Upper Domination and Upper Irredundance Perfect Graphs

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Abstract

Let $\beta(G)$, $\Gamma(G)$ and $IR(G)$ be the independence number, the upper domination number and the upper irredundance number, respectively. A graph G is called Γ-*perfect* if $\beta(H) = \Gamma(H)$, for every induced subgraph H of G. A graph G is called IR-*perfect* if $\Gamma(H) = IR(H)$, for every induced subgraph H of G. In this paper, we present a characterization of Γ-perfect graphs in terms of some family of forbidden induced subgraphs, and show that the class of Γ-perfect graphs is a subclass of IR-perfect graphs and that the class of absorbantly perfect graphs is a subclass of Γ-perfect graphs. These results imply a number of known theorems on Γ-perfect graphs and IR-perfect graphs. Moreover, we prove a sufficient condition for a graph to be Γ-perfect and IR-perfect which improves a known analogous result.

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1 Introduction

All graphs will be finite and undirected, without loops and multiple edges. If G is a graph, $V(G)$ denotes the set, and $|G|$ the number, of vertices in G. Let $N(x)$ denote the neighborhood of a vertex x, and let $\langle X\rangle$ denote the subgraph of G induced by $X \subseteq V(G)$. Also let $N(X) = \cup_{x\in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta(G)$ the minimal degree of vertices in G.

A set X is called a *dominating set* if $N[X] = V(G)$. The *independence number* $\beta(G)$ is the maximum cardinality of an independent set, and the *upper domination number* $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of G. A minimal dominating set of order $\Gamma(G)$ is called a Γ-*set.* A set X is *irredundant* if for every vertex $x \in X$,

$$I(x, X) = N[x] - N[X - \{x\}] \neq \emptyset.$$

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The maximum cardinality of an irredundant set is the *upper irredundance number* $IR(G)$.

It is well known [2] that for any graph G,

$$\beta(G) \leq \Gamma(G) \leq IR(G).$$

A graph G is called *upper domination perfect (*Γ*-perfect)* if $\beta(H) = \Gamma(H)$, for every induced subgraph H of G; G is *minimal* Γ*-imperfect* if G is not Γ-perfect and $\beta(H) = \Gamma(H)$, for every proper induced subgraph H of G. A graph G is called *upper irredundance perfect (*IR*-perfect)* if $\Gamma(H) = IR(H)$, for every induced subgraph H of G. The classes of Γ-perfect graphs and IR-perfect graphs in a sense are dual to the well known classes of domination perfect graphs (for a short survey, see [10]) and irredundance perfect graphs [5], respectively.

In this paper, we present a characterization of Γ-perfect graphs in terms of some family of forbidden induced subgraphs, and show that the class of Γ-perfect graphs is a subclass of IR-perfect graphs. We also show that the class of absorbantly perfect graphs introduced by Hammer and Maffray [4] is a subclass of Γ-perfect graphs. These results imply a number of known theorems on the above classes of graphs, for example, the theorem of Cheston and Fricke [1] and Jacobson and Peters [6] that any strongly perfect graph is Γ-perfect and IR-perfect and the theorem of Golumbic and Laskar [3] that any circular arc graph is Γ-perfect and IR-perfect. Moreover, we prove a sufficient condition for a graph to be Γ-perfect and IR-perfect which essentially improves a sufficient condition for a graph to be IR-perfect of Cockayne, Favaron, Payan and Thomason [2].

2 Main Results

We say that the graph G *belongs to the class* $\mathcal{W}$ if G is a connected graph, has $|G| \geq 10$ and $\delta(G) \geq 2$, and its vertex set $V(G)$ has a partition $V(G) = A \cup B$ such that $|A| = |B| = \beta(G) + 1$ and the only edges between A and B are a perfect matching.

Proposition 2.1 *If* $G \in \mathcal{W}$*, then* $\Gamma(G) = \beta(G) + 1$.

