Solvability of superlinear fractional parabolic equations

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Abstract. We study necessary conditions and sufficient conditions for the existence of local-in-time solutions of the Cauchy problem for superlinear fractional parabolic equations. Our conditions are sharp and clarify the relationship between the solvability of the Cauchy problem and the strength of the singularities of the initial measure.

1. Introduction

We consider the Cauchy problem for a superlinear fractional parabolic equation

$$\begin{cases} \partial_t u + (-\Delta)^{\frac{\theta}{2}} u = F(u), & x \in \mathbb{R}^N, \ t > 0, \\ u(0) = \mu & \text{in} \ \ \mathbb{R}^N, \end{cases} \tag{P}$$

where μ is a nonnegative Radon measure in $\mathbb{R}^N$. Throughout the paper, we assume that $N \geq 1$, $0 < \theta \leq 2$, and $F : [0, \infty) \to [0, \infty)$ is (at least) continuous.

In general, the existence of local-in-time nonnegative solutions of problem (P) depends crucially on the delicate interplay between the strength of the singularities of the initial measure μ and the behavior of $F(\tau)$ as $\tau \to \infty$. In this paper, for a large class of nonlinearities F, we obtain new necessary conditions and new sufficient conditions for the local solvability of problem (P). The prototypical example we have in mind is

$$F(\tau) = \tau^p [\log(L + \tau)]^q, \ \text{where} \ p > 1, q \in \mathbb{R}, \ \text{and} \ L \geq 1.$$

As a consequence of our more general results, we are then able to derive sharp results for classes of nonlinearities which include these prototypes as special cases, and quantify this interplay more precisely via 'optimal singularities.'

Throughout this paper, we use the following notations. For $T > 0$, we set $Q_T := \mathbb{R}^N \times (0, T)$ and let $B(x, \sigma)$ denote the Euclidean ball in $\mathbb{R}^N$ center x, radius σ. We use $\fint_B f \, \mathrm{d}x$ for the average value of f over B with respect to the Lebesgue measure $\mathrm{d}x$. The set of nonnegative Lebesgue measurable functions in $\mathbb{R}^N$ is denoted by $\mathcal{L}_0$, while $\mathcal{M}$ denotes the set of nonnegative Radon measures in $\mathbb{R}^N$. For $\mu \in \mathcal{L}_0$, we abuse terminology somewhat by speaking of 'measure μ' defined via $\mathrm{d}\mu = \mu(x)\mathrm{d}x$.

1.1. Background

The solvability of the Cauchy problem for superlinear parabolic equations has been studied in many papers since the pioneering work by Fujita [14]. The literature is now very extensive, and we refer to the comprehensive monograph [35]. We also mention the following works, some of which are directly related to this paper, others with a different emphasis (higher-order equations, systems, nonlinear boundary conditions): superlinear parabolic equations [2,6,7,14,29–31,33,36,38–41]; linear heat equation with nonlinear boundary conditions [10,15,20,27,28]; superlinear parabolic equations with a potential [1,3,9,22,23,39]; superlinear parabolic systems [11–13,26,34]; superlinear fractional parabolic equations [18,19,21,32,37]; superlinear higher-order parabolic equations [8,16,17,24,25].

In [19], the second and third authors of this paper considered problem (P) in the special case of the power law nonlinearity $F(u) = u^p$ with $p > 1$:

$$\begin{cases} \partial_t u + (-\Delta)^{\frac{\theta}{2}} u = u^p, & x \in \mathbb{R}^N, \ t > 0, \\ u(0) = \mu & \text{in} \ \mathbb{R}^N. \end{cases} \tag{1.1}$$

There, as here, the exponent $p_\theta := 1 + \theta/N$ plays a critical role. They proved the following necessary conditions for the local existence (cases (i) and (ii)).

(i) Let $\mu \in \mathcal{M}$. If problem (1.1) possesses a nonnegative solution in Q_T for some $T > 0$, then there exists $C_1 = C_1(N, \theta, p) > 0$ such that

$$\sup_{x \in \mathbb{R}^N} \mu(B(x, \sigma)) \le C_1 \sigma^{N - \frac{\theta}{p-1}}, \quad 0 < \sigma \le T^{\frac{1}{\theta}}. \tag{1.2}$$

In the case where $1 < p < p_\theta$, the function $(0, \infty) \ni \sigma \mapsto \sigma^{N - \theta/(p-1)}$ is decreasing so that relation (1.2) is equivalent to

$$\sup_{x \in \mathbb{R}^N} \mu\big(B\big(x, T^{\frac{1}{\theta}}\big)\big) \le C_1 T^{\frac{N}{\theta} - \frac{1}{p-1}}.$$

In the case where $p = p_\theta$, there exists $C_2 = C_2(N, \theta) > 0$ such that

$$\sup_{x \in \mathbb{R}^N} \mu(B(x, \sigma)) \le C_2 \left[\log\left(e + \frac{T^{\frac{1}{\theta}}}{\sigma}\right) \right]^{-\frac{N}{\theta}}, \quad 0 < \sigma \le T^{\frac{1}{\theta}}.$$

(See [2] for the Laplacian case $\theta = 2$.)

Condition (i) implies the following nonexistence result.

(ii) Let $p \ge p_\theta$. There exists $\gamma = \gamma(N, \theta, p) > 0$ such that if $\mu \in \mathcal{L}_0$ satisfies

$$\mu(x) \ge \gamma |x|^{-N} \left[\log\left(e + \frac{1}{|x|}\right) \right]^{-\frac{N}{\theta} - 1} \quad \text{if} \quad p = p_\theta,$$

$$\mu(x) \ge \gamma |x|^{-\frac{\theta}{p-1}} \quad \text{if} \quad p > p_\theta,$$

for almost all (a.a.) x in a neighborhood of the origin, then problem (1.1) possesses no local-in-time nonnegative solutions.

Regarding sufficiency, in [19] they obtained results (iii) and (iv) below.

(iii) Let $\mu \in \mathcal{M}$ and $1 < p < p_\theta$. There exists $c = c(N, \theta, p) > 0$ such that if

$$\sup_{x \in \mathbb{R}^N} \mu\left(B\left(x, T^{\frac{1}{\theta}}\right)\right) \leq cT^{\frac{N}{\theta} - \frac{1}{p-1}}$$

for some $T > 0$, then problem (1.1) possesses a nonnegative solution in Q_T.

(iv) Let $\mu \in \mathcal{L}_0$ and $p \geq p_\theta$. There exists $\varepsilon = \varepsilon(N, \theta, p) > 0$ such that if

$$0 \leq \mu(x) \leq \varepsilon |x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{N}{\theta}-1} + K \quad \text{if} \quad p = p_\theta,$$

$$0 \leq \mu(x) \leq \varepsilon |x|^{-\frac{\theta}{p-1}} + K \qquad\qquad\qquad \text{if} \quad p > p_\theta,$$

for a.a. $x \in \mathbb{R}^N$ for some $K > 0$, then problem (1.1) possesses a local-in-time nonnegative solution.

For $\mu \in \mathcal{L}_0$, the results in (ii) and (iv) demonstrate that the 'strength' of the singularity at the origin of the functions

$$\mu_c(x) = \begin{cases} |x|^{-\frac{\theta}{p-1}} & \text{if} \quad p > p_\theta, \\ |x|^{-N} |\log|x||^{-\frac{N}{\theta}-1} & \text{if} \quad p = p_\theta, \end{cases}$$

is the critical threshold for the local solvability of problem (1.1). We term such a singularity in the initial data an *optimal singularity* for the solvability for problem (1.1). Of course, by translation invariance the singularity could be located at any point of $\mathbb{R}^N$.

Subsequently, the results of [19] were extended to some related parabolic problems with a power law nonlinearity (see [20–25]). However, one cannot apply the arguments in these papers to problem (P) with a general nonlinearity F since they depend heavily upon the homogeneous structure of the power law nonlinearity.

1.2. The main result

In this paper, we improve the arguments in [19] to obtain necessary conditions and sufficient conditions for the existence of local-in-time solutions of problem (P) for a significantly larger class of nonlinearities F and determine the optimal singularities of the initial data for the solvability of problem (P).

Let f_1 and f_2 be real-valued functions defined in an interval $[L, \infty)$, where $L \in \mathbb{R}$. We write $f_1(t) \preceq f_2(t)$ as $t \to \infty$ if there exists $C > 0$ such that $f_1(t) \leq Cf_2(t)$ for all large enough $t \in [L, \infty)$. We define $\succeq$ in the obvious way, namely $f_2(t) \succeq f_1(t)$ as $t \to \infty$ if and only if $f_1(t) \preceq f_2(t)$ as $t \to \infty$. We write $f_1(t) \asymp f_2(t)$ as $t \to \infty$ whenever $f_1(t) \preceq f_2(t)$ and $f_1(t) \succeq f_2(t)$ as $t \to \infty$, i.e., there exists $C > 0$ such that $C^{-1} f_2(t) \leq f_1(t) \leq Cf_2(t)$ for large enough $t \in [L, \infty)$.

We consider nonlinearities which are asymptotic to the prototypical example (F), in this sense:

(F1) F is locally Lipschitz continuous in $[0, \infty)$;

(F2) $F(\tau) \asymp \tau^p [\log \tau]^q$ as $\tau \to \infty$ for some $p > 1$ and $q \in \mathbb{R}$.

Theorem 1.1. *Assume conditions* (F1) *and* (F2).

(i) *Let* $\mu \in \mathcal{M}$ *and either*

$$\text{(i)} \quad 1 < p < p_\theta \quad \text{or} \quad \text{(ii)} \quad p = p_\theta \text{ and } q < -1.$$

Problem (P) *possesses a local-in-time solution if and only if* $\sup_{z \in \mathbb{R}^N} \mu(B(z, 1)) < \infty$.

(ii) *Suppose* $\mu \in \mathcal{L}_0$.

