$$

which is a result obtained in Woodford (2003a), appendix C.

For the case of pure AU, with n fixed and equal to 0, using the monetary policy rule (76) leads to a second order generalisation of the model in lemma 2,

$$\pi_t = \left[\frac{\psi\kappa + (1-\rho_r)\delta\beta\theta_y}{(1-\rho_r)(\theta_y + \psi\kappa\theta_\pi)}\right]\pi_{t-1} - \left[\frac{\rho_r}{(1-\rho_r)(\theta_y + \psi\kappa\theta_\pi)}\right]\pi_{t-2}. \tag{57}$$

Using $z_t = \pi_{t-1}$ as before, we can re-write the model in (57) as,

$$\begin{bmatrix} \pi_t \\ z_t \end{bmatrix} = \begin{bmatrix} \frac{\psi\kappa + (1-\rho_r)\delta\beta\theta_y}{(1-\rho_r)(\theta_y + \psi\kappa\theta_\pi)} & -\frac{\rho_r}{(1-\rho_r)(\theta_y + \psi\kappa\theta_\pi)} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ z_{t-1} \end{bmatrix}. \tag{58}$$

Denoting the trace of the model in (58) by τ and the determinant by Δ, necessary and sufficient conditions for stability are,

1. $\Delta < 1$,
2. $1 - \tau + \Delta > 0$,
3. $1 + \tau + \Delta > 0$.

As τ and Δ are both positive the third condition is not binding, and for $\rho_r < 1$ condition 3 yields the familiar condition $\theta_\pi + \frac{1-\beta}{\kappa}\theta_y > 1$. But condition 1 adds a further restriction on persistence in the monetary policy rule, given by,

$$\rho_r < \frac{\theta_\pi\psi\kappa}{\theta_y + \psi\kappa(1+\theta_\pi)}. \tag{59}$$

Thus we have our fourth result:

Proposition 4: With persistence in the interest rate, the policy space (θ_π, θ_y) under rational expectations is increased to $\theta_\pi + \frac{1-\beta}{\kappa}\theta_y > 1 - \rho_r$. Under AU the policy space remains as $\theta_\pi + \frac{1-\beta}{\kappa}\theta_y > 1$ and persistence is constrained by (59).

By considering the limiting case of $\theta_y = 0$, one can see that (59) restricts the stability region of the model with $n = 0$ quite substantially. This is further illustrated by considering the limiting case of $\rho_r = 1$. By re-parameterising the rule as,