Proof: Since A is a minimal dominating set, we have $\Gamma(G) \geq |A|$. Let X be a Γ-set of G. If x is a non-isolated vertex of $\langle X \rangle$, then there exists a vertex $y \notin X$ such that y is not adjacent to any vertex of $X - x$. If x is an isolated vertex of $\langle X \rangle$, then there is a vertex $y \notin X$ such that xy is an edge of the perfect matching of G. Thus for each vertex of X we can indicate a vertex not in X and obviously different vertices of X result in different vertices of $V(G) - X$. Thus, $\Gamma(G) \leq \frac{1}{2}|G| = |A|$. ∎

The class $\mathcal{W}$ contains an infinite subclass consisting of minimal Γ-imperfect graphs. The graph $H(k, l, m)$ is constructed from two disjoint cycles $C = C_{4k+1}$ ($k \geq 1$) and $C' = C_{4l+1}$ ($l \geq 1$) by adding the chain $(v_1, v_2, ..., v_{2m})$ ($m \geq 1$) joining the vertex $v_1 \in V(C)$ and $v_{2m} \in V(C')$. Thus, $|H(k, l, m)| = 4k + 4l + 2m \geq 10$. It is not difficult to see that $H(k, l, m)$ belongs to the class $\mathcal{W}$. By Proposition 2.1, the graph $H(k, l, m)$ is not Γ-perfect. Moreover, it is possible to show that $H(k, l, m)$ is minimal Γ-imperfect graph for any k, l, and m.

The following theorem gives a characterization of Γ-perfect graphs in terms of the forbidden induced graphs in Figure 1 and the graphs from the class $\mathcal{W}$.

Theorem 2.2 *A graph G is Γ-perfect if and only if G does not contain the graphs G_1-G_{15} in Figure 1 and any member of $\mathcal{W}$ as induced subgraphs.*

Proof: The necessity follows from Proposition 2.1 and the fact that $\beta(G_i) < \Gamma(G_i)$, $1 \leq i \leq 15$. The dotted edges in Figure 1 mean the following: G_2 has none of the dotted edges, G_3 has one of the dotted edges, G_4 has both of the dotted edges, and so on. To prove the sufficiency, let F be a minimum counterexample, i.e., the graph F does not contain the graphs G_1-G_{15} and any graph from the family $\mathcal{W}$ as induced subgraphs, $\beta(F) < \Gamma(F)$, and F has minimum order. The graph F is connected, since otherwise one of the component F' satisfies $\beta(F') < \Gamma(F')$, contrary to the minimality of F. Let X be a Γ-set of F such that the number of edges in $\langle X\rangle$ is minimum, and let $Y = V(F) - X$. Denote all isolated vertices of the graph $\langle X\rangle$ by X_2 and let $X_1 = X - X_2$. Since X is a minimal dominating set, it follows that $I(x, X) \neq \emptyset$ for any $x \in X$. If $x \in X_1$, then $I(x, X) \subset Y$. For each vertex $x \in X_1$, take one vertex from the set $I(x, X)$ and form the set $Y_1 \subset Y$.

Suppose that $Y_2 = V(F) - (X \cup Y_1) \neq \emptyset$ and consider the graph $F - Y_2$. We have

$$\beta(F - Y_2) \leq \beta(F) < \Gamma(F) \leq \Gamma(F - Y_2),$$

a contradiction, since the graph F is a minimum counterexample. Therefore $V(F) = X \cup Y_1$. Now suppose that there is a vertex $x \in X_2$. Since $Y = Y_1$ and Y_1 consists of vertices from $I(x, X)$, it follows that x is an isolated vertex of F. This is a contradiction, since F is a connected graph and $F \neq K_1$.

Thus, the graph $\langle X\rangle$ does not contain isolated vertices and all edges between the sets X and Y form a perfect matching. If y is an isolated vertex of $\langle Y\rangle$, then form the set $(X - x) \cup \{y\}$, where x is the vertex from X adjacent to y. This set is a Γ-set and contains fewer edges than $\langle X\rangle$, contrary to hypothesis. Therefore, $\delta(F) \geq 2$.

Assume that $\beta(F) < |X| - 1$ and let uv be any edge of the perfect matching between X and Y. It is not difficult to see that

$$\beta(F - \{u, v\}) \leq \beta(F) < |X| - 1 \leq \Gamma(F - \{u, v\}),$$

contrary to the minimality of F. On the other hand, $\beta(F) < \Gamma(F) = |X|$. We get