(1) *Let* $p = p_\theta$ *and* $q = -1$. *There exists* $\gamma_1 > 0$ *such that if*

$$\mu(x) \geq \gamma_1 |x|^{-N} |\log |x||^{-1} [\log |\log |x||]^{-\frac{N}{\theta} - 1} \tag{1.3}$$

in a neighborhood of $x = 0$, *then problem* (P) *possesses no local-in-time solutions. On the other hand, for any* $R \in (0, 1)$, *there exists* $\varepsilon_1 > 0$ *such that if*

$$0 \leq \mu(x) \leq \varepsilon_1 |x|^{-N} |\log |x||^{-1} [\log |\log |x||]^{-\frac{N}{\theta} - 1} \chi_{B(0,R)}(x) + K_1, \quad x \in \mathbb{R}^N, \tag{1.4}$$

for some $K_1 > 0$, *then problem* (P) *possesses a local-in-time solution.*

(2) *Let* $p = p_\theta$ *and* $q > -1$. *There exists* $\gamma_2 > 0$ *such that if*

$$\mu(x) \geq \gamma_2 |x|^{-N} |\log |x||^{-\frac{N(q+1)}{\theta} - 1} \tag{1.5}$$

in a neighborhood of $x = 0$, *then problem* (P) *possesses no local-in-time solutions. On the other hand, for any* $R \in (0, 1)$, *there exists* $\varepsilon_2 > 0$ *such that if*

$$0 \leq \mu(x) \leq \varepsilon_2 |x|^{-N} |\log |x||^{-\frac{N(q+1)}{\theta} - 1} \chi_{B(0,R)}(x) + K_2, \quad x \in \mathbb{R}^N, \tag{1.6}$$

for some $K_2 > 0$, *then problem* (P) *possesses a local-in-time solution.*

(3) *Let* $p > p_\theta$. *There exists* $\gamma_3 > 0$ *such that if*

$$\mu(x) \geq \gamma_3 |x|^{-\frac{\theta}{p-1}} |\log |x||^{-\frac{q}{p-1}} \tag{1.7}$$

in a neighborhood of $x = 0$, *then problem* (P) *possesses no local-in-time solutions. On the other hand, for any* $R \in (0, 1)$, *there exists* $\varepsilon_3 > 0$ *such that if*

$$0 \leq \mu(x) \leq \varepsilon_3 |x|^{-\frac{\theta}{p-1}} |\log |x||^{-\frac{q}{p-1}} \chi_{B(0,R)}(x) + K_3, \quad x \in \mathbb{R}^N, \tag{1.8}$$

for some $K_3 > 0$, *then problem* (P) *possesses a local-in-time solution.*

While Theorem 1.1 provides sharp results on the identification of optimal singularities for the solvability of problem (P), we point out that we have obtained several other interesting and powerful results in this paper regarding necessary conditions and sufficient conditions for existence under very general conditions on F. We mention, in particular, Theorems 3.1, 4.1, 4.2, and 4.3.

Subject to mild assumptions on F (essentially that of majorizing a convex function with suitable monotonicity properties), we follow the strategy in [19] and obtain necessary conditions for the existence in Theorem 3.1. However, the iteration step in [19] to obtain the estimate for the optimal singularity relies on the homogeneity of the pure power law nonlinearity considered there. For the class of nonlinearities satisfying (F1)–(F2), we combine the arguments in [19] with the method introduced in [32], to obtain a sharper necessary condition in Corollary 3.1. Conversely, in order to derive sharp sufficient conditions we require delicate arguments for F satisfying (F1)–(F2). Indeed, the arguments are separated into three cases: (i) $1 < p < p_\theta$ (see Theorem 4.1), $p > p_\theta$ (see Theorem 4.3), and (iii) $p = p_\theta$ (see Theorem 4.2). The arguments in case (i) are somewhat standard but the other cases involve certain intricacies, in particular, for the critical case $p = p_\theta$.

The rest of this paper is organized as follows. In Sect. 2, we recall some properties of the fundamental solution Γ_θ and prove some preliminary lemmas. In Sect. 3, we obtain necessary conditions for the existence of local-in-time solutions of problem (P). In Sect. 4, we prove several theorems on sufficient conditions for the existence of local-in-time solutions of problem (P). In Sect. 4.4, we also provide a necessary and sufficient condition on the nonlinearity F for which problem (P) is solvable for the case of initial data a Dirac measure (Corollary 4.4). Finally, in Sect. 5 we complete the proof of our main result, Theorem 1.1, and outline some analogous results for nonlinearities which are asymptotic to further log-refinements of the cases above (see Remark 5.1).

2. Preliminaries

In this section, we prove some important technical lemmas, modifying the arguments in [19] for the more general nonlinearities considered here. We make precise our notion of solution used throughout this paper, which implicitly considers nonnegative functions only. The word 'solvability' for problem (P) is always used with respect to this solution concept. In all that follows, we will use C to denote generic positive constants which depend only on N, θ, and F and point out that C may take different values within a calculation. We begin by recalling some properties of the kernel for the fractional Laplacian.

Let $\Gamma_\theta = \Gamma_\theta(x, t)$ be the fundamental solution of

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty).$$

The function Γ_θ satisfies

$$\Gamma_\theta(x, t) = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right) \quad \text{if} \quad \theta = 2,$$

$$C^{-1}t^{-\frac{N}{\theta}}\left(1 + t^{-\frac{1}{\theta}}|x|\right)^{-N-\theta} \le \Gamma_\theta(x, t) \le Ct^{-\frac{N}{\theta}}\left(1 + t^{-\frac{1}{\theta}}|x|\right)^{-N-\theta} \quad \text{if} \quad 0 < \theta < 2,$$

(2.1)

for all $x \in \mathbb{R}^N$ and $t > 0$ and has the following properties:

- Γ_θ is positive and smooth in $\mathbb{R}^N \times (0, \infty)$,
- $\Gamma_\theta(x, t) = t^{-\frac{N}{\theta}} \Gamma_\theta\left(t^{-\frac{1}{\theta}}x, 1\right), \quad \int_{\mathbb{R}^N} \Gamma_\theta(x, t)\,\mathrm{d}x = 1,$ (2.2)
- $\Gamma_\theta(\cdot, 1)$ is radially symmetric and $\Gamma_\theta(x, 1) \le \Gamma_\theta(y, 1)$ if $|x| \ge |y|$,   (2.3)
- $\Gamma_\theta(x, t) = \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t - s)\Gamma_\theta(y, s)\,\mathrm{d}y,$ (2.4)

for all $x, y \in \mathbb{R}^N$ and $0 < s < t$ (see for example [4,5,37]). Furthermore, we have the following smoothing estimate for the semigroup associated with Γ_θ (see [19, Lemma 2.1]).

Lemma 2.1. *For any $\mu \in \mathcal{M}$, set*

$$[S(t)\mu](x) := \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t)\,\mathrm{d}\mu(y), \quad x \in \mathbb{R}^N, \quad t > 0.$$

Then, there exists $C = C(N, \theta) > 0$ such that

$$\|S(t)\mu\|_{L^\infty(\mathbb{R}^N)} \le Ct^{-\frac{N}{\theta}} \sup_{x \in \mathbb{R}^N} \mu\left(B\left(x, t^{\frac{1}{\theta}}\right)\right), \quad t > 0.$$

Remark 2.1. (i) $S(t)\mu$ is possibly infinite everywhere in $\mathbb{R}^N$; (ii) if $\mu \in \mathcal{M}$ is such that

$$\sup_{x \in \mathbb{R}^N} \mu(B(x, r)) < \infty$$

for some $r > 0$, then for any $R \ge r$ there exists $C \ge 1$ such that

$$\sup_{x \in \mathbb{R}^N} \mu(B(x, R)) \le C \sup_{x \in \mathbb{R}^N} \mu(B(x, r)) < \infty.$$

See for example [27, Lemma 2.1] or [12, Lemma 2.4].

We now make precise our solution concepts for problem (P).

Definition 2.1. Let $T > 0$ and u be a nonnegative, measurable, finite almost everywhere function in Q_T. Let F be a nonnegative and continuous function in $[0, \infty)$.

(i) We say that u satisfies

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = F(u)$$

(2.5)

in Q_T if, for a.a. $\tau \in (0, T)$, u satisfies

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t - \tau) u(y, \tau) \, dy + \int_\tau^t \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t - s) F(u(y, s)) \, dy \, ds$$

for a.a. $(x, t) \in \mathbb{R}^N \times (\tau, T)$.

(ii) Let $\mu \in \mathcal{M}$. We say that u is a solution of problem (P) in Q_T if u satisfies

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t) \, d\mu(y) + \int_0^t \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t - s) F(u(y, s)) \, dy \, ds \tag{2.6}$$

for a.a. $(x, t) \in Q_T$. If u satisfies (2.6) with '=' replaced by '≥,' then u is said to be a supersolution of problem (P) in Q_T.

Next, we recall a lemma on the existence of solutions of problem (P) in the presence of a supersolution (see [19, Lemma 2.2]).

Lemma 2.2. *Let F be an increasing, nonnegative continuous function in $[0, \infty)$. Let $\mu \in \mathcal{M}$ and $0 < T \leq \infty$. If there exists a supersolution v of problem (P) in Q_T, then there exists a solution u of problem (P) in Q_T such that $0 \leq u(x, t) \leq v(x, t)$ in Q_T.*

Combining Lemma 2.2 and parabolic regularity theory, we have:

Lemma 2.3. *Let $\mu \in \mathcal{M}$ be such that $\sup_{z \in \mathbb{R}^N} \mu(B(z, 1)) < \infty$. Suppose*

(i) *F_1 is nonnegative and locally Lipschitz continuous in $[0, \infty)$;*
(ii) *F_2 is an increasing and continuous function in $[0, \infty)$ such that $F_1(\tau) \leq F_2(\tau)$ for all $\tau \in [0, \infty)$.*

If there exists a supersolution v of (P) in Q_T with F replaced by F_2 such that for all $\tau \in (0, T)$

$$\sup_{\tau < t < T} \|v(t)\|_{L^\infty(\mathbb{R}^N)} < \infty, \tag{2.7}$$

then there exists a solution u of (P) in Q_T with F replaced by F_1, with u satisfying $0 \leq u(x, t) \leq v(x, t)$ in Q_T.