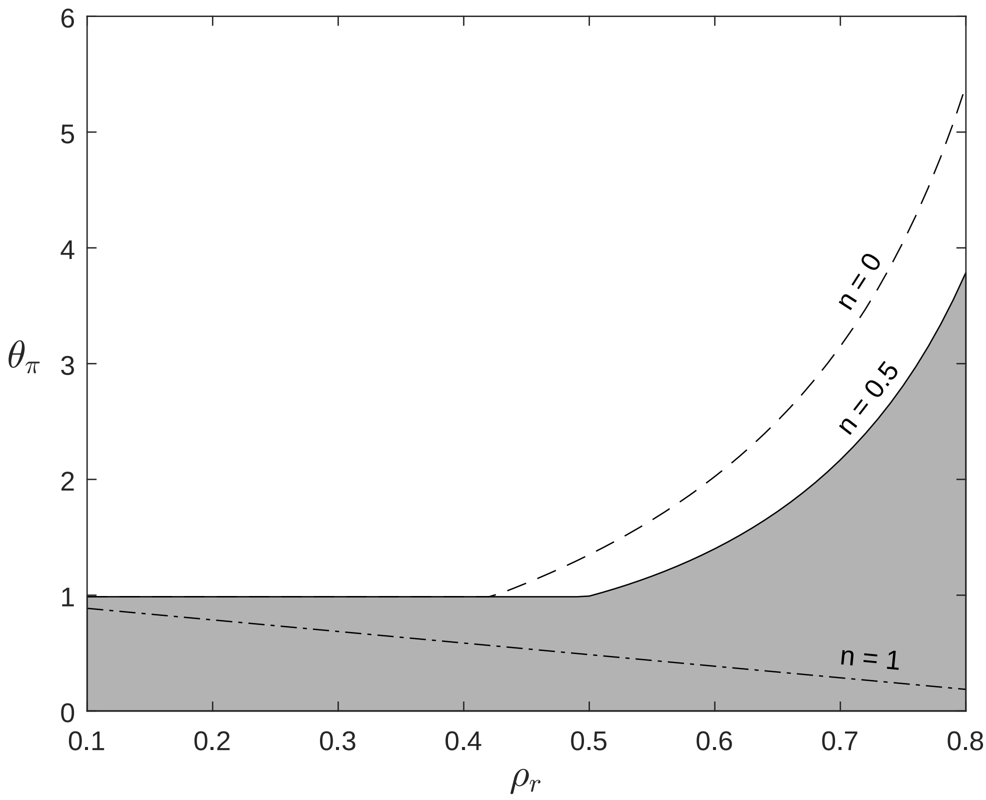

Figure 10: Stability for model with persistence in the monetary policy rule, as a function of θ_π and ρ_r, with $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $\theta_y = 0.5$, for $n = 0$ (dashed line), $n = 0.5$ (solid line), and $n = 1$ (dot-dashed line). (ρ_r, θ_π) combinations above the constraints imply saddle path stability; the white area is saddle path stable for $n = 0.5$.

$$r_{n,t} = r_{n,t-1} + \alpha_\pi \pi_t + \alpha_y y_t, \tag{60}$$

then the case $\alpha_y = 0$ gives $\Delta r_{n,t} = \theta_\pi \Delta p_t$, where $\pi_t = p_t - p_{t-1}$ and p_t is the price level. Thus $r_{n,t} = \theta_\pi p_t$, and (60) is a price level rule. Putting $\alpha_\pi = (1-\rho_r)\theta_\pi$ and $\alpha_y = (1-\rho_r)\theta_y$ into the previous result and letting $\rho_r \to 1$, the policy space (α_π, α_y) under rational expectations is $\alpha_\pi + \frac{1-\beta}{\kappa}\alpha_y > 0$ and the policy space under AU is $\alpha_\pi + \frac{(1-\beta\xi)}{\kappa}\alpha_y > 1$. Hence under rational expectations and $\rho_r = 1$, at least one slightly positive feedback from inflation and output is necessary and sufficient to result in saddle-path stability. Under AU and $\rho_r = 1$, the policy space is considerably reduced for plausible values of the Calvo contract parameter ξ. Thus proposition 4 qualifies propositions 1 and 2, and implies that the stability properties of the New Keynesian model with AU are sensitive to changes in the monetary policy rule. Again, this reinforces the existing results discussed in the introduction.

Finally, we consider the case with interest rate smoothing in which $n \in (0, 1)$. By substituting (30) into the Taylor rule with persistence in (55) and rearranging we get the equivalent equilibrium condition to (31), except now with interest rate smoothing:

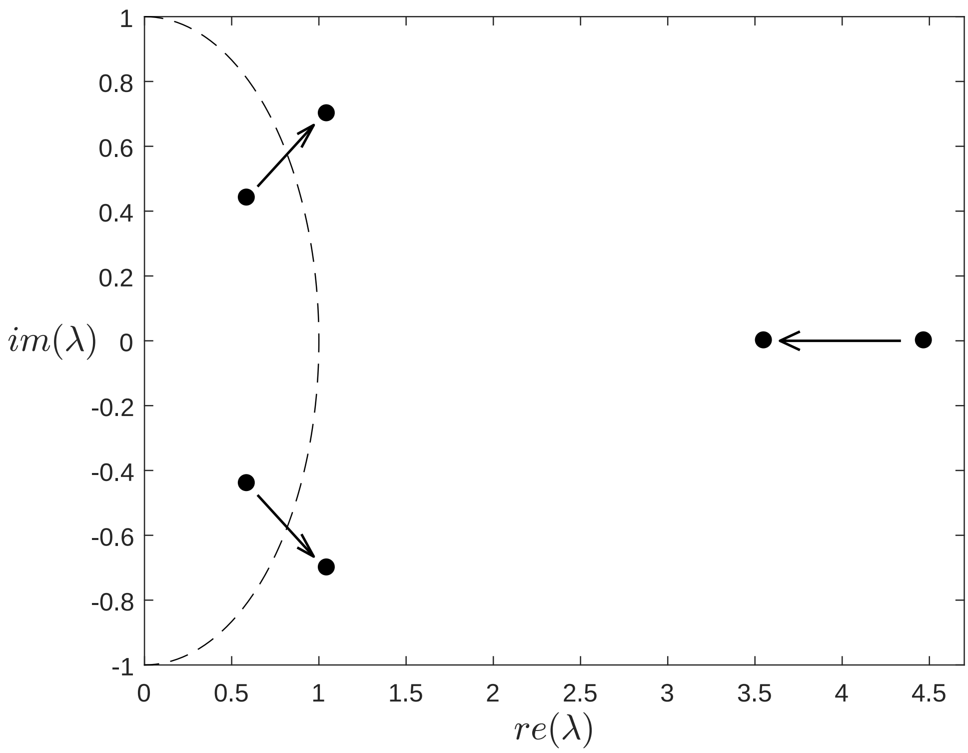

Figure 11: Root-locus plot for model with persistence in the monetary policy rule, with $\theta_\pi = 1.5$, $\theta_y = 0.75$, $\phi = 2$, $\alpha = 0.7$, $\beta = 0.99$, $\xi = 0.75$, $n = 0.5$, for $\rho_r = 0.5, 0.7$. The Hopf bifurcation when ρ_r increases from 0.5 to 0.7 is apparent.

$$y_t = -\left(\frac{\theta_\pi}{\theta_y}\right)\pi_t + \left(\frac{1}{\theta_y(1-\rho_r)}\right)\pi_{t-1} - \left(\frac{\rho_r}{\theta_y(1-\rho_r)}\right)\pi_{t-2}. \tag{61}$$

By substituting (61) into (32) and rearranging, we get the new reduced form model,

$$\mathbb{E}_t\pi_{t+1} = \left(A + \frac{B}{n}\right)\pi_t - \frac{1}{1-\rho_r}\left(C + \frac{D}{n} + \left(\frac{n-1}{n}\right)\delta\rho_r\right)\pi_{t-1} + \frac{\rho_r}{1-\rho_r}\left(E + \frac{F}{n}\right)\pi_{t-2}, \tag{62}$$

where the new composite parameters are,

$$E = (\kappa - \kappa\psi)(\beta\theta_y)^{-1},$$

$$F = \kappa\psi(\beta\theta_y)^{-1}.$$

Note that when $\rho_r = 0$, i.e. no interest rate smoothing, (62) is identical to (33).

The state space form of the model in (62) is given by,

$$\begin{bmatrix} \mathbb{E}_t\pi_{t+1} \\ z_{t+1} \\ zz_{t+1} \end{bmatrix} = \begin{bmatrix} A + \dfrac{B}{n} & -\dfrac{1}{1-\rho_r}\left(C + \dfrac{D}{n} + \left(\dfrac{n-1}{n}\right)\delta\rho_r\right) & \dfrac{\rho_r}{1-\rho_r}\left(E + \dfrac{F}{n}\right) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_t \\ z_t \\ zz_t \end{bmatrix},$$