$$\beta(F) = |X| - 1.$$

Now, if $|X| \geq 5$, then F is a member of $\mathcal{W}$, a contradiction. If $|X| = 2$, then $\beta(F) = 1$ and $F \cong K_4$ which is impossile. Consequently, $3 \leq |X| \leq 4$. Consider a maximum independent set U of the graph F. Clearly, U contains vertices in both X and Y, for otherwise some v has no neighbor in U. Since $\beta(F) < |X|$, there is an edge x_1y_1 of the perfect matching such that $x_1, y_1 \notin U$. In what follows, x_i and y_i denote vertices from X and Y, respectively. The set U is maximum independent, and thus there exist vertices $x_3 \in U$ and $y_2 \in U$ such that x_1 is adjacent to x_3 and y_1 is adjacent to y_2. Let x_2y_2 and x_3y_3 be edges of the perfect matching. Now consider the graph $F' = \langle\{x_1, x_2, x_3, y_1, y_2, y_3\}\rangle$. The only edges of F' whose existence is not known yet are x_1x_2, x_2x_3, y_1y_3 and y_2y_3. If all these edges are present in F, then F' is isomorphic to G_1, a contradiction. Therefore, one of the above edges is absent and we have 15 possible graphs resulting from F'. It is straightforward to check that each of the 15 graphs is isomorphic (with saving the partition) to one of the 8 graphs resulting from F' by taking any combination of only the three edges x_1x_2, y_1y_3 and y_2y_3. Hence we can suppose that $x_2x_3 \notin E$, where E is the edge set of F. Thus, there are 8 cases to consider. Before considering these cases we derive some facts common to all the cases. As $\{y_1, x_2, x_3\}$ is independent in F and $\beta(F) + 1 = |X| \leq 4$, we have $|X| = 4$, i.e., F contains one more edge x_4y_4 in the perfect matching. We shall often use the following simple but useful fact which will be called *β-argument*: if $x_ix_j \notin E$, then $y_ky_t \in E$ where $\{i,j,k,t\} = \{1,2,3,4\}$, for otherwise the set $\{x_i, x_j, y_k, y_t\}$ is independent which is impossible (and, analogously, if $y_iy_j \notin E$, then $x_kx_t \in E$ where $\{i,j,k,t\} = \{1,2,3,4\}$). Since $x_2x_3 \notin E$, by β-argument we immediately conclude that $y_1y_4 \in E$.

Case 1: $x_1x_2, y_1y_3, y_2y_3 \notin E$. Since $\delta(F) \geq 2$, we get $y_3y_4 \in E$. By β-argument, $x_1x_4 \in E$ and $x_2x_4 \in E$. If $x_3x_4 \notin E$, then $F \cong G_2$ or G_3 depending on the existence of the edge y_2y_4, a contradiction. Consequently, $x_3x_4 \in E$ and we have $F \cong G_3$ if $y_2y_4 \notin E$, and $F \cong G_{11}$ if $y_2y_4 \in E$, which is impossible.

Case 2: $x_1x_2 \in E$ and $y_1y_3, y_2y_3 \notin E$. Analogously to Case 1, $y_3y_4 \in E$ and $x_1x_4, x_2x_4 \in E$. We have $y_2y_4 \notin E$, since otherwise $F - \{x_3, y_3\} \cong G_1$, a contradiction. Now, depending on the existence of x_3x_4, $F \cong G_3$ or G_4, a contradiction.

Case 3: $y_1y_3 \in E$ and $x_1x_2, y_2y_3 \notin E$. This case is similar to Case 2.

Case 4: $y_2y_3 \in E$ and $x_1x_2, y_1y_3 \notin E$. By β-argument, $x_2x_4 \in E$ and $y_3y_4 \in E$. Suppose $y_2y_4 \notin E$. If $x_1x_4, x_3x_4 \notin E$, then $F \cong G_2$ (note that G_2 has two partitions $V(G_2) = A \cup B$ such that the only edges between A and B are a perfect matching). If only one edge from $\{x_1x_4, x_3x_4\}$ is present in F, then $F \cong G_5$. At last, $x_1x_4, x_3x_4 \in E$ implies $F \cong G_6$. All cases yield a contradiction. Hence, $y_2y_4 \in E$. Now

$$x_1x_4, x_3x_4 \notin E \Rightarrow F \cong G_7, \quad x_1x_4, x_3x_4 \in E \Rightarrow F \cong G_{12}.$$

If only one edge from $\{x_1x_4, x_3x_4\}$ is present, then $F \cong G_9$, which is impossible.

Case 5: $x_1x_2, y_1y_3 \in E$ and $y_2y_3 \notin E$. By β-argument, $x_1x_4 \in E$. Also, the set $\{x_1, y_2, y_3, y_4\}$ cannot be independent, and therefore we may assume w.l.o.g. that $y_3y_4 \in E$. We have, $x_3x_4 \notin E$, since otherwise $F - \{x_2, y_2\} \cong G_1$. The set $\{x_2, x_3, x_4, y_1\}$ cannot be independent, so $x_2x_4 \in E$. We get $F - \{x_3, y_3\} \cong G_1$ if $y_2y_4 \in E$, and $F \cong G_{11}$ if $y_2y_4 \notin E$, a contradiction.

