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TRANSVERSALITY AND SEPARATION OF ZEROS IN SECOND ORDER DIFFERENTIAL EQUATIONS

R. LAISTER AND R. E. BEARDMORE

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ABSTRACT. Sufficient conditions on the non-linearity f are given which ensure that non-trivial solutions of second order differential equations of the form $Lu = f(t, u)$ have a finite number of transverse zeros in a given finite time interval. We also obtain *a priori* lower bounds on the separation of zeros of solutions. In particular our results apply to non-Lipschitz non-linearities. Applications to non-linear porous medium equations are considered, yielding information on the existence and strict positivity of equilibrium solutions in some important classes of equations.

1. Introduction

We consider the second order, non-autonomous, non-linear differential equation

$$Lu := -\left(p(t)u'(t)\right)' + q(t)u(t) = f(t, u(t)), \qquad t \in (a, b), \tag{1.1}$$

where the non-linearity f is continuous but not necessarily Lipschitz continuous in u and $f(t, 0) \equiv 0$. The non-uniqueness of solutions of (1.1) which may occur when f is non-Lipschitz can manifest itself in a number of ways. For example the differential equation

$$-u'' = 12\sqrt{|u|}, \qquad u(0) = u'(0) = 0, \tag{1.2}$$

has at least two solutions, $u_1 \equiv 0$ and $u_2(t) = 0$ for $t \leq 0$, and $u_2(t) = -t^4$ for $t > 0$. Hence there exist non-unique, non-zero solutions possessing a non-transverse zero ($u(0) = u'(0) = 0$) and, in particular, infinitely many zeros on any open time interval containing $t = 0$. In fact, by a well-known result for ordinary differential equations, such non-uniqueness implies the existence of uncountably many solutions satisfying $u(0) = u'(0) = 0$ [18, Proposition 13.9, p. 567].

In this paper we will mainly be concerned with proving sufficient conditions on f which ensure that non-trivial solutions of (1.1) have a finite number of transverse zeros in a given finite time interval, ruling out equations such as (1.2). This is the content of our main result on transversality, Theorem 2.1. The conditions on f are required to hold only locally near $u = 0$ and are independent of the sign of q. In particular, non-Lipschitz f are permitted. Under minimal assumptions

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on f and g, Theorem 2.1 holds for the special cases $f(t,u) = f(u)$ and $f(t,u) = g(t)f(u)$. In Section 2 we see how non-Lipschitz forcing functions arise naturally when considering equilibrium solutions of non-linear porous medium (or degenerate diffusion) equations of the form

$$(\eta(v))_{xx} + g(x,v) = 0, \quad x \in (a,b), \tag{1.3}$$

subject to prescribed boundary conditions. We apply Theorem 2.1 to (1.3) in the case $g(x,v) = v(d(x)-v)$ used in modelling ecological populations, yielding a strong maximum principle for non-negative solutions.

Even under the conditions of Theorem 2.1 there still exist simple differential equations which have uniformly bounded solutions with arbitrarily many transverse zeros in a fixed time interval. In Section 3 we consider such an example and discuss how this behaviour is intimately related to non-Lipschitzian growth of f near $u = 0$. The main result of Section 3, Theorem 3.1, gives conditions on f which prevent this kind of behaviour by obtaining *a priori* lower bounds on the distance between zeros of solutions to (1.1). We apply this result to another important non-linear porous medium equation of the form (1.3), which is a generalisation of Nagumo's equation. In particular, Theorem 3.1 yields necessary lower bounds on the domain size for non-trivial equilibrium solutions to exist.

2. Transversality of zeros

We begin with some terminology. Throughout $C^r[a,b]$ $(r \geq 0)$ denotes the Banach space of real-valued functions which are r-times continuously differentiable on $[a,b]$, endowed with its usual norm. We write $L^p(a,b)$ $(p \geq 1)$ for the Lebesgue space of real-valued p-th power integrable functions on (a,b). For a real-valued function $G = G(t,u)$ where $t \in [a,b]$ and $u \in \mathbb{R}$ we write $G \in C^{r,k}([a,b] \times \mathbb{R})$ if G is r-times continuously differentiable in t and k-times continuously differentiable in u. For $G \in C^{1,1}([a,b] \times \mathbb{R})$ we denote the first partial derivatives of $G(t,u)$ by G_t and G_u. A function u is said to be a *solution* of (1.1) if $u \in C^2[a,b]$ and u satisfies (1.1) for all $t \in (a,b)$. A solution u is *non-trivial* if $u \not\equiv 0$ on $[a,b]$.