where $z_t = \pi_{t-1}$ and $zz_t = z_{t-1} = \pi_{t-2}$ as before. Figure 10 plots the stability condition for this model alongside the stability conditions for the special cases $n = 0$ and $n = 1$ discussed above. This stability condition has been computed numerically, and lies in between the stability conditions for $n = 0$ and $n = 1$. The $(\rho_r,\, \theta_\pi)$ combinations shaded in white are saddle path stable for the case $n = 0.5$ in figure 1, and the $(\rho_r,\, \theta_\pi)$ combinations shaded in grey are explosive. Thus, as in the case of pure anticipated utility, the Taylor condition is insufficient for saddle path stability in the case with $n \in (0, 1)$, although the constraint on ρ_r is not as severe as in the case of pure anticipated utility. The analysis in this section suggests that current central banking practice may not ensure saddle path stability if the degree of interest rate smoothing is high. In fact, as figure 11 demonstrates, a Hopf bifurcation occurs in this model as ρ_r increases past some critical value, even when the standard Taylor condition holds. Again, this result implies that the stability properties of the New Keynesian model with AU are sensitive to changes in the monetary policy rule, reinforcing the existing results discussed in the introduction.

6 Concluding Remarks

This paper constructs and explores the monetary policy consequences of the workhorse New Keynesian model with AU learning and heterogeneous agents. First, we derive the model with a fixed proportion n of fully rational agents and a fixed proportion $1 - n$ of anticipated utility agents, in a similar manner to Massaro (2013). We then extend the model to include reinforcement learning along the lines of Branch and McGough (2010). Using this model, we demonstrate four propositions. First, the rational expectations determinacy condition is sufficient for determinacy and stability when n is fixed. Second, the rational expectations determinacy condition is sufficient for local determinacy and stability when n varies according to reinforcement learning. Third, when monetary policy is such that the dynamics are indeterminate, limit cycles can exist, and may be followed by a rational route to randomness. Fourth, the rational expectations determinacy condition not is sufficient for determinacy and stability in the presence of interest rate smoothing.

These results are consistent with the general message of the behavioural New Keynesian literature, i.e. qualified support for the existing monetary policy orthodoxy. Nevertheless, it is worth highlighting that while some papers in the literature find that the Taylor condition is sufficient for determinacy and stability (e.g. Pecora and Spelta, 2017), others find that this is not the case (e.g. Branch and McGough, 2010). There does not appear to be a straightforward answer as to why this is the case, with part of the problem being the inherent complexity of the models - and therefore a lack of analytical results - and part of the problem being the diversity of monetary policy rules used in the literature. For example, while our model uses a Taylor rule that conditions on observable output and inflation, Branch and McGough (2010) use a Taylor rule that conditions on forecasts of output and inflation, and this difference is likely to affect the stability results. The only obvious resolution to this problem is to study the stability properties of an "inventory of monetary policy rules" in the major bounded rational approaches - including *at least* Euler learning, anticipated utility, and internal rationality - along the lines of Lubik and Marzo (2007).

On the other hand, our results concerning interest rate smoothing are dissimilar to the standard results on interest rate smoothing - e.g. Bullard and Mitra (2007) - in which monetary policy inertia improves the stability properties of rational expectations models and promotes the learnability of rational expectations equilibria. Our result, in which the stability region is substantially reduced in the presence of interest rate smoothing, is similar to that of Gasteiger (2014), who finds that monetary policy inertia reduces the space of determinate optimal policy rules in a New Keynesian model with heterogeneous expectations. Again, differing results may partly be due to differences over the type of policy rule used, with Bullard and Mitra (2007) using Taylor rules with lagged observations and forecasts of output and inflation, rather than contemporary observations. But more straightforwardly, the benefits of interest rate smoothing in rational expectations models are usually attributed to the ability of current interest rates to signal future interest rate movements when policy inertia is present. Thus,

> [A]n effective response by the Fed to inflationary pressures, say, requires that the private sector be able to believe that the entire future path of short rates has changed. A policy that maintains interest rates at a higher level for a period of time once they are raised . . . is one that, if understood by the private sector, will allow a moderate adjustment of current short rates to have a significant effect on long rates. Such a policy offers the prospect of significant effects of central-bank policy upon aggregate demand, without requiring excessively volatile short-term interest rates. (Woodford, 2003b, pp.863).