Proof. For any $m, n \in \mathbb{N}$ set

$$F_{1,m}(\tau) := \min\{F_1(\tau), m\} \quad \text{for } \tau \geq 0,$$
$$\mu_n(x) := \int_{\mathbb{R}^N} \Gamma_\theta(x - y, 2n^{-1}) \, d\mu(y) \quad \text{for } x \in \mathbb{R}^N.$$

It follows from Lemma 2.1 that $S(n^{-1})\mu \in L^\infty(\mathbb{R}^N)$. Also, since $\mu_n = S(n^{-1})S(n^{-1})\mu$, we have that $\mu_n \in BC(\mathbb{R}^N)$. For each $m, n \in \mathbb{N}$ define the sequence

$\{u_{m,n,k}\}_{k=0}^{\infty}$ by

$$u_{m,n,0}(x,t) := \int_{\mathbb{R}^N} \Gamma_\theta(x-y,t)\mu_n(y)\,dy,$$

$$u_{m,n,k+1}(x,t) := \int_{\mathbb{R}^N} \Gamma_\theta(x-y,t)\mu_n(y)\,dy$$

$$+ \int_0^t \int_{\mathbb{R}^N} \Gamma_\theta(x-y,t-s)F_{1,m}(u_{m,n,k}(y,s))\,dy\,ds.$$

By (2.4) and Definition 2.1 (ii), we have

$$u_{m,n,0}(x,t) = \int_{\mathbb{R}^N} \Gamma_\theta(x-y,t)\left(\int_{\mathbb{R}^N} \Gamma_\theta(y-z,2n^{-1})\,d\mu(z)\right)dy$$

$$= \int_{\mathbb{R}^N} \Gamma_\theta(x-z,t+2n^{-1})\,d\mu(z)$$

$$\leq v(x,t+2n^{-1})$$

for $x \in \mathbb{R}^N$ and $t \in [0, T-2n^{-1})$. Since $F_1(\tau) \leq F_2(\tau)$ for $\tau \in [0,\infty)$, by induction, we obtain

$$0 \leq u_{m,n,k}(x,t) \leq v(x,t+2n^{-1}) \tag{2.8}$$

for all $x \in \mathbb{R}^N$, $t \in [0, T-2n^{-1})$, and $k \geq 0$. Here, we used the assumption that F_2 is increasing. Since $F_{1,m}$ is globally Lipschitz in $[0,\infty)$, we may apply the standard theory of evolution equations to see that the pointwise limit

$$u_{m,n}(x,t) := \lim_{k\to\infty} u_{m,n,k}(x,t)$$

exists in $\mathbb{R}^N \times [0,\infty)$ and satisfies

$$u_{m,n}(x,t) = \int_{\mathbb{R}^N} \Gamma_\theta(x-y,t)\mu_n(y)\,dy$$

$$+ \int_0^t \int_{\mathbb{R}^N} \Gamma_\theta(x-y,t-s)F_{1,m}(u_{m,n}(y,s))\,dy\,ds \tag{2.9}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Furthermore, by (2.8) we see that

$$0 \leq u_{m,n}(x,t) \leq v(x,t+2n^{-1}) \tag{2.10}$$

for all $x \in \mathbb{R}^N$ and $t \in [0, T-2n^{-1})$. Then, by (2.7), for any $\tau \in (0, T-2n^{-1})$ we have

$$\sup_{\tau < t < T-2n^{-1}} \|u_{m,n}(t)\|_{L^\infty(\mathbb{R}^N)} \leq \sup_{\tau < t < T} \|v(t)\|_{L^\infty(\mathbb{R}^N)} < \infty.$$

Applying the standard parabolic regularity theory to integral equation (2.9), we find $\alpha \in (0,1)$ such that

$$\sup_n \|u_{m,n}\|_{C^{\alpha;\alpha/2}(K)} < \infty \tag{2.11}$$

for any compact set $K \subset Q_T$. By the Ascoli–Arzelá theorem and the diagonal argument we obtain a subsequence $\{u_{m,n'}\}$ of $\{u_{m,n}\}$ and a function $u_m \in C(Q_T)$ such that

$$\lim_{n' \to \infty} u_{m,n'}(x, t) = u_m(x, t) \quad \text{in} \quad Q_T. \tag{2.12}$$

Since $F_{1,m}$ is bounded and continuous in $(0, \infty)$, by (2.9), we have

$$u_m(x, t) = \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t)\mu(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t - s)F_{1,m}(u_m(y, s))\,\mathrm{d}y\,\mathrm{d}s \tag{2.13}$$

in Q_T. Furthermore, by (2.10) and (2.12), we see that

$$0 \le u_m(x, t) \le v(x, t) \tag{2.14}$$

for all $x \in \mathbb{R}^N$ and $t \in [0, T)$.

Similarly to (2.11), using (2.14), instead of (2.10), we have

$$\sup_m \|u_m\|_{C^{\alpha; \alpha/2}(K)} < \infty$$

for any compact set $K \subset Q_T$. By the Ascoli–Arzelá theorem and the diagonal argument we obtain a subsequence $\{u_{m'}\}$ of $\{u_m\}$ and a function $u \in C(Q_T)$ such that

$$\lim_{m' \to \infty} u_{m'}(x, t) = u(x, t) \quad \text{in} \quad Q_T. \tag{2.15}$$

Since $F_{1,m}(\tau) \le F_1(\tau) \le F_2(\tau)$ for $\tau \in (0, \infty)$, by (2.14) we see that

$$\sup_{m'} \int_0^t \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t - s)F_{1,m'}(u_{m'}(y, s))\,\mathrm{d}y\,\mathrm{d}s$$
$$\le \int_0^t \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t - s)F_2(v(y, s))\,\mathrm{d}y\,\mathrm{d}s \le v(x, t) < \infty$$

for a.a. $(x, t) \in Q_T$. Then, by (2.13) and (2.15) we apply Lebesgue's dominated convergence theorem to see that u is a solution of problem (P) in Q_T with F replaced by F_1 and $0 \le u(x, t) \le v(x, t)$ in Q_T. Thus, Lemma 2.3 follows. $\qquad\square$

Next, we provide two lemmas on the relationship between the initial measure and the initial trace for problem (P).

Lemma 2.4. *Let F be a nonnegative continuous function in $[0, \infty)$.*

(i) *Let u satisfy (2.5) in Q_T for some $T > 0$. Then,*

$$\operatorname*{ess\,sup}_{0 < t < T - \varepsilon} \int_{B(0,R)} u(y, t)\,\mathrm{d}y < \infty$$

for all $R > 0$ and $0 < \varepsilon < T$. Furthermore, there exists a unique $v \in \mathcal{M}$ as an initial trace of the solution u; that is,

$$\operatorname*{ess\,lim}_{t \to +0} \int_{\mathbb{R}^N} u(y, t)\eta(y)\, dy = \int_{\mathbb{R}^N} \eta(y)\, dv(y)$$

for all $\eta \in C_0(\mathbb{R}^N)$.

(ii) *Let u be a solution of problem* (P) *in Q_T for some $T > 0$. Then, assertion* (i) *holds with v replaced by μ.*

The proof of Lemma 2.4 is the same as in [19, Lemma 2.3]. Furthermore, by assertion (i) we can apply the same argument as in the proof of [19, Theorem 1.2] to obtain the following lemma.

Lemma 2.5. *Let F be a nonnegative continuous function in $[0, \infty)$ and $T > 0$. Let u satisfy* (2.5) *in Q_T. Let $\mu \in \mathcal{M}$ be the unique initial trace of u guaranteed by Lemma 2.4. If $\sup_{z \in \mathbb{R}^N} \mu(B(z, 1)) < \infty$, then u is a solution of problem* (P) *in Q_T.*

In the rest of this section, we prepare preliminary lemmas.

Lemma 2.6. *Let $a > 0$ and $b, c \in \mathbb{R}$. Set*

$$\varphi(\tau) := \tau^a (\log \tau)^b (\log \log \tau)^c, \quad \tau \in (e, \infty).$$

Then, there exists $L \in (e, \infty)$ such that $\varphi' > 0$ in (L, ∞) and the inverse function $\varphi^{-1} : (\varphi(L), \infty) \to (L, \infty)$ exists. Furthermore,

$$\varphi^{-1}(\tau) \asymp \tau^{\frac{1}{a}} (\log \tau)^{-\frac{b}{a}} (\log \log \tau)^{-\frac{c}{a}} \tag{2.16}$$

as $\tau \to \infty$.

Proof. Since $a > 0$, we can find $L \in (e, \infty)$ such that

$$\varphi'(\tau) = \tau^{a-1} (\log \tau)^b (\log \log \tau)^c \left[a + b(\log \tau)^{-1} + c(\log \tau)^{-1}(\log \log \tau)^{-1} \right] > 0$$

for all $\tau \in (L, \infty)$. Since $\varphi(\tau) \to \infty$ as $\tau \to \infty$, it follows that $\varphi^{-1} : (\varphi(L), \infty) \to (L, \infty)$ exists and satisfies $\varphi^{-1}(\tau) \to \infty$ as $\tau \to \infty$. Now,

$$\log \tau = \log \varphi(\varphi^{-1}(\tau)) = a \log \varphi^{-1}(\tau) + b \log \log \varphi^{-1}(\tau) + c \log \log \log \varphi^{-1}(\tau)$$
$$= a \log \varphi^{-1}(\tau)(1 + o(1)) \tag{2.17}$$

as $\tau \to \infty$, so that

$$\log \varphi^{-1}(\tau) = \frac{1}{a}(\log \tau)(1 + o(1))$$

as $\tau \to \infty$. Then, by (2.17) we have

$$a \log \varphi^{-1}(\tau) = \log \tau - b \log \log \varphi^{-1}(\tau) - c \log \log \log \varphi^{-1}(\tau)$$
$$= \log \tau - b \log \left(\frac{1}{a}(\log \tau)(1 + o(1)) \right) - c \log \log \left(\frac{1}{a}(\log \tau)(1 + o(1)) \right)$$

as $\tau \to \infty$. Hence,

$$\log \varphi^{-1}(\tau) = \log \left[\left(\tau (\log \tau)^{-b} (\log \log \tau)^{-c} \right)^{\frac{1}{a}} \right] - \frac{b}{a} \log \left(\frac{1}{a} (1 + o(1)) \right) + o(1)$$

as $\tau \to \infty$, from which (2.16) follows and completes the proof of Lemma 2.6.   □

Lemma 2.7. *Let $a > 0$, $b \geq 0$, and $c \in \mathbb{R}$.*

(i) *There exists $C_1 > 0$ such that*

$$\int_A^B \tau^{a-b-1} (\log \tau)^c \, d\tau \geq C_1 A^a B^{-b} (\log A)^c \log \frac{B}{A}$$

for all A, $B \in [2, \infty)$ with $A \leq B$.

(ii) *There exists $C_2 \in [1, \infty)$ such that*

$$C_2^{-1} \tau^a [\log(e + \tau)]^c \leq \int_{\tau/2}^\tau s^{a-1} [\log(e + s)]^c \, ds$$

$$\leq \int_0^\tau s^{a-1} [\log(e + s)]^c \, ds \leq C_2 \tau^a [\log(e + \tau)]^c$$

for all $\tau \in [0, \infty)$.