We label the following hypotheses on the inhomogeneous coefficients p and q and the non-linearity f:

- (C) $p, q \in C^1[a,b]$ and $p(t) > 0$ for all $t \in [a,b]$.
- (N) $f \in C^{1,0}([a,b] \times \mathbb{R})$ and $f(t,0) = 0$ for all $t \in [a,b]$. Furthermore, there exists an $r > 0$ such that for all $t \in [a,b]$, $f(t,u)$ is strictly increasing in u for $|u| < r$.

We now present our main result on the transversality of zeros of solutions of (1.1).

Theorem 2.1. *Let* (C) *and* (N) *hold. Define the primitive* $F \in C^{1,1}([a,b] \times \mathbb{R})$ *by* $F(t,u) = \int_0^u f(t,v)\ dv$ *and suppose there exists a constant* $c > 0$ *such that* $|F_t(t,u)| \leq cuf(t,u)$ *for all* $t \in [a,b]$ *and* $|u| < r$*. If* u *is any solution of* (1.1) *satisfying* $u(\alpha) = u'(\alpha) = 0$ *for some* $\alpha \in [a,b]$*, then* $u \equiv 0$ *on* $[a,b]$.

Proof. Suppose initially that $q \geq 0$ on $[a,b]$. We first show that if $a < \alpha$, then $u \equiv 0$ on $[a,\alpha]$.

Clearly $F(t,0) \equiv 0$ and $F(t,u) > 0$ for all $t \in [a,b]$ and $0 < |u| < r$. Let $v(t) := F(t,u(t)) \geq 0$. Then for $t \in (a,\alpha)$,

$$\begin{aligned} v(\alpha) - v(t) &= \int_t^\alpha v'(s)\ ds = \int_t^\alpha F_t(s,u(s)) + f(s,u(s))u'(s)\ ds \\ &= \int_t^\alpha F_t(s,u(s)) - (p(s)u'(s))'u'(s) + q(s)u(s)u'(s)\ ds. \end{aligned}$$

Since $v(\alpha) = F(\alpha, u(\alpha)) = F(\alpha, 0) = 0$, integrating by parts (twice) yields

$$\begin{aligned} -v(t) &= \int_t^\alpha F_t(s,u(s))\ ds - \Big[p(s)u'^2(s)\Big]_t^\alpha + \int_t^\alpha p(s)u'(s)u''(s)\ ds \\ &\quad + \left[\frac{1}{2}q(s)u^2(s)\right]_t^\alpha - \frac{1}{2}\int_t^\alpha q'(s)u^2(s)\ ds \\ &= \int_t^\alpha F_t(s,u(s))\ ds + p(t)u'^2(t) + \left[\frac{1}{2}p(s)u'^2(s)\right]_t^\alpha - \frac{1}{2}\int_t^\alpha p'(s)u'^2(s)\ ds \\ &\quad - \frac{1}{2}q(t)u^2(t) - \frac{1}{2}\int_t^\alpha q'(s)u^2(s)\ ds. \end{aligned}$$

Hence

(2.1)

$$2v(t) = q(t)u^2(t) - p(t)u'^2(t) + \int_t^\alpha p'(s)u'^2(s) + q'(s)u^2(s) - 2F_t(s,u(s))\ ds.$$

Now note that there exists a sequence s_n such that $a < s_n < \alpha$, $u(s_n) = 0$ and $s_n \to \alpha$ as $n \to \infty$. For if not there exists an $\varepsilon > 0$ such that $u > 0$ or $u < 0$ on $(\alpha - \varepsilon, \alpha)$. If $u > 0$, then $Lu = f(t,u) > 0$ and $u(\alpha) = 0$ imply $u'(\alpha) < 0$ by the maximum principle and Hopf's Lemma (see [14, Theorem 4, p. 7] for example), a contradiction. A similar argument holds for $u < 0$. Integrating (2.1) from $t = s_n$ to $t = \alpha$,

$$\begin{aligned} 0 \leq 2\int_{s_n}^\alpha v(t)\ dt &= \int_{s_n}^\alpha q(t)u^2(t) - p(t)u'^2(t)\ dt - 2\int_{s_n}^\alpha \int_t^\alpha F_t(s,u(s))\ dsdt \\ &\quad + \int_{s_n}^\alpha \int_t^\alpha p'(s)u'^2(s) + q'(s)u^2(s)\ dsdt \\ &=: I_1 + I_2 + I_3. \qquad (2.2) \end{aligned}$$