In other words, under rational expectations, the interest rate in period t provides information about the interest rate in period $t + 1$ when policy follows a rule like (55), and can therefore stabilise expectations. As noted by Eusepi and Preston (2018), this stabilising function of interest rate smoothing is lost when expectations are backwards looking, and the backwards looking dynamic introduced by policy inertia may instead interact with the backwards looking dynamic introduced by expectational inertia to generate instability. This simple observation illustrates the complexity with which demand management interacts with private sector behaviour, and the consequent importance of testing policy rules on a variety of private sector behaviours, rather than relying exclusively on rational expectations.

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Appendix

A Deriving the reduced form model

In this appendix we explain how to arrive at the reduced form equation (30), which leads in a straightforward manner to the reduced form equation for output (31), and thus the reduced form model analysed in the main body of the text.

There are two ways to derive (30), the first of which is the most straightforward and is emphasised in the main body of the text. Substituting expectations into (13) we have,

$$\alpha_1 c_t^{AU} = \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_{n,t-1} - \pi_t) + \alpha_4 \omega_{1,t}, \tag{A.1}$$

with,

$$\omega_{1,t} = \frac{1}{1-\beta}\left(\alpha_5 w_t + \alpha_6 \pi_{t-1}\right) - \alpha_6\left(r_{n,t} + \frac{\beta}{1-\beta} r_{n,t}\right),$$

$$\omega_{2,t} = (1-\beta)\gamma_t^{AU} + \beta\gamma_t^{AU} - \left(r_{n,t-1} + \beta r_{n,t} + \frac{\beta^2}{1-\beta} r_{n,t}\right) + \pi_t + \left(\frac{\beta}{1-\beta}\right)\pi_{t-1},$$

hence,

$$\alpha_1 c_t^{AU} = \alpha_2 w_t + \alpha_3\left[\gamma_t^{AU} - \frac{\beta}{1-\beta}(r_{n,t} - \pi_{t-1})\right] + \frac{\alpha_4}{1-\beta}[\alpha_5 w_t - \alpha_6(r_{n,t} - \pi_{t-1})]. \tag{A.2}$$

Collecting terms in (A.2) gives us,

$$\alpha_1 c_t^{AU} = \left(\alpha_2 + \frac{\alpha_4\alpha_5}{1-\beta}\right) w_t + \alpha_3\gamma_t^{AU} - \left(\frac{\alpha_3\beta + \alpha_4\alpha_6}{1-\beta}\right)(r_{n,t} - \pi_{t-1}). \tag{A.3}$$

To proceed, we can either assume that profit is distributed in proportion to economic activity, as in the main body of the text, or we can assume that profit is distributed equally. In the first case we have,

$$\gamma_t^{RE} = \frac{1}{1-\alpha} c_t^{RE} - \frac{\alpha}{1-\alpha}(w_t + h_t^{RE}), \tag{A.4}$$

$$\gamma_t^{AU} = \frac{1}{1-\alpha} c_t^{AU} - \frac{\alpha}{1-\alpha}(w_t + h_t^{AU}), \tag{A.5}$$

with the associated labour supply functions given by,

$$h_t^{AU} = \frac{w_t - c_t^{AU}}{\phi}, \tag{A.6}$$

$$h_t^{RE} = \frac{w_t - c_t^{RE}}{\phi}. \tag{A.7}$$

Substituting (A.6) into (A.5) and rearranging, we have,

$$\gamma_t^{AU} = \left(\frac{\alpha + \phi}{\phi(1-\alpha)}\right) c_t^{AU} - \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\phi}{\phi}\right) w_t. \tag{A.8}$$