Proof. We first prove assertion (i). Thanks to $a > 0$, by Lemma 2.6 we find $R_1 \in [2, \infty)$ such that the function $(R_1, \infty) \ni \tau \mapsto \tau^a (\log \tau)^c$ is increasing. Then, we have

$$\begin{cases} \tau^a (\log \tau)^c \geq A^a (\log A)^c & \text{if } R_1 \leq A \leq \tau, \\ \tau^a (\log \tau)^c \geq R_1^a (\log R_1)^c \geq C A^a (\log A)^c & \text{if } A \leq R_1 \leq \tau, \\ \tau^a (\log \tau)^c \geq C A^a (\log A)^c & \text{if } A \leq \tau < R_1, \end{cases}$$

for all A, $B \in [2, \infty)$ with $A \leq B$. We notice that

$$\inf_{A \in [2, R_1]} \frac{R_1^a (\log R_1)^c}{A^a (\log A)^c} > 0, \quad \inf_{A \in [2, R_1], \tau \in [A, R_1]} \frac{\tau^a (\log \tau)^c}{A^a (\log A)^c} > 0.$$

We observe that

$$\int_A^B \tau^{a-b-1} (\log \tau)^c \, d\tau \geq B^{-b} \int_A^B \tau^{a-1} (\log \tau)^c \, d\tau$$

$$\geq C B^{-b} A^a (\log A)^c \int_A^B \tau^{-1} \, d\tau = C A^a B^{-b} (\log A)^c \log \frac{B}{A}$$

for all A, $B \in [2, \infty)$ with $A \leq B$. Then, assertion (i) follows.

Next, we prove assertion (ii). Let $\varepsilon \in (0, a)$. By Lemma 2.6 we find $R_2 > 0$ such that the function $(R_2, \infty) \ni \tau \mapsto (e + \tau)^\varepsilon (\log(e + \tau))^c$ is increasing. Then, we have

$$\int_0^\tau s^{a-1} [\log(e + s)]^c \, ds \leq C + \int_{R_2}^\tau s^{a-1} (e + s)^{-\varepsilon} (e + s)^\varepsilon [\log(e + s)]^c \, ds$$

$$\leq C + (e + \tau)^{\varepsilon} [\log(e + \tau)]^c \int_{R_2}^{\tau} s^{a-1-\varepsilon} \, ds$$

$$\leq C + C\tau^{a-\varepsilon} (e + \tau)^{\varepsilon} [\log(e + \tau)]^c \leq C\tau^a [\log(e + \tau)]^c$$

for all $\tau \in [R_2, \infty)$. On the other hand,

$$\int_0^{\tau} s^{a-1} [\log(e + s)]^c \, ds \leq C \int_0^{\tau} s^{a-1} \, ds \leq C\tau^a \leq C\tau^a [\log(e + \tau)]^c$$

for all $\tau \in (0, R_2)$. These imply that

$$\int_0^{\tau} s^{a-1} [\log(e + s)]^c \, ds \leq C\tau^{\varepsilon} [\log(e + \tau)]^c \int_0^{\tau} s^{a-\varepsilon-1} \, ds \leq C\tau^a [\log(e + \tau)]^c \tag{2.18}$$

for all $\tau \in [0, \infty)$. On the other hand, since

$$\inf_{\tau \in (0,\infty)} \frac{\log(e + \tau/2)}{\log(e + \tau)} > 0,$$

we have

$$C^{-1} \log(e + \tau) \leq \log(e + \tau/2) \leq \inf_{\xi \in (\tau/2,\tau)} \log(e + \xi)$$

$$\leq \sup_{\xi \in (\tau/2,\tau)} \log(e + \xi) \leq \log(e + \tau)$$

for $\tau > 0$. This yields

$$\int_{\tau/2}^{\tau} s^{a-1} [\log(e + s)]^c \, ds \geq C[\log(e + \tau)]^c \int_{\tau/2}^{\tau} s^{a-1} \, ds \geq C\tau^a [\log(e + \tau)]^c \tag{2.19}$$

for all $\tau \in [0, \infty)$. By (2.18) and (2.19), we have assertion (ii). The proof is complete. $\qquad\square$

Lemma 2.8. *Let $p > 1$, $d \in [1, p)$, $q \in \mathbb{R}$, and $R \geq 0$. Define a function f in $[0, \infty)$ by*

$$f(\tau) := \begin{cases} 0 & \text{for} \quad \tau \in [0, R], \\ \tau^d \int_R^{\tau} s^{-d} \left(\int_R^s \xi^{p-2} [\log(e + \xi)]^q \, d\xi \right) ds & \text{for} \quad \tau \in (R, \infty). \end{cases}$$

Then,

(i) *the function $(0, \infty) \ni \tau \mapsto \tau^{-d} f(\tau)$ is increasing;*

(ii) *f is convex in $[0, \infty)$;*

(iii) *$f(\tau) \asymp \tau^p (\log \tau)^q$ as $\tau \to \infty$.*

Proof. By the definition of f, we easily obtain property (i). Since

$$f'(\tau) = d\tau^{d-1} \int_R^\tau s^{-d} \left(\int_R^s \xi^{p-2} [\log(e+\xi)]^q \, d\xi \right) ds + \int_R^\tau \xi^{p-2} [\log(e+\xi)]^q \, d\xi$$

for $\tau \in (R, \infty)$, we observe that f' is increasing in $[0, \infty)$, so that property (ii) holds.

We prove property (iii). Since $d \in (1, p)$, by Lemma 2.7 (ii) we have

$$\begin{aligned}
f(\tau) &\leq \tau^d \int_0^\tau s^{-d} \left(\int_0^s \xi^{p-2} [\log(e+\xi)]^q \, d\xi \right) ds \\
&\leq C\tau^d \int_0^\tau s^{p-1-d} [\log(e+s)]^q \, ds \leq C\tau^p [\log(e+\tau)]^q
\end{aligned} \tag{2.20}$$

for all $\tau > R$ and

$$\begin{aligned}
f(\tau) &\geq \tau^d \int_{\tau/2}^\tau s^{-d} \left(\int_{s/2}^s \xi^{p-2} [\log(e+\xi)]^q \, d\xi \right) ds \\
&\geq C\tau^d \int_{\tau/2}^\tau s^{p-1-d} [\log(e+s)]^q \, ds \geq C\tau^p [\log(e+\tau)]^q
\end{aligned} \tag{2.21}$$

for all $\tau > 4R$. By (2.20) and (2.21), we obtain assertion (iii). Thus, Lemma 2.8 follows. □

3. Necessary conditions for solvability

In this section, we establish necessary conditions for the solvability of problem (P). We begin in Theorem 3.1 by imposing only weak constraints on the nonlinearity F, before specializing to the case where F satisfies (F1) and (F2) in Corollary 3.1.

Theorem 3.1. *Let F be a continuous function in $[0, \infty)$. Assume that there exists a convex function f in $[0, \infty)$ with the following properties:*

(f1) $F(\tau) \geq f(\tau) \geq 0$ *in $[0, \infty)$;*

(f2) *the function $(0, \infty) \ni \tau \mapsto \tau^{-d} f(\tau)$ is increasing for some $d > 1$.*

Let u satisfy (2.5) in Q_T for some $T > 0$ and let μ be the initial trace of u. Then, there exists $\gamma = \gamma(N, \theta, f) \geq 1$ such that

$$\int_{\gamma^{-1} T^{-\frac{N}{\theta}} m_\sigma(z)}^{\gamma^{-1} \sigma^{-N} m_\sigma(z)} s^{-p_\theta - 1} f(s) \, ds \leq \gamma^{p_\theta + 1} m_\sigma(z)^{-\frac{\theta}{N}} \tag{3.1}$$

for all $z \in \mathbb{R}^N$ and $\sigma \in (0, T^{\frac{1}{\theta}})$, where $m_\sigma(z) := \mu(B(z, \sigma))$.

Proof. It follows from Definition 2.1 (i) and property (f1) that, for a.a. $\tau \in (0, T)$,

$$\infty > u(x, t) \geq \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t - \tau) u(y, \tau) \, dy + \int_\tau^t \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t - s) f(u(y, s)) \, dy \, ds$$

for a.a. $x \in \mathbb{R}^N$ and a.a. $t \in (\tau, T)$. This implies that

$$\infty > u(x, 2t) \geq \int_{\mathbb{R}^N} \Gamma_\theta(x - y, t) u(y, t) \, dy \tag{3.2}$$

for a.a. $x \in \mathbb{R}^N$ and a.a. $t \in (0, T/2)$.

Let $0 < \rho < (T/2)^{\frac{1}{\theta}}$. It follows from Definition 2.1 (i), property (f1), and (2.4) that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Gamma_\theta(z - x, t) u(x, t) \, dx \\
&\geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma_\theta(z - x, t) \Gamma_\theta(x - y, t) \, d\mu(y) \, dx \\
&\quad + \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma_\theta(z - x, t) \Gamma_\theta(x - y, t - s) f(u(y, s)) \, dy \, ds \, dx \\
&= \int_{\mathbb{R}^N} \Gamma_\theta(z - y, 2t) \, d\mu(y) + \int_0^t \int_{\mathbb{R}^N} \Gamma_\theta(z - y, 2t - s) f(u(y, s)) \, dy \, ds
\end{aligned}
\tag{3.3}
$$

for all $z \in \mathbb{R}^N$ and a.a. $t \in (0, T)$. On the other hand, by (2.1) we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} \Gamma_\theta(z - y, 2t) \, d\mu(y) &\geq \int_{B(z,\sigma)} \Gamma_\theta(z - y, 2t) \, d\mu(y) \\
&\geq \min_{x \in B(0,\sigma)} \Gamma_\theta(x, 2t) \mu(B(z, \sigma)) \geq C t^{-\frac{N}{\theta}} \mu(B(z, \sigma)) \quad (3.4)
\end{aligned}
$$

for all $z \in \mathbb{R}^N$ and $t \geq \rho^\theta$, where $\sigma := 2^{\frac{1}{\theta}} \rho \in (0, T^{\frac{1}{\theta}})$. Furthermore, by (2.2) and (2.3) we see that

$$
\begin{aligned}
\Gamma_\theta(z - y, 2t - s) &= (2t - s)^{-\frac{N}{\theta}} \Gamma_\theta\left((2t - s)^{-\frac{1}{\theta}}(z - y), 1\right) \\
&\geq \left(\frac{s}{2t}\right)^{\frac{N}{\theta}} s^{-\frac{N}{\theta}} \Gamma_\theta\left(s^{-\frac{1}{\theta}}(z - y), 1\right) = \left(\frac{s}{2t}\right)^{\frac{N}{\theta}} \Gamma_\theta(z - y, s) \quad (3.5)
\end{aligned}
$$

for all $y, z \in \mathbb{R}^N$ and $0 < s < t$. Combining (3.2), (3.3), (3.4), and (3.5), we find $C_* \geq 1$ such that

$$
\begin{aligned}
\infty > w(t) &:= \int_{\mathbb{R}^N} \Gamma_\theta(z - x, t) u(x, t) \, dx \\
&\geq C_*^{-1} t^{-\frac{N}{\theta}} \mu(B(z, \sigma)) + C_*^{-1} t^{-\frac{N}{\theta}} \int_{\rho^\theta}^t \int_{\mathbb{R}^N} s^{\frac{N}{\theta}} \Gamma_\theta(z - y, s) f(u(y, s)) \, dy \, ds
\end{aligned}
$$

for a.a. $z \in \mathbb{R}^N$ and a.a. $t \in (\rho^\theta, T/2)$. Thanks to the convexity of f, by (2.2) we may apply Jensen's inequality to obtain

$$
\infty > w(t) \geq C_*^{-1} t^{-\frac{N}{\theta}} \mu(B(z, \sigma)) + C_*^{-1} t^{-\frac{N}{\theta}} \int_{\rho^\theta}^t s^{\frac{N}{\theta}} f\left(\int_{\mathbb{R}^N} \Gamma_\theta(z - y, s) u(y, s) \, dy\right) ds
$$

$$= C_*^{-1} t^{-\frac{N}{\theta}} \mu(B(z, \sigma)) + C_*^{-1} t^{-\frac{N}{\theta}} \int_{\rho^\theta}^t s^{\frac{N}{\theta}} f(w(s)) \, ds \tag{3.6}$$

for a.a. $z \in \mathbb{R}^N$ and a.a. $t \in (\rho^\theta, T/2)$.