Since $u(s_n) = u(\alpha) = 0$ we may apply Poincaré's inequality

$$\int_{s_n}^\alpha u^2(s)\ ds \leq \frac{(\alpha - s_n)^2}{\pi^2}\int_{s_n}^\alpha u'^2(s)\ ds$$

to I_1 to give

$$I_1 \leq \int_{s_n}^\alpha \left(\frac{K(\alpha - s_n)^2}{\pi^2} - p(s)\right) u'^2(s)\ ds \qquad (2.3)$$

where

$$K = \max\{\|q\|_\infty, \|p'\|_\infty, \|q'\|_\infty\}$$

and $\|\cdot\|_\infty$ denotes the sup-norm on $[a,b]$. Applying Fubini's Theorem and Poincaré's inequality to I_3, one has

$$\begin{aligned} I_3 &= \int_{s_n}^{\alpha}\int_{s_n}^{s} p'(s){u'}^2(s) + q'(s)u^2(s)\ dtds \\ &= \int_{s_n}^{\alpha}(s-s_n)\left(p'(s){u'}^2(s) + q'(s)u^2(s)\right)\ ds \\ &\le K(\alpha-s_n)\int_{s_n}^{\alpha}\left({u'}^2(s)+u^2(s)\right)\ ds \\ &\le K(\alpha-s_n)\left(1+\frac{(\alpha-s_n)^2}{\pi^2}\right)\int_{s_n}^{\alpha}{u'}^2(s)\ ds. \end{aligned} \tag{2.4}$$

For $|t-\alpha|$ sufficiently small, $|u(s)|<r$ for all $s\in[t,\alpha]$. For such t,

$$|I_2| \le 2\int_{s_n}^{\alpha}\int_{t}^{\alpha}|F_t(s,u(s))|\ dsdt \le 2c\int_{s_n}^{\alpha}\int_{t}^{\alpha}u(s)f(s,u(s))\ dsdt.$$

But

$$\begin{aligned} \int_t^{\alpha} u(s)f(s,u(s))\ ds &= \int_t^{\alpha} -\left(p(s)u'(s)\right)'u(s) + q(s)u^2(s)\ ds \\ &= p(t)u(t)u'(t) + \int_t^{\alpha} p(s){u'}^2(s) + q(s)u^2(s)\ ds \end{aligned}$$

and so, applying Fubini's Theorem and Poincaré's inequality once more,

$$\begin{aligned} |I_2| &\le c\int_{s_n}^{\alpha} p(s)(u^2)'(s)\ ds + 2c\int_{s_n}^{\alpha}\int_t^{\alpha} p(s){u'}^2(s) + q(s)u^2(s)\ dsdt \\ &= -c\int_{s_n}^{\alpha} p'(s)u^2(s)\ ds + 2c\int_{s_n}^{\alpha}\int_{s_n}^{s} p(s){u'}^2(s)+q(s)u^2(s)\ dtds \\ &= c\int_{s_n}^{\alpha} -p'(s)u^2(s) + 2(s-s_n)\left(p(s){u'}^2(s)+q(s)u^2(s)\right)\ ds \\ &\le Kc\left[\frac{(\alpha-s_n)^2}{\pi^2} + 2(\alpha-s_n)\left(1+\frac{(\alpha-s_n)^2}{\pi^2}\right)\right]\int_{s_n}^{\alpha}{u'}^2(s)\ ds. \end{aligned} \tag{2.5}$$

By (2.2)-(2.5) and the mean value theorem for integration,

$$\begin{aligned} 0 &\le 2\int_{s_n}^{\alpha} v(t)\ dt \\ &\le (\alpha-s_n){u'}^2(\theta)\left[K_1(\alpha-s_n)+K_2(\alpha-s_n)^2+K_3(\alpha-s_n)^3-p(\theta)\right] \end{aligned} \tag{2.6}$$

for some $\theta\in(s_n,\alpha)$, where the K_i are constant multiples of K. Clearly as $n\to\infty$ the term in square brackets in (2.6) tends to $-p(\alpha)$ which is negative by (C). It follows that

$$0 \le 2\int_{s_n}^{\alpha} v(t)\ dt \le 0$$

for all n sufficiently large. Consequently v, and hence u, are identically zero on $[s_n,\alpha]$ for large n.