Case 6: $x_1x_2, y_2y_3 \in E$ and $y_1y_3 \notin E$. By β-argument, $x_2x_4 \in E$. Suppose $y_2y_4 \in E$. Then $x_1x_4 \notin E$, since otherwise $F - \{x_3, y_3\} \cong G_1$. We have

$$x_3x_4 \notin E \Rightarrow F \cong G_3 \text{ or } G_9, \quad x_3x_4 \in E \Rightarrow F \cong G_6 \text{ or } G_{14}.$$

This contradiction implies $y_2y_4 \notin E$.

Now, if $y_3y_4 \notin E$, then

$$x_1x_4, x_3x_4 \notin E \Rightarrow F \cong G_2, \quad x_1x_4 \in E, x_3x_4 \notin E \Rightarrow F \cong G_3,$$

$$x_1x_4, x_3x_4 \in E \Rightarrow F \cong G_9, \quad x_1x_4 \notin E, x_3x_4 \in E \Rightarrow F \cong G_5.$$

If $y_3y_4 \in E$, then

$$x_1x_4, x_3x_4 \notin E \Rightarrow F \cong G_5, \quad x_1x_4 \in E, x_3x_4 \notin E \Rightarrow F \cong G_6,$$

$$x_1x_4, x_3x_4 \in E \Rightarrow F \cong G_{14}, \quad x_1x_4 \notin E, x_3x_4 \in E \Rightarrow F \cong G_{13}.$$

Both subcases yield a contradiction.

Case 7: $y_1y_3, y_2y_3 \in E$ and $x_1x_2 \notin E$. Since $\delta(F) \geq 2$, we have $x_2x_4 \in E$. By β-argument, $y_3y_4 \in E$. Now $x_1x_4, x_3x_4 \notin E$ implies $F \cong G_7$ or G_8. If only one edge from $\{x_1x_4, x_3x_4\}$ is present, then $F \cong G_9$ or G_{10}. At last, $x_1x_4, x_3x_4 \in E$ implies $F - \{x_2, y_2\} \cong G_1$, a contradiction.

Case 8: $x_1x_2, y_1y_3, y_2y_3 \in E$. The set $\{x_2, x_3, x_4, y_1\}$ is not independent, and so w.l.o.g $x_2x_4 \in E$. Suppose $y_2y_4 \in E$. Since $F - \{x_3, y_3\} \ncong G_1$, we have $x_1x_4 \notin E$. Now, if $x_3x_4 \notin E$, then $F \cong G_4$ or G_{10}, and if $x_3x_4 \in E$, then $F \cong G_{14}$ or G_{15}. This contradiction implies $y_2y_4 \notin E$.

If $y_3y_4 \in E$, then

$$x_1x_4, x_3x_4 \notin E \Rightarrow F \cong G_9, \qquad x_1x_4 \in E, x_3x_4 \notin E \Rightarrow F \cong G_{12},$$

$$x_1x_4, x_3x_4 \in E \Rightarrow F - \{x_2, y_2\} \cong G_1, \qquad x_1x_4 \notin E, x_3x_4 \in E \Rightarrow F \cong G_{14}.$$

If $y_3y_4 \notin E$, then

$$x_1x_4, x_3x_4 \notin E \ \Rightarrow \ F \cong G_3, \qquad x_1x_4 \in E, x_3x_4 \notin E \Rightarrow F \cong G_{11},$$
$$x_1x_4, x_3x_4 \in E \ \Rightarrow \ F \cong G_{12}, \qquad x_1x_4 \notin E, x_3x_4 \in E \Rightarrow F \cong G_6.$$

This contradiction completes the proof of Theorem 2.2. ∎

It turns out that the class of Γ-perfect graphs is a subclass of IR-perfect graphs.