Since f is convex in $[0, \infty)$, it is Lipschitz continuous in any compact subinterval of $[0, \infty)$. We may then let ζ denote the unique local solution of the integral equation

$$\zeta(t) = \mu(B(z, \sigma)) + \int_{\rho^\theta}^t s^{\frac{N}{\theta}} f(C_*^{-1} s^{-\frac{N}{\theta}} \zeta(s)) \, ds, \quad t \geq \rho^\theta. \tag{3.7}$$

Hence, ζ is the unique local solution of

$$\zeta'(t) = t^{\frac{N}{\theta}} f\left(C_*^{-1} t^{-\frac{N}{\theta}} \zeta(t)\right), \qquad \zeta(\rho^\theta) = \mu(B(z, \sigma)). \tag{3.8}$$

By (3.6), applying the standard theory for ordinary differential equations to (3.7), we see that the solution ζ exists in $[\rho^\theta, T/2)$ and satisfies

$$\zeta(t) \leq C_* t^{\frac{N}{\theta}} w(t) < \infty, \quad t \in [\rho^\theta, T/2).$$

It follows from (3.8) and property (f2) that

$$\zeta'(t) = t^{\frac{N}{\theta}} \left[C_*^{-1} t^{-\frac{N}{\theta}} \zeta(t)\right]^d \left[C_*^{-1} t^{-\frac{N}{\theta}} \zeta(t)\right]^{-d} f\left(C_*^{-1} t^{-\frac{N}{\theta}} \zeta(t)\right)$$

$$\geq t^{\frac{N}{\theta}} \left[C_*^{-1} t^{-\frac{N}{\theta}} \zeta(t)\right]^d \left[C_*^{-1} t^{-\frac{N}{\theta}} \zeta(\rho^\theta)\right]^{-d} f\left(C_*^{-1} t^{-\frac{N}{\theta}} \zeta(\rho^\theta)\right)$$

$$\geq \zeta(\rho^\theta)^{-d} t^{\frac{N}{\theta}} \zeta(t)^d f\left(C_*^{-1} t^{-\frac{N}{\theta}} \zeta(\rho^\theta)\right)$$

for all $t \in [\rho^\theta, T/2)$. Then, we have

$$\frac{1}{d-1} \zeta(\rho^\theta)^{1-d} \geq \int_{\rho^\theta}^{T/2} \frac{\zeta'(s)}{\zeta(s)^d} \, ds \geq \zeta(\rho^\theta)^{-d} \int_{\rho^\theta}^{T/2} s^{\frac{N}{\theta}} f\left(C_*^{-1} s^{-\frac{N}{\theta}} \zeta(\rho^\theta)\right) \, ds.$$

Recalling (3.8) and setting $\eta := C_*^{-1} \mu(B(z, \sigma)) s^{-\frac{N}{\theta}}$, we take large enough C_* if necessary so that

$$\mu(B(z, \sigma)) \geq (d-1) \int_{\rho^\theta}^{T/2} s^{\frac{N}{\theta}} f\left(C_*^{-1} s^{-\frac{N}{\theta}} \zeta(\rho^\theta)\right) \, ds$$

$$= (d-1) \int_{\rho^\theta}^{T/2} s^{\frac{N}{\theta}} f\left(C_*^{-1} \mu(B(z, \sigma)) s^{-\frac{N}{\theta}}\right) \, ds$$

$$= \frac{(d-1)\theta}{N} C_*^{-p_\theta} \mu(B(z, \sigma))^{p_\theta} \int_{C_*^{-1} (T/2)^{-\frac{N}{\theta}} \mu(B(z,\sigma))}^{C_*^{-1} \rho^{-N} \mu(B(z,\sigma))} \eta^{-p_\theta - 1} f(\eta) \, d\eta$$

$$\geq \gamma^{-p_\theta - 1} \mu(B(z, \sigma))^{p_\theta} \int_{\gamma^{-1} T^{-\frac{N}{\theta}} \mu(B(z,\sigma))}^{\gamma^{-1} \sigma^{-N} \mu(B(z,\sigma))} \eta^{-p_\theta - 1} f(\eta) \, d\eta$$

for all $z \in \mathbb{R}^N$ and $\sigma \in (0, T^{\frac{1}{\theta}})$, where $\gamma = 2^{-\frac{N}{\theta}} C_*$. Here, we used the relation $\sigma = 2^{\frac{1}{\theta}} \rho \in (0, T^{\frac{1}{\theta}})$. Thus, inequality (3.1) holds, and the proof of Theorem 3.1 is complete. $\qquad\square$

Corollary 3.1. *Assume conditions* (F1) *and* (F2). *Let u satisfy*

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = F(u)$$

in Q_T for some $T > 0$. Then, there exists a unique $v \in \mathcal{M}$ as the initial trace of u. Furthermore,

(i) *u is a solution of problem* (P) *in Q_T with $\mu = v$;*
(ii) *there exists $C = C(N, \theta, F) > 0$ such that*

$$\sup_{z \in \mathbb{R}^N} \mu(B(z, \sigma)) \leq \begin{cases} C\sigma^{N - \frac{\theta}{p-1}} |\log \sigma|^{-\frac{q}{p-1}} & \text{if } p \neq p_\theta, \\ C|\log \sigma|^{-\frac{N(q+1)}{\theta}} & \text{if } p = p_\theta, \ q \neq -1, \\ C[\log |\log \sigma|]^{-\frac{N}{\theta}} & \text{if } p = p_\theta, \ q = -1, \end{cases}$$

for all small enough $\sigma > 0$.

Proof. The existence and uniqueness of the initial trace of u follows from Lemma 2.4. Let $d \in (1, p)$, $R > 0$, and $\kappa > 0$. Set

$$f(\tau) := \begin{cases} 0 & \text{for } 0 \leq \tau < R, \\ \kappa \tau^d \int_R^\tau s^{-d} \left(\int_R^s \xi^{p-2} [\log(e + \xi)]^q \, d\xi \right) ds & \text{for } \tau \geq R. \end{cases}$$

By Lemma 2.8 (i) and (ii), we see that f is convex in $(0, \infty)$ and (f2) in Theorem 3.1 holds. Furthermore, thanks to Lemma 2.8 (iii), taking small enough $\kappa > 0$ and large enough $R > 0$, by (F2) we can ensure that $F(\tau) \geq f(\tau)$ in $[0, \infty)$ and consequently (f1) in Theorem 3.1 also holds. In particular, we find $L \in (R, \infty)$ such that

$$F(\tau) \geq f(\tau) \geq C\tau^p (\log \tau)^q, \quad \tau \in (L, \infty). \tag{3.9}$$

By Theorem 3.1, we also find $\gamma \geq 1$ such that

$$\gamma^{p_\theta + 1} m_\sigma(z)^{-\frac{\theta}{N}} \geq \int_{\gamma^{-1} T^{-\frac{N}{\theta}} m_\sigma(z)}^{\gamma^{-1} \sigma^{-N} m_\sigma(z)} s^{-p_\theta - 1} f(s) \, ds \tag{3.10}$$

for all $z \in \mathbb{R}^N$ and $\sigma \in (0, T^{\frac{1}{\theta}})$.

We show that

$$\sup_{z \in \mathbb{R}^N} m_\sigma(z) < \infty \quad \text{for all } \sigma \in \left(0, T^{\frac{1}{\theta}}\right). \tag{3.11}$$

For then by Remark 2.1 (ii), we have

$$\sup_{z \in \mathbb{R}^N} \mu(B(z, 1)) \leq C \sup_{z \in \mathbb{R}^N} \mu(B(z, T^{\frac{1}{\theta}}/2)) = C \sup_{z \in \mathbb{R}^N} m_{T^{\frac{1}{\theta}}/2}(z) < \infty, \tag{3.12}$$

and assertion (i) will follow from Lemma 2.5.

Suppose that $\sigma \in (0, T^{\frac{1}{\theta}})$ but (3.11) does not hold. Then, there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ such that $m_\sigma(z_n) \to \infty$ as $n \to \infty$. Consequently,

$$\gamma^{-1} T^{-\frac{N}{\theta}} m_\sigma(z_n) \geq \max\{L, 2\} \tag{3.13}$$

for all n large enough. By (3.9), (3.10), (3.13), and Lemma 2.7 (i) (with $a = p - 1$, $b = \theta/N$, and $c = q$), we obtain

$$m_\sigma(z_n)^{-\frac{\theta}{N}} \geq C\gamma^{-p_\theta - 1} \int_{\gamma^{-1} T^{-\frac{N}{\theta}} m_\sigma(z_n)}^{\gamma^{-1} \sigma^{-N} m_\sigma(z_n)} s^{p - p_\theta - 1} (\log s)^q \, ds$$

$$\geq C \left(\gamma^{-1} T^{-\frac{N}{\theta}} m_\sigma(z_n)\right)^{p-1} \left(\gamma^{-1} \sigma^{-N} m_\sigma(z_n)\right)^{-\frac{\theta}{N}} \left(\log\left(\gamma^{-1} T^{-\frac{N}{\theta}} m_\sigma(z_n)\right)\right)^q \log\left(\frac{\sigma^{-N}}{T^{-\frac{N}{\theta}}}\right)$$

$$= C\sigma^\theta T^{-\frac{N(p-1)}{\theta}} \log\left(\sigma^{-N} T^{\frac{N}{\theta}}\right) m_\sigma(z_n)^{p-1-\frac{\theta}{N}} \left(\log\left(\gamma^{-1} T^{-\frac{N}{\theta}} m_\sigma(z_n)\right)\right)^q.$$

Hence,

$$1 \geq C\sigma^\theta T^{-\frac{N(p-1)}{\theta}} \log\left(\sigma^{-N} T^{\frac{N}{\theta}}\right) m_\sigma(z_n)^{p-1} \left(\log\left(\gamma^{-1} T^{-\frac{N}{\theta}} m_\sigma(z_n)\right)\right)^q. \tag{3.14}$$

Letting $n \to \infty$ in (3.14) yields a contradiction and thus (3.11) holds.