We claim that in fact $u\equiv 0$ on $[a,\alpha]$. For suppose this is not the case. Define

$$\delta = \inf\left\{t\in[a,\alpha]\ :\ u\equiv 0 \text{ on } [\delta,\alpha] \text{ and } u\not\equiv 0 \text{ on } [a,\delta]\right\}.$$

By the above inequality $a \le \delta < \alpha$. If $\delta > a$, then since $u \in C^2[a,b]$ it follows that $u(\delta) = u'(\delta) = 0$. Applying the above argument with α replaced by δ then provides an $\varepsilon > 0$ sufficiently small such that $u \equiv 0$ on $[\delta - \varepsilon, \delta]$. Consequently $u \equiv 0$ on $[\delta - \varepsilon, \alpha]$, contradicting the definition of δ. Thus $\delta = a$, proving the claim.

In exactly the same way one may also show that $u \equiv 0$ on $[\alpha, b]$ if $\alpha < b$. Again there must exist a sequence $t_n \in (\alpha, b)$ of zeros of u by the maximum principle. Integration now takes place in the same way as before but from $s = \alpha$ to $s = t$ and $t = \alpha$ to $t = t_n$. Since the argument is a repetition of the above we omit the details.

Finally, suppose that $q \not\ge 0$ on $[a,b]$. Defining $q_1(t) = q(t) + \|q\|_\infty$ and $f_1(t,u) = f(t,u) + \|q\|_\infty u$, we see that $Lu = f(t,u)$ is equivalent to $L_1 u = f_1(t,u)$, where $L_1 u := -(p(t)u')' + q_1(t)u$. It is straightforward to check that all the hypotheses on f are also met by f_1 and, since $q_1 \ge 0$, we may apply the above proof to show that the solution of $L_1 u = f_1(t,u)$ satisfies $u \equiv 0$ on $[a,b]$. Consequently the same is true for the solution of $Lu = f(t,u)$. This completes the proof. □

Remark 2.1. The proof of Theorem 2.1 remains valid under weaker assumptions on f. Specifically, suppose (C) holds and let α be as in Theorem 2.1. Let f satisfy

(N′) $f \in C^{1,0}([a,b] \times \mathbb{R})$ and $f(t,0) = 0$ for all $t \in [a,b]$. Furthermore there exist $\varepsilon, r > 0$ such that for all $t \in (\alpha - \varepsilon, \alpha + \varepsilon) \cap [a,b]$, $f(t,u)$ is strictly increasing in u for $|u| < r$.

If $|F_t(t,u)| \le cuf(t,u)$ for all $t \in (\alpha - \varepsilon, \alpha + \varepsilon) \cap [a,b]$ and $|u| < r$, then $u \equiv 0$ on $[a,b]$.

Corollary 2.1. *Let the hypotheses of Theorem 2.1 hold. If u is any non-trivial solution of* (1.1)*, then u has a finite number of zeros in* $[a,b]$.

Proof. Suppose that u has an infinite number of zeros $t_n \in [a,b]$. Then by Bolzano-Weierstrass and the continuity of u there exists a subsequence t_{n_j} such that $t_{n_j} \to \alpha$ as $j \to \infty$ and $u(\alpha) = 0$ for some $\alpha \in [a,b]$. Applying Rolle's Theorem to u on $[\alpha, t_{n_j}]$ (or $[t_{n_j}, \alpha]$) and letting $j \to \infty$ shows that $u'(\alpha) = 0$. Hence $u \equiv 0$ on $[a,b]$ by Theorem 2.1, as required. □

In [16] the transversality of zeros of solutions to second order non-linear differential inequalities is proved in the case $q \equiv 0$ under different assumptions on f. Crucially, the results in [16] require the solution of the differential inequality to be of one sign on an open interval, thereby excluding the possibility that a solution may oscillate arbitrarily often in the neighbourhood of a non-transverse zero. Results similar to Corollary 2.1 for differential inequalities appear in [17, Corollary 2, Corollary 3]. We point out however that the proofs are clearly incorrect since they require the existence of an open interval on which the solution is of one sign in order to apply [17, Theorem 1].