Theorem 2.3 *Any Γ-perfect graph is IR-perfect.*

Proof: Let G be a Γ-perfect graph and let H be arbitrary induced subgraph of the graph G. Clearly, H is also a Γ-perfect graph. Let X be a maximum irredundant set of the graph H. Consider the induced subgraph $F = \langle N[X] \rangle$ of the graph H. Obviously, the set X is a dominating set of the graph F. The set X is an irredundant set of H, therefore $I(x, X) \neq \emptyset$ for each vertex $x \in X$ in H. Since $I(x, X) \subseteq N[X]$ for all $x \in X$ in H, we see that $I(x, X) \neq \emptyset$ for each vertex $x \in X$ in the graph F, i.e., the set X is an irredundant set in F. Consequently, X is a minimal dominating set of the graph F. Thus,

$$\Gamma(F) \geq |X| = IR(H).$$

Since H is a Γ-perfect graph, we have

$$\beta(H) = \Gamma(H) \quad \text{and} \quad \beta(F) = \Gamma(F).$$

We get

$$IR(H) \leq \Gamma(F) = \beta(F) \leq \beta(H) = \Gamma(H) \leq IR(H).$$

Therefore, $\Gamma(H) = IR(H)$. Thus, the graph G is an IR-perfect graph. The proof is complete. ∎

Theorem 2.2 implies a characterization of Γ-perfect graphs in terms of Property A defined below. Two vertex subsets A, B of a graph *independently match* each other if $A \cap B = \emptyset$, $|A| = |B|$, and all edges between A and B in $\langle A \cup B \rangle$ form a perfect matching. We say that a graph G satisfies *Property A* if for any vertex subsets $A, B \subset V(G)$ that independently match each other, the graph $\langle A \cup B \rangle$ has an independent set of order $|A|$.

Corollary 2.4 *A graph G is Γ-perfect if and only if G satisfies Property A.*

Proof: Let A and B be vertex subsets of a Γ-perfect graph G independently matching each other. Since A is a minimal dominating set of the graph $F = \langle A \cup B \rangle$, we have $\beta(F) = \Gamma(F) \geq |A|$, i.e., G satisfies Property A.

Let G possess Property A. The graphs $G_1 - G_{15}$ in Figure 1 and the graphs from $\mathcal{W}$ do not satisfy Property A, and so they cannot be induced subgraphs of the graph G. By Theorem 2.2, the graph G is Γ-perfect. ∎

Jacobson and Peters [6] considered the class of graphs G such that $\beta(H) = IR(H)$ for all induced subgraphs H of G. Clearly, this class is the intersection of Γ-perfect graphs and IR-perfect graphs. The next result follows directly from Theorem 2.3 and Corollary 2.4.

Corollary 2.5 (Jacobson and Peters [6]) *A graph G is both Γ-perfect and IR-perfect if and only if G satisfies Property A.*

We complete this section with the next simple observations following immediately from Theorem 2.2 and the definition of $\mathcal{W}$.

Corollary 2.6 *Let m be fixed. The class of Γ-perfect graphs having $\beta(G) \leq m$ can be characterized in terms of a finite number of forbidden induced subgraphs.*

As an illustration of Corollary 2.6, we have the following result.

Corollary 2.7 *A $\overline{K}_4$-free graph is Γ-perfect if and only if it does not contain the graphs $G_1 - G_{15}$ in Figure 1 as induced subgraphs.*

3 Subclasses of Γ-perfect and IR-perfect graphs

A number of well known classes of graphs are subclasses of Γ-perfect and IR-perfect graphs. Hammer and Maffray [4] define a graph G to be *absorbantly perfect* if every induced subgraph H of G contains a minimal dominating set that meets all maximal cliques of H.

Theorem 3.1 *An absorbantly perfect graph is Γ-perfect and IR-perfect.*

Proof: Let G be an absorbantly perfect graph and suppose that the sets $A, B \subset V(G)$ independently match each other. The graph $H = \langle A \cup B \rangle$ contains a minimal dominating set X that meets all maximal cliques of H. Since all the edges of the perfect matching P of H are maximal cliques, we have $|X| \geq |A|$, and for any edge ab of the perfect matching P at least one of the vertices a, b belongs to X. Let Z denote all isolated vertices in $\langle X \rangle$ and $Y = X - Z$. Since X is a minimal dominating set of H, we have $I(y, X) \neq \emptyset$ for any vertex $y \in Y$. Denote $I = \cup_{y \in Y} I(y, X)$. Suppose that there is an edge uv such that $u \in Y$, $v \in I$ and uv is not an edge of P. Then there exists an edge vw of P such that $w \in X$, contrary to the definition of $I(u, X)$. Thus, the edges between Y and I are edges of P, and $|I| = |Y|$. By the definition of I, there are no edges between I and Z. Suppose now that the set I is not independent, i.e., there is an edge e in $\langle I \rangle$, and consider the maximal clique C containing e. The set X meets all maximal cliques, so $X \cap C \neq \emptyset$. Consequently, a vertex $x \in X \cap C$ is incident to e, contrary to the definition of I. Thus, the set $I \cup Z$ is an independent set and

$$|I \cup Z| = |I| + |Z| = |Y| + |Z| = |X| \geq |A|.$$

Therefore, G satisfies Property A and the result now follows from Corollary 2.4 and Theorem 2.3. ∎