Substituting (A.8) into (A.3), and substituting out for the composite parameters α_1 - α_5 using table 1, yields,

$$\left(\frac{\alpha+\phi}{\phi}\right) c_t^{AU} = \alpha\left(\frac{1+\phi}{\phi}\right) w_t + \left(\frac{\alpha+\phi}{\phi}\right) c_t^{AU} - \alpha\left(\frac{1+\phi}{\phi}\right) w_t - \left(\frac{\alpha_3\beta + \alpha_4\alpha_6}{1-\beta}\right)(r_{n,t} - \pi_{t-1}), \tag{A.9}$$

and we are thus left with (30) in the main body of the text. Given (30) and the monetary policy rule, aggregate output y_t is determined by (31), i.e.,

$$y_t = -\left(\frac{\theta_\pi}{\theta_y}\right)\pi_t + \left(\frac{1}{\theta_y}\right)\pi_{t-1}, \tag{A.10}$$

the consumption of rational agents is determined by (12), i.e.,

$$c_t^{RE} = \mathbb{E}_t\left[c_{t+1}^{RE} - r_{t+1}\right], \tag{A.11}$$

and the consumption of AU agents is determined by the aggregation relationship (20), i.e.,

$$c_t^{AU} = \frac{y_t - nc_t^{RE}}{1-n}. \tag{A.12}$$

However, we can ignore (A.11) and (A.12) in the reduced form model, as neither y_t nor c_t^{RE} nor c_t^{AU} are state variables; it is precisely this fact which lets us derive the analytical results in the main body of the paper.

An alternative way to arrive at (30) is to assume that profit is distributed equally across households. Then,

$$\gamma_t^{AU} = \gamma_t^{RE} = \gamma_t = \frac{1}{1-\alpha}c_t - \frac{\alpha}{1-\alpha}(w_t + h_t), \tag{A.13}$$

with household labour supply as before. Aggregating household labour supply yields,

$$h_t = \frac{w_t - c_t}{\phi}, \tag{A.14}$$

and making use of the production function and the fact that $c_t = y_t$ yields $y_t = \alpha h_t$. We then arrive at,

$$w_t = \left(\frac{\alpha + \phi}{\alpha}\right) y_t, \tag{A.15}$$

$$\gamma_t = -\left(\frac{\alpha + \phi}{1 - \alpha}\right) y_t, \tag{A.16}$$

hence (A.3) becomes,

$$\begin{aligned} \alpha_1 c_t^{AU} = \left(\alpha_2 + \frac{\alpha_4 \alpha_5}{1-\beta}\right)\left(\frac{\alpha + \phi}{\alpha}\right) y_t - \alpha_3 \left(\frac{\alpha+\phi}{1-\alpha}\right) y_t \\ - \left(\frac{\alpha_3 \beta + \alpha_4 \alpha_6}{1-\beta}\right)(r_{n,t} - \pi_{t-1}). \end{aligned} \tag{A.17}$$

Rearranging (A.17) and substituting out α_1 - α_6 then yields,

$$\left(\frac{\alpha+\phi}{\phi}\right)(c_t^{AU} - y_t) = -\left(\frac{\beta}{1-\beta}\right)\left(\frac{\alpha+\phi}{\phi}\right)(r_{n,t} - \pi_{t-1}), \tag{A.18}$$

or,

$$c_t^{AU} = y_t - \left(\frac{\beta}{1-\beta}\right)(r_{n,t} - \pi_{t-1}), \tag{A.19}$$

which, interestingly, is a type of "Old Keynesian" (or "textbook Keynesian") consumption function.