We now prove assertion (ii). Consider first the case where $p \neq p_\theta$. We show that there exist $C > 0$ and $\sigma_* > 0$ such that

$$\sigma^{\frac{\theta}{p-1} - N} |\log \sigma|^{\frac{q}{p-1}} m_\sigma(z) \leq C \tag{3.15}$$

for all $z \in \mathbb{R}^N$ and $\sigma \in (0, \sigma_*)$. Suppose, for contradiction, that there exist sequences $\{z_n\} \subset \mathbb{R}^N$ and $\{\sigma_n\} \subset (0, \infty)$ such that

$$\sigma_n \to 0 \quad \text{and} \quad \sigma_n^{\frac{\theta}{p-1} - N} |\log \sigma_n|^{\frac{q}{p-1}} m_{\sigma_n}(z_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{3.16}$$

Set $M_n := m_{\sigma_n}(z_n)$. It follows from (3.12) that

$$M_n \leq \sup_{z \in \mathbb{R}^N} \mu(B(z, 1)) \leq C \tag{3.17}$$

for all n large enough. By (3.16), we necessarily have

$$\sigma_n^{-N} M_n \to \infty \quad \text{as} \quad n \to \infty,$$

so that for n large enough,

$$\gamma^{-1} (2\sigma_n)^{-N} M_n \geq \max\{L, 2\} \quad \text{and} \quad (2\sigma_n)^{-N} > T^{-\frac{N}{\theta}}. \tag{3.18}$$

Similar to the proof of part (i), it follows from (3.9), (3.10), and (3.18) that

$$\gamma^{p_\theta + 1} \sigma_n^{-\theta} \geq C \left(\sigma_n^{-N} M_n\right)^{\frac{\theta}{N}} \int_{\gamma^{-1} (2\sigma_n)^{-N} M_n}^{\gamma^{-1} \sigma_n^{-N} M_n} s^{p - p_\theta - 1} (\log s)^q \, ds.$$

Applying Lemma 2.7 (i) (with $a = p - 1$, $b = \theta/N$, and $c = q$), we obtain (for n large enough)

$$C\sigma_n^{-\theta} \geq \tau_n^{p-1}(\log(C\tau_n))^q, \tag{3.19}$$

where $\tau_n := \sigma_n^{-N} M_n$. For n large enough, and rescaling with $s_n = C\tau_n$ in (3.19), we can apply Lemma 2.6 (with $a = p - 1 > 0$, $b = q$, and $c = 0$) to obtain (after rescaling back to τ_n)

$$\sigma_n^{-N} M_n = \tau_n \leq C \left(C\sigma_n^{-\theta}\right)^{\frac{1}{p-1}} \left(\log\left[C\sigma_n^{-\theta}\right]\right)^{-\frac{q}{p-1}}.$$

Consequently, for such n,

$$\sigma_n^{\frac{\theta}{p-1}-N} |\log\sigma_n|^{\frac{q}{p-1}} M_n \leq C,$$

contradicting (3.16). Thus, (3.15) holds, as required.

Now consider the case when $p = p_\theta$ and $q \neq -1$. We show that there exist $C > 0$ and $\sigma_* > 0$ such that

$$|\log\sigma|^{\frac{N(q+1)}{\theta}} m_\sigma(z) \leq C \tag{3.20}$$

for all $z \in \mathbb{R}^N$ and $\sigma \in (0, \sigma_*)$. Suppose, for contradiction, that there exist sequences $\{z_n\} \subset \mathbb{R}^N$ and $\{\sigma_n\} \subset (0, \infty)$ such that

$$\sigma_n \to 0 \quad \text{and} \quad |\log\sigma_n|^{\frac{N(q+1)}{\theta}} m_{\sigma_n}(z_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{3.21}$$

Set $M_n := m_{\sigma_n}(z_n)$. Since $\sigma^{-N/2} \geq |\log\sigma|^{\frac{N(q+1)}{\theta}}$ for all $\sigma > 0$ small enough, by (3.21), we necessarily have

$$\sigma_n^{-\frac{N}{2}} M_n \to \infty \quad \text{as} \quad n \to \infty,$$

so that for n large enough,

$$\gamma^{-1}\sigma_n^{-\frac{N}{2}} M_n \geq \max\{L, 2\} \quad \text{and} \quad \sigma_n^{-\frac{N}{2}} > T^{-\frac{N}{\theta}}. \tag{3.22}$$

Similar to the proof of part (i), it follows from (3.9), (3.10), and (3.22) that

$$\gamma^{p_\theta+1} \geq CM_n^{\frac{\theta}{N}} \int_{\gamma^{-1}\sigma_n^{-\frac{N}{2}} M_n}^{\gamma^{-1}\sigma_n^{-N} M_n} s^{-1}(\log s)^q \, ds. \tag{3.23}$$

Now set $c_q := 1/2$ if $q \geq 0$ and $c_q := 1$ if $q < 0$. Then, by (3.23) we have

$$1 \geq CM_n^{\frac{\theta}{N}} \left(\log(\gamma^{-1}\sigma_n^{-Nc_q} M_n)\right)^q \int_{\gamma^{-1}\sigma_n^{-\frac{N}{2}} M_n}^{\gamma^{-1}\sigma_n^{-N} M_n} \tau^{-1} \, d\tau$$

$$= CM_n^{\frac{\theta}{N}} \left(\log\left(\gamma^{-1}\sigma_n^{-Nc_q} M_n\right)\right)^q \log\left(\sigma_n^{-\frac{N}{2}}\right),$$

so that

$$\left(\sigma_n^{-Nc_q} M_n\right)^{\frac{\theta}{N}} \left(\log\left(\gamma^{-1}\sigma_n^{-Nc_q} M_n\right)\right)^q \leq C\sigma_n^{-c_q\theta} |\log\sigma_n|^{-1}. \qquad (3.24)$$

Setting $\tau_n := \gamma^{-1}\sigma_n^{-Nc_q} M_n$, (3.24) can be written as

$$\tau_n^{\frac{\theta}{N}} (\log\tau_n)^q \leq C\sigma_n^{-c_q\theta} |\log\sigma_n|^{-1}. \qquad (3.25)$$

Applying Lemma 2.6 to (3.25) (with $a = \theta/N$, $b = q$, and $c = 0$) then yields

$$\sigma_n^{-Nc_q} M_n \leq C \left(\sigma_n^{-c_q\theta}|\log\sigma_n|^{-1}\right)^{\frac{N}{\theta}} \left(\log\left(C\sigma_n^{-c_q\theta}|\log\sigma_n|^{-1}\right)\right)^{-\frac{Nq}{\theta}}$$

$$\leq C\sigma_n^{-Nc_q}|\log\sigma_n|^{-\frac{N}{\theta}} \left(C\log\left(\sigma_n^{-1}\right)\right)^{-\frac{Nq}{\theta}}$$

$$\leq C\sigma_n^{-Nc_q}|\log\sigma_n|^{-\frac{N(q+1)}{\theta}}$$

for all n large enough. Consequently for such n,

$$|\log\sigma_n|^{\frac{N(q+1)}{\theta}} M_n \leq C,$$

contradicting (3.21). Thus, (3.20) holds, as required.

Finally, consider the case when $p = p_\theta$ and $q = -1$. We show that there exist $C > 0$ and $\sigma_* > 0$ such that

$$(\log|\log\sigma|)^{\frac{N}{\theta}} m_\sigma(z) \leq C \qquad (3.26)$$

for all $z \in \mathbb{R}^N$ and $\sigma \in (0, \sigma_*)$. Suppose, for contradiction, that there exist sequences $\{z_n\} \subset \mathbb{R}^N$ and $\{\sigma_n\} \subset (0, \infty)$ such that

$$\sigma_n \to 0 \quad \text{and} \quad (\log|\log\sigma_n|)^{\frac{N}{\theta}} m_{\sigma_n}(z_n) \to \infty \quad \text{as} \quad n \to \infty. \quad (3.27)$$

Set $M_n := m_{\sigma_n}(z_n)$. Since $\sigma^{-N} \geq (\log|\log\sigma|)^{\frac{N}{\theta}}$ for all $\sigma > 0$ small enough, by (3.27), we necessarily have

$$\sigma_n^{-N} M_n \to \infty \quad \text{as} \quad n \to \infty.$$

Then, combining (3.17), we find $L' > 0$ such that

$$\max\{\gamma^{-1}T^{-\frac{N}{\theta}} M_n, L, 2\} \leq L' < \gamma^{-1}\sigma_n^{-N} M_n \qquad (3.28)$$

for all n large enough. Once again, by (3.9), (3.10), and (3.28) we have

$$\sigma_n^{-\theta} \geq C\gamma^{-p_\theta-1} \left(\sigma_n^{-N} M_n\right)^{\frac{\theta}{N}} \int_{L'}^{\gamma^{-1}\sigma_n^{-N} M_n} \tau^{-1}(\log\tau)^{-1}\, d\tau$$

$$= C\tau_n^{\frac{\theta}{N}} \log\left(\frac{\log\tau_n}{\log L'}\right) \geq C\tau_n^{\frac{\theta}{N}} \log\log\tau_n$$

for all n large enough, where $\tau_n := \gamma^{-1}\sigma_n^{-N} M_n$. By (3.22) and Lemma 2.6 (with $a = \theta/N$, $b = 0$, and $c = 1$), we have

$$\gamma^{-1}\sigma_n^{-N} M_n = \tau_n \leq \left(C\sigma_n^{-\theta}\right)^{\frac{N}{\theta}} \left(\log\log\left[C\sigma_n^{-\theta}\right]\right)^{-\frac{N}{\theta}} \leq C\sigma_n^{-N} \left(\log|\log\sigma_n|\right)^{-\frac{N}{\theta}},$$

so that

$$(\log|\log\sigma_n|)^{\frac{N}{\theta}} M_n \leq C,$$

contradicting (3.27). Hence, (3.26) holds, as required. The proof of Corollary 3.1 is complete. $\qquad\square$

4. Sufficient conditions for solvability

In this section, we establish sufficient conditions for the existence of a supersolution, and consequently of a local-in-time solution of problem (P), for three general classes of nonlinearity F (see Theorems 4.1, 4.2, and 4.3). As corollaries, we obtain the corresponding results when specializing to nonlinearities satisfying (F1) and (F2) (Corollaries 4.1, 4.2, and 4.3). Indeed, for F satisfying (F1) and (F2) the classification of initial data for which problem (P) is locally solvable separates naturally into the following three cases:

(A): either   (i) $1 < p < p_\theta$ and $q \in \mathbb{R}$ or   (ii) $p = p_\theta$ and $q < -1$;
(B): $p = p_\theta$ and $q \geq -1$;
(C): $p > p_\theta$.