A set of vertices S in a graph G is called a *stable transversal* if $|S \cap C| = 1$ for any maximal clique C of G. Obviously, a stable transversal is a maximal independent set. A graph G is *strongly perfect* if every induced subgraph of G has a stable transversal. Since any maximal independent set is a minimal dominating set, strongly perfect graphs form a subclass of absorbantly perfect graphs, and the inclusion is strict (see [4]). A graph G is called *strongly* Γ*-perfect* if G is both perfect and Γ-perfect. It is proved in [4] that every absorbantly perfect graph is perfect. Using Theorem 3.1 we get that absorbantly perfect graphs form a subclass of strongly Γ-perfect graphs. Take the graph G_1 in Figure 1 and make a subdivision by two vertices of an edge not belonging to a C_3. The resulting graph shows that the above inclusion is strict. By the definition, strongly Γ-perfect graphs are a subclass of Γ-perfect graphs and this inclusion is strict, since C_5 is Γ-perfect but not strongly Γ-perfect. Using Theorem 2.3 and taking into account that G_1 in Figure 1 is IR-perfect and is not Γ-perfect, we get the following chain of strict inclusions:

$$\{\text{Strongly perfect graphs}\} \subset \{\text{Absorbantly perfect graphs}\} \subset$$
$$\{\text{Strongly } \Gamma\text{-perfect graphs}\} \subset \{\Gamma\text{-perfect graphs}\} \subset \{IR\text{-perfect graphs}\}.$$

Corollary 3.2 (Cheston and Fricke [1], Jacobson and Peters [6]) *A strongly perfect graph is* Γ*-perfect and* IR*-perfect.*

The same result is valid for bipartite graphs [2] and chordal graphs [7], since they are strongly perfect. Moreover, the class of strongly perfect graphs contains perfectly orderable graphs, comparability graphs, peripheral graphs, complements of chordal graphs, Meyniel graphs, parity graphs, i-triangulated graphs, cographs, permutation graphs, and thus graphs in all these classes are Γ-perfect and IR-perfect.

Recall that a graph G is called *circular arc* if G can be represented as the intersection graph of arcs on a circle.

Corollary 3.3 (Golumbic and Laskar [3]) *A circular arc graph is* Γ*-perfect and* IR*-perfect.*

Proof: Let G be a minimal Γ-imperfect graph and suppose that G is a circular arc graph. By Theorem 2.2, $G \in \mathcal{W}$ or $G \cong G_i$, $1 \leq i \leq 15$. In both cases there is a partition $V(G) = A \cup B$ as in the definition of $\mathcal{W}$. The graph G contains an induced odd cycle C_m, since otherwise G is a bipartite graph and hence $\beta(G) = |A|$, a contradiction. The cycle C_m is odd, and hence C_m contains consecutive vertices u, v, w such that $\{u, v, w\} \subset A$ (w.l.o.g). Let I_u, I_v and I_w be circular arcs corresponding to u, v, w. Assume that $m \geq 5$.

Clearly, the arcs of C_m cover the circle and $I_u \not\subseteq I_v$, $I_w \not\subseteq I_v$. By the definition of $\mathcal{W}$, v is adjacent to $b \in B$ not adjacent to u and w, so $I_b \subseteq I_v$. This is a contradiction, since $\delta(G) \geq 2$ and b is adjacent to $b' \in B$ not adjacent to v. It remains to consider the case when $m = 3$ and the arcs of C_m do not cover the circle, i.e., $I_u \cap I_v \cap I_w \neq \emptyset$. Clearly, one of the arcs, say I_v, is contained in $I_u \cup I_w$. This is a contradiction, since v is adjacent to $b \in B$ not adjacent to u and w. ∎

Volkmann [9] generalized the above mentioned result from [2] that every bipartite graph is Γ-perfect and IR-perfect, and also the result of Topp [8] that each unicycle graph is Γ-perfect and IR-perfect.