Now, from the aggregation relationship and output equilibrium we have (A.12), which gives us,

$$c_t^{AU} = \frac{c_t - n_t c_t^{RE}}{1 - n_t} = \frac{y_t - n_t c_t^{RE}}{1 - n_t} = y_t + \frac{n_t (y_t - c_t^{RE})}{1 - n_t}. \tag{A.20}$$

Substituting this term into (A.19) and rearranging, we arrive at,

$$r_{n,t} - \pi_{t-1} = \frac{(1-\beta)n_t}{\beta(1-n_t)}(c_t^{RE} - y_t). \tag{A.21}$$

This should be compared to the expression in (30), and holds when profit income is equally split across households rather than being split in proportion to activity as in the main body of the paper. However, we note that $\beta \approx 1$, and therefore,

$$\frac{(1-\beta)n}{\beta(1-n)} \approx 0 \text{ for } n \ll 1. \tag{A.22}$$

In fact, for $\beta = 0.99$ as in the numerical examples in the text, $\frac{(1-\beta)n}{\beta(1-n)} < 0.1$ for $n < 0.9$, then rises rapidly as n is increased past 0.9. We therefore expect the dynamics of the model with (30) to be a reasonable approximation to the dynamics of the model with (A.21), particularly in the case with n fixed (and less than 0.9), or close to the steady state in the case with n variable.

B Deriving the Linearized Consumption Function

Solving (1) for a symmetric equilibrium forward in time and using the law of iterated expectation we have for $i \geq 1$

$$\frac{1}{C_t} = \beta^i \mathbb{E}_t \left[\frac{R_{t+1,t+i}}{C_{t+i}} \right] \; ; \; i \geq 1 \tag{B.1}$$

We now express the solution to the household optimization problem for C_t and H_t that are functions of *point expectations* $\{\mathbb{E}_t W_{t+i}\}_{i=1}^{\infty}$, $\{\mathbb{E}_t R_{t+1,t+i}\}_{i=1}^{\infty}$ and $\{\mathbb{E}_t \Gamma_{t+i}\}_{i=0}^{\infty}$ treated as exogenous processes given at time t. With point expectations we use (B.1) to obtain the following optimal decision for C_{t+i} given point expectations $\mathbb{E}_t R_{t+1,t+i}$

$$\mathbb{E}_t C_{t+i} = C_t \beta^i \mathbb{E}_t R_{t+1,t+i} \; ; \; i \geq 1 \tag{B.2}$$

$$\mathbb{E}_t(W_{t+i} H_{t+i}) = \frac{(\mathbb{E}_t W_{t+i})^{1+\frac{1}{\phi}}}{C_{t+i}^{\frac{1}{\phi}}} \tag{B.3}$$

Substituting (B.2) and (B.3) into the forward-looking household budget constraint, using $\sum_{i=0}^{\infty} \beta^i = \frac{1}{1-\beta}$ and $\mathbb{E}_t R_{t,t+i} = R_t \mathbb{E}_t R_{t+1,t+i}$ for $i \geq 1$, we arrive at

$$\frac{C_t}{(1-\beta)} = \frac{1}{C_t^{\frac{1}{\phi}}} \left(W_t^{1+\frac{1}{\phi}} + \sum_{i=1}^{\infty} (\beta^{\frac{1}{\phi}})^{-i} \left(\frac{\mathbb{E}_t W_{t+i}}{\mathbb{E}_t R_{t+1,t+i}} \right)^{1+\frac{1}{\phi}} \right) + \Gamma_t + \sum_{i=1}^{\infty} \frac{\mathbb{E}_t \Gamma_{t+i})}{\mathbb{E}_t R_{t+1,t+i}}$$