Corollary 2.2. *Assume the hypotheses of Theorem 2.1 hold. Let u_n be any sequence of solutions of* (1.1) *and let $\zeta(u_n)$ denote the number of zeros of u_n in $[a,b]$. Suppose that $u_n \to u$ in $C^2[a,b]$ as $n \to \infty$. If $\zeta(u_n) \to \infty$ as $n \to \infty$, then $u \equiv 0$ on $[a,b]$.*

Proof. Necessarily u must have an infinite number of zeros in $[a,b]$. The result follows by Corollary 2.1. □

Remark 2.2. If L has a continuous inverse $L^{-1} : C[a,b] \to C^2[a,b]$ when (1.1) is supplied with boundary conditions, then the conclusion of Corollary 2.2 follows if

$u_n \to u$ in $C[a,b]$ and $\zeta(u_n) \to \infty$ as $n \to \infty$. In particular, since L^{-1} is compact on $C[a,b]$, if $\zeta(u_n) \to \infty$ as $n \to \infty$ and u_n is uniformly bounded in $C[a,b]$, then there exists a subsequence u_{n_j} such that $u_{n_j} \to 0$ in $C[a,b]$ as $j \to \infty$. This is the case, for example, when $q \geq 0$ and the Dirichlet conditions $u(a) = u(b) = 0$ are imposed.

Example 2.1. Consider the following degenerate diffusion model of an ecological population density $v = v(x,t) \geq 0$, [5, 8]:

$$v_t = v(d(x) - v) + (v^m)_{xx}, \qquad x \in (a,b), \quad t > 0, \tag{2.7}$$

$$0 = v(a) = v(b). \tag{2.8}$$

Here $m > 1$, $d \in C^1[a,b]$ and $d > 0$ on $[a,b]$, where d represents the spatially dependent natural growth rate of the population. It is known that bounded solutions of equations such as (2.7)-(2.8) converge to the equilibrium set E of time-independent solutions of (2.7)-(2.8), [1, 10]. Setting $u = v^m$, (2.7)-(2.8), then become

$$-u'' = f(x,u), \qquad x \in (a,b), \tag{2.9}$$

$$0 = u(a) = u(b), \tag{2.10}$$

where $f(x,u) := u^{1/m}\left(d(x) - u^{1/m}\right)$ for $u \geq 0$. For $u \leq 0$ we simply take the odd extension of f. Clearly all the hypotheses of Theorem 2.1 hold, except possibly the bound on $|F_x|$. But it can easily be checked that this is satisfied for some $c > 0$ and $r > 0$ if d is any positive function satisfying the differential inequality $|d'| < \gamma d$ on $[a,b]$, for some $\gamma > 0$. Hence by Theorem 2.1, any non-negative equilibrium solution must satisfy $u > 0$ on (a,b), $u'(a) > 0$ and $u'(b) < 0$. Thus there can exist no interior region in the domain where the population density is zero (sometimes known as a 'dead core' in the porous medium literature [6]). In the case of zero-flux boundary conditions where $(v^m)_x = u_x = 0$ at $x = a, b$, non-negative equilibrium solutions satisfy $u > 0$ on $[a,b]$.

An important special case of Theorem 2.1 and Corollary 2.1 is where the nonlinearity f is autonomous, i.e. $f(t,u) = f(u)$. For then the conditions on F given in Theorem 2.1 are trivially satisfied since $F_t \equiv 0$. We therefore have the following results.

Theorem 2.2. *Let* (C) *hold and $f \in C(\mathbb{R})$ be autonomous. Suppose $f(0) = 0$ and there exists an $r > 0$ such that f is strictly increasing for $|u| < r$. Then the conclusions of Theorem 2.1 and Corollary 2.1 hold.*

Corollary 2.3. *Let* (C) *hold and let $f(t,u) = g(t)f(u)$ where $g \in C^1[a,b]$ and f satisfies the hypotheses of Theorem 2.2. If $g(t) > 0$ for all $t \in [a,b]$, then the conclusions of Theorem 2.1 and Corollary 2.1 hold.*

Proof. The result easily follows on rescaling by $t \mapsto \int_a^t \sqrt{g(s)}\, ds$, dividing (1.1) by $g(t)$ and applying Theorem 2.2 with q replaced by q/g. □