Corollary 3.4 (Volkmann [9]) *If G is a graph such that all cycles of odd length contain a common vertex, then G is Γ-perfect and IR-perfect.*

Proof: Suppose that $A, B \subset V(G)$ independently match each other. If $H = \langle A \cup B \rangle$ is bipartite, then H has an independent set of order $|A|$. If H is not bipartite, then it contains a vertex v, a common vertex of all odd cycles. Now the graph $H' = H - \{v\}$ is bipartite and we have

$$\beta(H) \geq \beta(H') \geq \frac{1}{2}(|V(H)| - 1) = |A| - \frac{1}{2}.$$

Thus, G satisfies Property A and the result follows from Corollary 2.4 and Theorem 2.3. ∎

Let $\mathcal{P}$ be a family of connected graphs of Figure 2 having independence number four.

Theorem 3.5 *If a graph G does not contain the graphs $G_1 - G_{15}$ in Figure 1 and any member of $\mathcal{P}$ as induced subgraphs, then G is Γ-perfect and IR-perfect.*

Proof: Let G not contain the graphs $G_1 - G_{15}$ and any member of $\mathcal{P}$ as an induced subgraph. Suppose that G contains a graph $H \in \mathcal{W}$ as an induced subgraph, and consider a maximum independent set U of the graph H. By the definition of $\mathcal{W}$, $\beta(H) < |A| = |B|$, and therefore there is an edge a_1b_1 of the perfect matching ($a_1 \in A$, $b_1 \in B$) such that $a_1 \notin U$ and $b_1 \notin U$. The set U is maximum independent, and thus there exist vertices $a_2 \in U \cap A$ and $b_3 \in U \cap B$ such that a_1 is adjacent to a_2 and b_1 is adjacent to b_3. Let a_2b_2 and a_3b_3 be edges of the perfect matching. Now consider the graph $H' = \langle\{a_1, a_2, a_3, b_1, b_2, b_3\}\rangle$. The only edges whose existence is not known yet are a_1a_3, a_2a_3, b_1b_2 and b_2b_3. If all these edges are present in H, then $H' \cong G_1$, a contradiction. Using the same argument as in the proof of Theorem 2.2, we can suppose that $a_2a_3 \notin E(H)$. Since $H \in \mathcal{W}$, we have $|H| \geq 10$, and H is a connected graph. Hence there is the edge a_4b_4 of the perfect matching, and a_4b_4 is not an isolated edge in the graph $F = \langle\{a_i, b_i : 1 \leq i \leq 4\}\rangle$. Clearly, $\Gamma(F) = 4$ and F is a connected graph. If $\beta(F) < 4$, then F is not Γ-perfect, and by Theorem 2.2, F contains an induced subgraph from $G_1 - G_{15}$, a contradiction. Therefore, $\beta(F) = 4$ and $F \in \mathcal{P}$, a contradiction. Thus, G does not contain any member of $\mathcal{W}$ as an induced subgraph and also does not have the induced $G_1 - G_{15}$. By Theorems 2.2 and 2.3, G is Γ-perfect and IR-perfect. ∎

Theorem 3.5 essentially improves the known sufficient condition for a graph to be IR-perfect (Corollary 3.7). To show this, we weaken the conditions of Theorem 3.5:

Corollary 3.6 *If G does not contain the graphs P_5 and G_1 in Figure 1 as induced subgraphs, then G is Γ-perfect and IR-perfect.*

Proof: All the graphs $G_2 - G_{15}$ in Figure 1 contain P_5 as an induced subgraph. Let us show that any graph of the family $\mathcal{P}$ contains induced P_5. In fact, the family $\mathcal{P}$ is determined by the connected graph F in the proof of Theorem 3.5. Suppose that F does not contain induced P_5. We know that $a_2a_3 \notin E(F)$. If $a_1a_3 \notin E(F)$, then $\langle\{a_2, a_1, b_1, b_3, a_3\}\rangle \cong P_5$, and hence $a_1a_3 \in E(F)$. We have $b_2b_3 \in E(F)$, for otherwise $\langle\{b_2, a_2, a_1, a_3, b_3\}\rangle \cong P_5$. Now $b_3b_4 \notin E(F)$, for otherwise $\langle\{a_2, a_1, a_3, b_3, b_4\}\rangle \cong P_5$, and $b_2b_4 \notin E(F)$, for otherwise $\langle\{a_3, a_1, a_2, b_2, b_4\}\rangle \cong P_5$. Also, $\langle\{b_2, b_3, a_3, a_4, b_4\}\rangle \ncong P_5$ implies $a_3a_4 \notin E(F)$, $\langle\{b_3, b_2, a_2, a_4, b_4\}\rangle \ncong P_5$ implies $a_2a_4 \notin E(F)$, $\langle\{b_2, b_3, a_3, a_1, a_4\}\rangle \ncong P_5$ implies $a_1a_4 \notin E(F)$, and $\langle\{a_4, b_4, b_1, b_3, a_3\}\rangle \ncong P_5$ implies $b_1b_4 \notin E(F)$. Hence the edge a_4b_4 is isolated in F, a contradiction. Thus, if G does not contain induced P_5, then G also does not contain the graphs $G_2 - G_{15}$ and any member of $\mathcal{P}$ as induced subgraphs. The result now follows by Theorem 3.5. ∎