4.1. Sufficiency: case (A)

We begin with nonlinearities F which generalize case (A).

Theorem 4.1. *Let F be a nonnegative continuous function in $[0, \infty)$ and assume the following conditions:*

(A1) *there exists $R \geq 0$ such that the function $(R, \infty) \ni \tau \mapsto \tau^{-1}F(\tau)$ is increasing;*
(A2) $\int_1^\infty \tau^{-p_\theta - 1} F(\tau)\, d\tau < \infty.$
If $\mu \in \mathcal{M}$ satisfies

$$\sup_{x \in \mathbb{R}^N} \mu(B(x, 1)) < \infty, \tag{4.1}$$

then problem (P) *possesses a solution u in Q_T for some $T > 0$, with u satisfying*

$$0 \leq u(x, t) \leq 2[S(t)\mu](x) + R \leq Ct^{-\frac{N}{\theta}}$$

in Q_T for some $C > 0$.

Proof. Let $T \in (0, 1)$ be chosen later. Set $w(x, t) := R + 2[S(t)\mu](x)$. It follows from Lemma 2.1 and (4.1) that

$$R \leq w(x, t) \leq R + Ct^{-\frac{N}{\theta}} \sup_{z \in \mathbb{R}^N} \mu\big(B(z, t^{\frac{1}{\theta}})\big) \leq R + Mt^{-\frac{N}{\theta}} \leq 2Mt^{-\frac{N}{\theta}} \quad (4.2)$$

for $0 < t \leq T$ and small enough T, where $M := C\sup_{x \in \mathbb{R}^N} \mu(B(x, 1)) + 1 < \infty$. Then, by (A1) and (4.2) we have

$$0 \leq \frac{F(w(x, t))}{w(x, t)} \leq (2M)^{-1} t^{\frac{N}{\theta}} F\big(2Mt^{-\frac{N}{\theta}}\big), \quad (x, t) \in Q_T. \quad (4.3)$$

Noting that

$$S(t - s)w(s) = S(t - s)[R + 2S(s)\mu] = R + 2S(t)\mu = w(t),$$

then by (A2) and (4.3) we obtain

$$
\begin{aligned}
&[S(t)\mu](x) + \int_0^t S(t - s)F(w(s)) \, ds \\
&\leq \frac{1}{2}w(x, t) + \int_0^t \left\| \frac{F(w(s))}{w(s)} \right\|_{L^\infty(\mathbb{R}^N)} S(t - s)w(s) \, ds \\
&\leq \frac{1}{2}w(x, t) + (2M)^{-1} w(x, t) \int_0^t s^{\frac{N}{\theta}} F\left(2Ms^{-\frac{N}{\theta}}\right) ds \\
&\leq w(x, t) \left[\frac{1}{2} + CM^{\frac{\theta}{N}} \int_{2MT^{-\frac{N}{\theta}}}^\infty \tau^{-p_\theta - 1} F(\tau) \, d\tau \right] \leq w(x, t), \quad (x, t) \in Q_T,
\end{aligned}
$$

for small enough T. This means that w is a supersolution in Q_T and the desired result follows from Lemma 2.2 and (4.2). $\qquad \square$

Corollary 4.1. *Assume conditions* (F1) *and* (F2) *with*

$$\textit{either} \quad (i) \ \ 1 < p < p_\theta \ \textit{and} \ q \in \mathbb{R} \quad \textit{or} \quad (ii) \ \ p = p_\theta \ \textit{and} \ q < -1.$$

If $\mu \in \mathcal{M}$ satisfies

$$\sup_{z \in \mathbb{R}^N} \mu(B(z, 1)) < \infty,$$

then problem (P) *possesses a solution u in Q_T for some $T > 0$, with u satisfying*

$$0 \leq u(x, t) \leq 2[S(t)\mu](x) + R \leq Ct^{-\frac{N}{\theta}}$$

in Q_T for some $R > 0$ and $C > 0$.

Proof. Set

$$g(\tau) := \tau \int_0^\tau s^{-1} \left(\int_0^s \xi^{p-2}[\log(e + \xi)]^q \, d\xi \right) ds, \quad \tau \geq 0.$$

It follows from Lemma 2.8 (iii) (with $d = 1$ and $R = 0$) that $g(\tau) \asymp \tau^p[\log \tau]^q$ as $\tau \to \infty$. Hence, since either $1 < p < p_\theta$, or $p = p_\theta$ and $q < -1$, we have

$$\int_1^\infty \tau^{-p_\theta-1} g(\tau) \, d\tau \le C \int_1^\infty \tau^{p-p_\theta-1} [\log \tau]^q \, d\tau < \infty. \tag{4.4}$$

Let $\kappa > 0$ and $L > 0$. Set

$$f(\tau) := \kappa g(\tau) + L, \qquad \tau \ge 0. \tag{4.5}$$

Clearly, $f(\tau) \asymp g(\tau) \asymp \tau^p[\log \tau]^q$ as $\tau \to \infty$ and so by (F1)–(F2) we may choose κ and L large enough such that

$$F(\tau) \le f(\tau), \quad \tau \ge 0. \tag{4.6}$$

Now,

$$\left(\frac{f(\tau)}{\tau} \right)' = \kappa \tau^{-1} \left(\int_0^\tau \xi^{p-2} [\log(e + \xi)]^q \, d\xi \right) ds - L\tau^{-2} > 0$$

for all τ large enough ($\tau > R = R(\kappa, L)$). Hence, f satisfies hypothesis (A1) of Theorem 4.1. Furthermore, by (4.4) and (4.5), f also satisfies hypothesis (A2) of Theorem 4.1.

Hence, by Theorem 4.1, there exists $T > 0$ and a solution v in Q_T of problem (P) with F replaced by f, with v satisfying

$$0 \le v(x, t) \le 2[S(t)\mu](x) + R \le Ct^{-\frac{N}{\theta}}$$

in Q_T for some $C > 0$. This together with Lemma 2.3 implies that problem (P) possesses a solution u in Q_T such that

$$0 \le u(x, t) \le v(x, t) \le 2[S(t)\mu](x) + R \le Ct^{-\frac{N}{\theta}}$$

in Q_T. Thus, Corollary 4.1 follows. $\qquad\square$

4.2. Sufficiency: case (B)

We consider nonlinearities F which generalize case (B).

Theorem 4.2. *Let* $\mu \in \mathcal{L}_0$ *and let* F *be an increasing, nonnegative continuous function in* $[0, \infty)$. *Assume that there exist* $R > 0$, $\alpha > 0$, *and positive functions* $G \in C([R, \infty))$ *and* $H \in C^1([R, \infty))$ *satisfying the following conditions* (B1)–(B5)*:*
(B1) $\tau^{-p_\theta} F(\tau) \asymp G(\tau)$ *as* $\tau \to \infty$;
(B2) (i) *for any* $a \ge 1$ *and* $b > 0$, $G(a\tau^b) \asymp G(\tau)$ *as* $\tau \to \infty$. *Furthermore,* (ii) $\lim_{\tau\to\infty} \tau^{-\delta} G(\tau) = 0$ *for all* $\delta > 0$;
(B3) (i) $H'(\tau) \asymp \tau^{-1} G(\tau) > 0$ *and* (ii) $G(\tau H(\tau)^{-1}) \asymp G(\tau)$ *as* $\tau \to \infty$. *Furthermore,* (iii) $\lim_{\tau\to\infty} H(\tau) = \infty$ *and* (iv) $\lim_{\tau\to\infty} \tau^{-\delta} H(\tau) = 0$ *for all* $\delta > 0$;

(B4) *there exists a strictly increasing and convex function Φ_α in $[R, \infty)$ such that*

$$\Phi_\alpha^{-1}(\tau) = \tau H(\tau)^{-\alpha}$$

for all $\tau \in [\Phi_\alpha(R), \infty)$;

(B5) *there exists $\eta \in (0, \theta/N)$ such that the function $P : (R, \infty) \ni \tau \mapsto \tau^\eta H(\tau)^{-\alpha} G(\tau)$ is increasing.*

Then, there exists $\varepsilon > 0$ such that if μ satisfies

$$\sup_{x \in \mathbb{R}^N} \Phi_\alpha^{-1} \left[\fint_{B(x,\sigma)} \Phi_\alpha(\mu(y) + R) \, \mathrm{d}y \right] \leq \varepsilon \sigma^{-N} H(\sigma^{-1})^{-\frac{N}{\theta}} \tag{4.7}$$

for all small enough $\sigma > 0$, then problem (P) possesses a solution u in Q_T for some $T > 0$, with u satisfying

$$0 \leq u(x,t) \leq \Phi_\alpha^{-1}[S(t)\Phi_\alpha(\mu + C)] \leq Ct^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}$$

in Q_T for some $C > 0$.

We prepare a preliminary lemma.

Lemma 4.1. *Let $R > 0$ and $\alpha > 0$. Let G and H be positive functions in $[R, \infty)$ such that $G \in C([R, \infty))$ and $H \in C^1([R, \infty))$. Assume also that conditions (B2)-(i), (B3)-(i), (iii), (iv), and (B4) in Theorem 4.2 hold. Then, for any $a > 0$, $b > 0$, and $c \in \mathbb{R}$,*

$$H(a\tau^b) \asymp H(\tau), \tag{4.8}$$

$$H(\tau^b H(\tau)^c) \asymp H(\tau), \tag{4.9}$$

$$\Phi_\alpha(\tau) \asymp \tau H(\tau)^\alpha, \tag{4.10}$$

as $\tau \to \infty$.