Example 2.2. Suppose one seeks radially symmetric solutions of the elliptic problem $\Delta u + f(u) = 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is an annulus in $\mathbb{R}^2$ consisting of points x such that $0 < a < |x| < b$, [7, 9]. The problem then reduces to solving $-(ru')' = rf(u)$ for $a < r < b$, $u(a) = u(b) = 0$, where $r = |x|$ and $'$ denotes d/dr. The hypotheses of Corollary 2.3 are satisfied with $p(r) = r$, $q \equiv 0$ and $g(r) = r$, provided f is continuous, $f(0) = 0$ and is strictly increasing near zero. In particular, one again has a strong maximum principle result for non-negative solutions, similar to that in Example 2.1.

3. Lower bounds on the separation of zeros

The results of Section 2 show that under suitable conditions on p, q and f, non-trivial solutions of (1.1) can have at most finitely many zeros on a given time interval $[a, b]$. However, given $[a, b]$, it is possible that there exist non-trivial, uniformly bounded solutions of (1.1) with arbitrarily many zeros in $[a, b]$. The following example is instructive in this regard.

Example 3.1. Consider the second order differential equation

$$-u'' = u^{1/3}, \qquad t > 0, \tag{3.1}$$

for which the theory of Section 2 applies. Writing this as a pair of first order differential equations $u' = v$, $v' = -u^{1/3}$ it is easy to see via phase plane arguments that non-trivial solutions of (3.1) are periodic. These solutions are represented in the (u, v)-phase plane by the closed curves $2v^2 = 3(k^{4/3} - u^{4/3})$, where $u(0) = k > 0$ and $v(0) = u'(0) = 0$. The time taken, T, for a solution u to first reach zero (equal to one quarter of the period by symmetry) is given via the usual 'time-map'

$$T = \int_0^k \frac{\sqrt{2}\, du}{\sqrt{3(k^{4/3} - u^{4/3})}} = k^{1/3} \int_0^1 \frac{\sqrt{2}\, ds}{\sqrt{3(1 - s^{4/3})}} \qquad (u = ks). \tag{3.2}$$

Hence the period of a solution can be chosen to be arbitrarily small by accordingly taking k arbitrarily small. Without loss of generality we take the time interval $[a, b] = [0, 1]$. By (3.2), given any positive integer n there exists a $k_n > 0$ (non-unique) such that (3.1) has a non-trivial solution satisfying $u(0) = k_n$ and $u'(0) = 0$ and having precisely n zeros in $[0, 1]$. Furthermore $k_n \to 0$ as $n \to \infty$ (recall Corollary 2.2). Note that k_n may be chosen uniquely by fixing $u(1) = 0$.

This example demonstrates the existence of non-Lipschitz non-linearities f for which no *a priori* lower bound for the distance between consecutive zeros of uniformly bounded solutions to (1.1) can exist. In fact, when $q \geq 0$, f must necessarily be non-Lipschitz near $u = 0$ for this behaviour to occur. To see this, suppose that u_n is a uniformly bounded sequence of non-trivial solutions of $Lu = f(t, u)$, $u(a) = u(b) = 0$ such that $\zeta(u_n) \to \infty$ as $n \to \infty$. Passing to a subsequence if necessary we may assume $u_n \to 0$ in $C[a, b]$ as $n \to \infty$ by Remark 2.2. Clearly there exist $[a_n, b_n] \subset [a, b]$ such that $Lu_n = f(t, u_n)$, $u_n(a_n) = u_n(b_n) = 0$ and $|a_n - b_n| \to 0$ as $n \to \infty$. Rescaling by $t \mapsto (t - a_n)/(b_n - a_n)$ then gives $Lu_n = \varepsilon_n^2 f(t, u_n)$, $u_n(0) = u_n(1) = 0$, where $\varepsilon_n := b_n - a_n$. Now rewrite this in the form $u_n = \varepsilon_n^2 S(u_n)$ where $S(u) := L^{-1}(f(t, u))$. Using standard properties of Green's function for L and the local Lipschitz bound on f, it follows that there exists an $M > 0$ (independent of n) such that $\|S(u_n)\|_\infty \leq M \|u_n\|_\infty$ for all n sufficiently large. On taking the sup-norm of the equation $u_n = \varepsilon_n^2 S(u_n)$ it follows that $u_n = 0$ for n large, a contradiction.