Corollary 3.7 (Cockayne, Favaron, Payan and Thomason [2]) *If G does not contain P_5, C_5, $G_1 - v$ and the 5-vertex graph with edge set $\{ab, bc, cd, de, bd\}$ as induced subgraphs, then G is IR-perfect.*

Proof: This follows directly from Corollary 3.6. ∎

Notice that the list of forbidden subgraphs in Corollary 3.7 consists of four Γ-perfect graphs while Corollary 3.6 contains only one Γ-perfect graph from this list and one minimal Γ-imperfect graph.

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UPPER DOMINATION AND UPPER IRREDUNDANCE PERFECT GRAPHS
G.Gutin and V.Zverovich

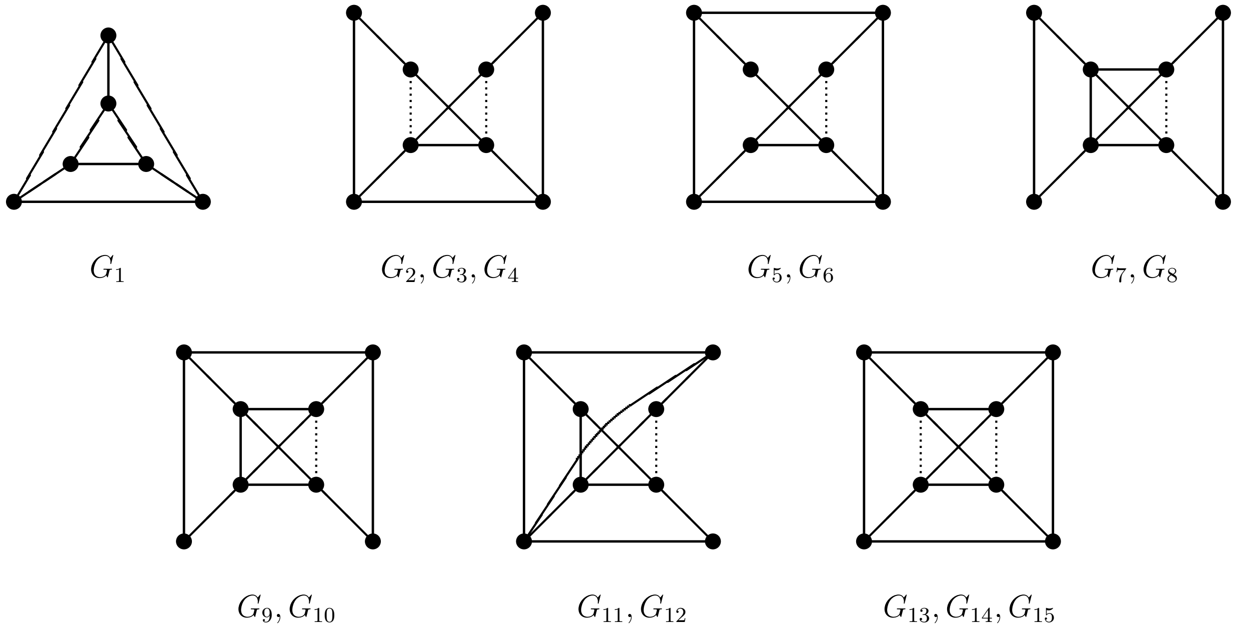

FIGURE 1. Minimal Γ-imperfect graphs $G_1 - G_{15}$

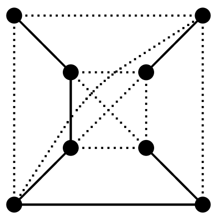

FIGURE 2