which can be written in recursive form as

$$\begin{aligned}\frac{C_t}{(1-\beta)} &= \frac{1}{C_t^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}}+\Omega_{2,t}\right)+\Gamma_t+\Omega_{1,t} \qquad \text{(B.4)}\\ \Omega_{1,t} &\equiv \sum_{i=1}^{\infty}\frac{\mathbb{E}_t\Gamma_{t+i}}{\mathbb{E}_tR_{t+1,t+i}}=\frac{\mathbb{E}_t\Gamma_{t+1}}{\mathbb{E}_tR_{t+1,t+1}}+\frac{\Omega_{1,t+1}}{\mathbb{E}_tR_{t+1}}\\ \Omega_{2,t} &\equiv \sum_{i=1}^{\infty}(\beta^{\frac{1}{\phi}})^{-i}\left(\frac{\mathbb{E}_tW_{t+i}}{\mathbb{E}_tR_{t+1,t+i}}\right)^{1+\frac{1}{\phi}}=(\beta^{\frac{1}{\phi}})^{-1}\left(\frac{\mathbb{E}_tW_{t+1}}{\mathbb{E}_tR_{t+1,t+1}}\right)^{1+\frac{1}{\phi}}+\frac{\Omega_{2,t+1}}{\beta^{\frac{1}{\phi}}\mathbb{E}_tR_{t+1}}\end{aligned}$$

Consumption is then given by (B.4) assuming point expectations or by the symmetric form of the Euler equation (1) under full rationality (i.e. households know symmetric nature of equilibrium with $C_t(j) = C_t$). C_t is a function of *rational point expectations* $\{\mathbb{E}_tW_{t+i}\}_{i=1}^{\infty}$, $\{\mathbb{E}_tR_{t,t+i}\}_{i=i}^{\infty}$ and $\{\mathbb{E}_t\Gamma_{t+i}\}_{i=1}^{\infty}$ which can be treated as exogenous processes given at time t or as rational model-consistent expectations.

The linearized consumption function (2) where $x_t = \log X_t/X$ is the log of each variable $X_t = C_t$, W_t, R_t, $\Omega_{1,t}$, $\Omega_{2,t}$, Γ_t, H_t relative to its deterministic steady state X, derived in a straightforward manner as the first-order Taylor series expansion about that steady state. Since $E_tf(X_t) \approx f(E_t(X_t))$; $E_tf(X_tY_t)) \approx f(E_t(X_t)E_t(Y_t))$ up to a first-order Taylor-series expansion, assuming point expectations is equivalent to using this linear approximation.

C Deriving the Recursive Form of Optimal Price Setting

In the first order conditions for Calvo contracts and expressions for value functions we are confronted with expected discounted sums of the general form

$$\Omega_t = \mathbb{E}_t\left[\sum_{k=0}^{\infty}\beta^kX_{t,t+k}Y_{t+k}\right] \qquad \text{(C.1)}$$

where $X_{t,t+k}$ has the property $X_{t,t+k} = X_{t,t+1}X_{t+1,t+k}$ and $X_{t,t} = 1$ (for example an inflation, interest or discount rate over the interval $[t, t+k]$).

Lemma

Ω_t can be expressed as

$$\Omega_t = Y_t + \beta\mathbb{E}_t\left[X_{t,t+1}\Omega_{t+1}\right] \qquad \text{(C.2)}$$

Proof

$$\begin{aligned}\Omega_t &= X_{t,t}Y_t+\mathbb{E}_t\left[\sum_{k=1}^{\infty}\beta^kX_{t,t+k}Y_{t+k}\right]\\ &= Y_t+\mathbb{E}_t\left[\sum_{k'=0}^{\infty}\beta^{k'+1}X_{t,t+k'+1}Y_{t+k'+1}\right]\\ &= Y_t+\beta\mathbb{E}_t\left[\sum_{k'=0}^{\infty}\beta^{k'}X_{t,t+1}X_{t+1,t+k'+1}Y_{t+k'+1}\right]\\ &= Y_t+\beta\mathbb{E}_t\left[X_{t,t+1}\Omega_{t+1}\right] \qquad \square\end{aligned}$$

Then summations $\Omega_{3,t}$ and $\Omega_{4,t}$ are of the form considered in the Lemma above. Applying the Lemma gives (5)-(5) in the main text.