If one performs an analysis similar to that in Example 3.1 for the differential equation $-u'' = f(u) := u^3$, one still obtains non-trivial solutions u_n possessing n zeros for any $n \geq 1$. This time however, $\|u_n\|_\infty \to \infty$ as $n \to \infty$. We will see that the absence of an *a priori* lower bound on the zeros in cases like this is due to the superlinear growth of f as $|u| \to \infty$.

We now prove sufficient conditions on f which ensure that an *a priori* lower bound on the separation of zeros of (1.1) exists. Results along these lines for *linear* second order differential equations already exist in the literature. See for example [4, 11, 12, 13] and the references therein.

The following lemma is a simple application of a result due to Boyd [3] and can be found in [4] (following Theorem B).

Lemma 3.1. *If u is absolutely continuous on $[a,b]$ with $u(a) = u(b) = 0$ and $1 \leq \lambda \leq 2$, then*

$$\int_a^b |u(t)|^\lambda |u'(t)|^\lambda \, dt \leq K(\lambda)\frac{(b-a)}{2}\left(\int_a^b |u'(t)|^2 \, dt\right)^\lambda,$$

where

$$K(\lambda) = \begin{cases} \frac{1}{2}, & \lambda = 1, \\ \frac{4}{\pi^2}, & \lambda = 2, \\ \frac{2-\lambda}{2\lambda}\left(\frac{1}{\lambda}\right)^{2\lambda-2} I^{-\lambda}, & 1 < \lambda < 2, \end{cases}$$

and

$$I = \int_0^1 \left(1 + \frac{2t(\lambda-1)}{2-\lambda}\right)^{-2} (1+(\lambda-1)t)^{\frac{1}{\lambda}-1} \, dt.$$

Theorem 3.1. *Let u be any non-trivial solution of (1.1) with $u(a) = u(b) = 0$. Let $p, q \in C[a,b]$, $f \in C([a,b] \times \mathbb{R})$ and $p(t) > 0$ for all $t \in [a,b]$. If there exists a $c \geq 0$ such that $uf(t,u) \leq cu^2$ for all $t \in [a,b]$ and $u \in \mathbb{R}$, then*

$$\text{(3.3)} \qquad \frac{c(b-a)^2}{\pi^2} + 2K(\lambda)^{\frac{1}{\lambda}}\left(\frac{b-a}{2}\right)^{\frac{1}{\lambda}} \|Q\|_{L^\mu(a,b)} \geq p_0$$

where $p_0 := \min_{a \leq t \leq b} p(t) > 0$, $1 \leq \lambda \leq 2$, $\frac{1}{\mu} + \frac{1}{\lambda} = 1$ and Q is any antiderivative of q.

Proof. Multiplying (1.1) by u and integrating by parts,

$$\int_a^b p(t){u'}^2(t) \, dt - 2\int_a^b Q(t)u(t)u'(t) \, dt = \int_a^b u(t)f(t,u(t)) \, dt$$

$$\Rightarrow \quad p_0 \int_a^b {u'}^2(t) \, dt \leq \int_a^b p(t){u'}^2(t) \, dt \leq c\int_a^b u^2(t) \, dt + 2\int_a^b |Q(t)||u(t)u'(t)| \, dt$$

$$\leq \quad \frac{c(b-a)^2}{\pi^2}\int_a^b {u'}^2(t) \, dt + 2\|Q\|_{L^\mu(a,b)}\|u(t)u'(t)\|_{L^\lambda(a,b)}$$

$$\leq \quad \frac{c(b-a)^2}{\pi^2}\int_a^b {u'}^2(t) \, dt + 2K(\lambda)^{\frac{1}{\lambda}}\left(\frac{b-a}{2}\right)^{\frac{1}{\lambda}} \|Q\|_{L^\mu(a,b)} \int_a^b |u'(t)|^2 \, dt$$

by Poincaré's inequality, Hölder's inequality and Lemma 3.1. Then dividing by $\int_a^b |u'(t)|^2 \, dt$ gives the desired bound. □

When $\lambda = \mu = 2$ (3.3) becomes

$$\text{(3.4)} \qquad \frac{c(b-a)^2}{\pi^2} + \frac{2\sqrt{2}}{\pi}(b-a)^{\frac{1}{2}}\|Q\|_{L^2(a,b)} \geq p_0.$$

In particular the bounds (3.3) and (3.4) hold for *any* antiderivative Q. We can therefore seek to minimise $\|Q+k\|_{L^2(a,b)}$ over all real constants k in order to obtain a sharper bound than (3.4). It is easily shown that

$$\inf_k \int_a^b (Q(t)+k)^2 \, dt = \int_a^b Q^2(t) \, dt - \frac{1}{(b-a)}\left(\int_a^b Q(t) \, dt\right)^2,$$

the minimum being attained at $k = \dfrac{-1}{(b-a)}\displaystyle\int_a^b Q(t)\ dt$. We therefore have:

Corollary 3.1. *If the hypotheses of Theorem 3.1 hold, then*

$$\frac{c(b-a)^2}{\pi^2} + \frac{2\sqrt{2}}{\pi}\left((b-a)\int_a^b Q^2(t)\ dt - \left(\int_a^b Q(t)\ dt\right)^2\right)^{\frac{1}{2}} \geq p_0.$$

In particular if $q(t) \equiv q_0$, a constant, then

$$\left(\frac{c}{\pi^2} + \frac{|q_0|\sqrt{2}}{\pi\sqrt{3}}\right)(b-a)^2 \geq p_0.$$

Example 3.2. Consider the following quasilinear parabolic partial differential equation equipped with Dirichlet boundary conditions:

$$v_t = (g(x,v) - q(x)v^m) + l^{-2}(v^m)_{xx}, \qquad x \in (0,1), \quad t > 0, \tag{3.5}$$

$$0 = v(0) = v(1). \tag{3.6}$$

Here m is an odd positive integer, l is proportional to domain size and $g(x,v) = v(v-\beta(x))(1-v)$, where $0 < \beta(x) < 1$ for all $x \in [0,1]$. Equation (3.5) is a generalised version of Nagumo's equation used in modelling nerve impulse propagation and population genetics [1, 2, 15]. As in Example 2.1 the equilibrium solutions satisfy an elliptic problem

$$-l^{-2}u'' + q(x)u = f(x,u), \qquad x \in (0,1), \tag{3.7}$$

$$0 = u(0) = u(1) \tag{3.8}$$

after setting $u = v^m$, where $f(x,u) := g(x,u^{1/m})$ and $'$ denotes d/dx. In order to apply Theorem 3.1 to (3.7)-(3.8) it is sufficient to prove the existence of a $c > 0$ such that $f(x,u) \geq cu$ for $u \leq 0$ and $f(x,u) \leq cu$ for $u \geq 0$, for all $x \in [0,1]$. It is straightforward to see that for fixed $x \in [0,1]$ the graphs of cu and $f(x,u)$ are tangent at a unique positive value of u for a unique positive value $c(x)$ of c. Moreover $c(x)$ is continuous. Hence the hypotheses of Theorem 3.1 are satisfied with $c = c_0 := \max_{0\leq x\leq 1} c(x) > 0$. For the special case where $q = q_0$ is constant, Corollary 3.1 gives the bound

$$\left(\frac{c_0}{\pi^2} + \frac{|q_0|\sqrt{2}}{\pi\sqrt{3}}\right) l^2 \geq 1. \tag{3.9}$$

Thus non-trivial solutions only exist for sufficiently large spatial domains l.

If we apply Theorem 3.1 with $\lambda = 1$ and $\mu = \infty$ we obtain the inequality

$$\frac{c_0}{\pi^2} + \frac{1}{2}\|q_0 x + k\|_{L^\infty(0,1)} \geq l^{-2}$$

for any real k. But

$$\inf_k \|q_0 x + k\|_{L^\infty(0,1)} = \frac{|q_0|}{2}$$

(the minimum being attained at $k = -q_0/2$) yielding the bound

$$\left(\frac{c_0}{\pi^2} + \frac{|q_0|}{4}\right) l^2 \geq 1.$$

This gives a larger lower bound for l than that obtained in (3.9) for $\lambda = \mu = 2$.

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School of Mathematical Sciences, University of the West of England, Frenchay Campus, Bristol, England BS16 1QY

E-mail address: `Robert.Laister@uwe.ac.uk`

Department of Mathematics, Imperial College, London, England SW7 2BZ

E-mail address: `R.Beardmore@ic.ac.uk`