Proof. We first prove (4.8). Consider the case where $a \geq 1$ and $b \geq 1$. By (B3)-(i), we see that H is increasing for large enough τ. Then, we take large enough $R' \in (R, \infty)$ so that $a\tau^b \geq \tau \geq R'$ for $\tau \in [R', \infty)$, $(R'/a)^{1/b} \geq R$, and

$$H(\tau) \leq H(a\tau^b) = \int_{R'}^{a\tau^b} H'(s) \, \mathrm{d}s + H(R') \leq C \int_{R'}^{a\tau^b} s^{-1} G(s) \, \mathrm{d}s + H(R')$$

$$= C \int_{(R'/a)^{1/b}}^{\tau} \xi^{-1} G(a\xi^b) \, \mathrm{d}\xi + H(R') \leq C \int_{R}^{\tau} \xi^{-1} G(a\xi^b) \, \mathrm{d}\xi + H(\tau)$$

for all $\tau \in [R', \infty)$, where $\xi = (s/a)^{1/b}$. Then, by (B2)-(i) and (B3)-(i), (iii) we take large enough $R'' \in (R', \infty)$ so that

$$H(\tau) \leq H(a\tau^b) \leq C \int_{R}^{R''} \xi^{-1} G(a\xi^b) \, \mathrm{d}\xi + C \int_{R''}^{\tau} \xi^{-1} G(a\xi^b) \, \mathrm{d}\xi + H(\tau)$$

$$\leq C + C \int_{R''}^{\tau} \xi^{-1} G(\xi) \, \mathrm{d}\xi + H(\tau) \leq C + C \int_{R''}^{\tau} H'(\xi) \, \mathrm{d}\xi$$

$$+ H(\tau) \leq CH(\tau) + C \leq CH(\tau)$$

for large enough τ. Thus, (4.8) holds for $a \geq 1$ and $b \geq 1$. In particular, we have

$$H(\tau) \asymp H(a\tau) \asymp H(\tau^b) \tag{4.11}$$

as $\tau \to \infty$ for $a \geq 1$ and $b \geq 1$. Then, we see that

$$H(a^{-1}\tau) \asymp H(a \cdot a^{-1}\tau) = H(\tau), \quad H(\tau^{1/b}) \asymp H((\tau^{1/b})^b) = H(\tau), \tag{4.12}$$

as $\tau \to \infty$ for $a \geq 1$ and $b \geq 1$. By (4.11) and (4.12), for any $a > 0$ and $b > 0$, we obtain

$$H(a\tau^b) \asymp H(\tau^b) \asymp H(\tau)$$

as $\tau \to \infty$, and (4.8) holds.

Next, we prove (4.9). Let $\delta > 0$ be such that $b - \delta|c| > 0$. By (B3)-(iii), (iv), we see that $1 \leq H(\tau) \leq \tau^\delta$ for large enough τ. Since H is increasing for large enough τ, we have

$$H(\tau^{b-|c|\delta}) \leq H(\tau^b H(\tau)^c) \leq H(\tau^{b+|c|\delta})$$

as $\tau \to \infty$. This together with (4.8) implies that $H(\tau^b H(\tau)^c) \asymp H(\tau)$ as $\tau \to \infty$, that is, (4.9) holds.

Furthermore, we observe from (B4) and (4.9) (with $b = 1$ and $c = \alpha$) that

$$\Phi_\alpha^{-1}(\tau H(\tau)^\alpha) = \tau H(\tau)^\alpha H(\tau H(\tau)^\alpha)^{-\alpha} \asymp \tau$$

as $\tau \to \infty$. Then, we find $C \geq 1$ such that

$$C^{-1}\tau \leq \Phi_\alpha^{-1}(\tau H(\tau)^\alpha) \leq C\tau$$

for large enough τ, which together with (B4) implies that

$$\Phi_\alpha(C^{-1}\tau) \leq \tau H(\tau)^\alpha \leq \Phi_\alpha(C\tau)$$

for large enough τ. Then, by (4.8) we see that

$$\Phi_\alpha(\tau) \leq C\tau H(C\tau)^\alpha \leq C\tau H(\tau)^\alpha, \qquad \Phi_\alpha(\tau) \geq C^{-1}\tau H(C^{-1}\tau)^\alpha \geq C\tau H(\tau)^\alpha$$

as $\tau \to \infty$, yielding (4.10). The proof of Lemma 4.1 is complete. $\qquad\square$

Proof of Theorem 4.2. Let $\varepsilon \in (0, 1)$ and $L \in (R, \infty)$ be chosen later. Set

$$v(x, t) := [S(t)\Phi_\alpha(\mu + L)](x), \quad w(x, t) := 2\Phi_\alpha^{-1}(v(x, t)), \quad \rho(\tau) := \tau^{-N} H(\tau^{-1})^{-\frac{N}{\theta}}.$$

It follows from (4.8) that

$$\rho(t^{\frac{1}{\theta}}) = t^{-\frac{N}{\theta}} H(t^{-\frac{1}{\theta}})^{-\frac{N}{\theta}} \leq C t^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}} \tag{4.13}$$

for all $t \in (0, T)$ and small enough T. Furthermore, by (B3)-(iv) we see that $\rho(t^{\frac{1}{\theta}}) \to \infty$ as $t \to 0$. We apply Lemmas 2.1 and 4.1 to obtain

$$\Phi_\alpha(L) \le v(x, t) \le Ct^{-\frac{N}{\theta}} \sup_{z \in \mathbb{R}^N} \int_{B(z, t^{\frac{1}{\theta}})} \Phi_\alpha(\mu(y) + L)\, dy \qquad \text{[by Lemma 2.1]}$$

$$\le C\Phi_\alpha\left(\varepsilon\rho(t^{\frac{1}{\theta}})\right) \le C\varepsilon\rho(t^{\frac{1}{\theta}})H\left(\varepsilon\rho(t^{\frac{1}{\theta}})\right)^\alpha \qquad \text{[by 4.7, 4.10]}$$

$$\le C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}}H\left(C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}}\right)^\alpha \quad \text{[by (B3)-(i), 4.13]}$$

$$(4.14)$$

in Q_T for small enough T. Since

$$Ct^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha} \ge C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha} \ge t^{-\frac{N}{2\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha} \to \infty \quad (4.15)$$

as $t \to 0$ (see (B3)-(iii), (iv)), by (B3)-(i), (4.8), (4.9), and (4.14), we have

$$\Phi_\alpha(L) \le v(x, t) \le C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}}H\left(Ct^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}}\right)^\alpha \le C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha}$$

$$(4.16)$$

in Q_T for small enough T. By (B4) and (4.16), we have

$$2L \le w(x, t) \le 2\Phi_\alpha^{-1}\left(C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha}\right)$$

$$= C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha}H\left(C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha}\right)^{-\alpha} \qquad (4.17)$$

in Q_T for small enough T. Since $H^{-\alpha}$ is monotone decreasing for large enough τ, by (4.15) we have

$$H\left(C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha}\right)^{-\alpha} \le H\left(t^{-\frac{N}{2\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha}\right)^{-\alpha}$$

for all $t \in (0, T)$ and small enough T. This together with (4.9) implies that

$$H\left(C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}+\alpha}\right)^{-\alpha} \le CH(t^{-1})^{-\alpha} \qquad (4.18)$$

for all $t \in (0, T)$ and small enough T. By (4.17) and (4.18), we obtain

$$2L \le w(x, t) \le C\varepsilon t^{-\frac{N}{\theta}}H(t^{-1})^{-\frac{N}{\theta}} \qquad (4.19)$$

in Q_T for small enough T. Then, taking large enough L if necessary, by (B1) and (4.10) we have

$$\frac{F(w(x, t))}{v(x, s)} = \frac{F(w(x, t))}{\Phi_\alpha(w(x, t)/2)} \le C\frac{w(x, t)^{p_\theta}G(w(x, t))}{w(x, t)H(w(x, t))^\alpha} = Cw(x, t)^{\frac{\theta}{N}-\eta}P(w(x, t))$$

$$(4.20)$$

in Q_T, where P is as in (B5). Furthermore, by (B5) and (4.19) we obtain

$$P(w(x,t)) \le P\left(C\varepsilon t^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}\right) \le P\left(Ct^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}\right)$$

$$= \left(Ct^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}\right)^{\eta} H\left(Ct^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}\right)^{-\alpha} G\left(Ct^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}\right) \quad (4.21)$$

in Q_T for small enough T. On the other hand, by (B3)-(iv) we see that $t^{-1}H(t^{-1})^{-1} \to \infty$ as $t \to 0$. Then, by (B2)-(i) and (B3)-(ii) we see that

$$G\left(Ct^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}\right) \le CG\left(t^{-1}H(t^{-1})^{-1}\right) \le CG(t^{-1})$$

for all $t \in (0,T)$ and small enough T. This together with (4.8), (4.9), and (4.21) implies that

$$P(w(x,t)) \le C\left(t^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}\right)^{\eta} H(t^{-1})^{-\alpha} G(t^{-1}) \quad (4.22)$$

in Q_T. Since $0 < \eta < \theta/N$ (see (B5)), by (4.19), (4.20), and (4.22) we obtain

$$\frac{F(w(x,t))}{v(x,s)} \le C\left(C\varepsilon t^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}\right)^{\frac{\theta}{N}-\eta} \left(t^{-\frac{N}{\theta}} H(t^{-1})^{-\frac{N}{\theta}}\right)^{\eta} H(t^{-1})^{-\alpha} G(t^{-1})$$

$$\le C\varepsilon^{\frac{\theta}{N}-\eta} t^{-1} H(t^{-1})^{-1-\alpha} G(t^{-1}) \quad (4.23)$$

in Q_T. Therefore, we deduce from (B3)-(i) and (4.23) that

$$\int_0^t \left\| \frac{F(w(s))}{v(s)} \right\|_{\infty} ds \le C\varepsilon^{\frac{\theta}{N}-\eta} \int_0^t s^{-1} H(s^{-1})^{-1-\alpha} G(s^{-1}) ds$$

$$\le C\varepsilon^{\frac{\theta}{N}-\eta} \int_{t^{-1}}^{\infty} \tau^{-1} H(\tau)^{-1-\alpha} G(\tau) d\tau$$

$$\le C\varepsilon^{\frac{\theta}{N}-\eta} \int_{t^{-1}}^{\infty} H(\tau)^{-1-\alpha} H'(\tau) d\tau \quad (4.24)$$

$$\le C\varepsilon^{\frac{\theta}{N}-\eta} H(t^{-1})^{-\alpha}$$

for all $t \in (0,T)$. Similarly, by (4.14) and Lemma 4.1, we have

$$\frac{[S(t)\Phi_\alpha(\mu+L)](x)}{w(x,t)} = \frac{v(x,t)}{2\Phi_\alpha^{-1}(v(x,t))} = \frac{1}{2} H(v(x,t))^{\alpha} \le CH(t^{-1})^{\alpha} \quad (4.25)$$

in Q_T. Therefore, taking small enough $\varepsilon \in (0,1)$, by (4.24) and (4.25) we obtain

$$[S(t)\mu](x) + \int_0^t S(t-s)F(w(s)) ds$$

$$\le \frac{1}{2} w(x,t) + \int_0^t \left\| \frac{F(w(s))}{v(s)} \right\|_{\infty} S(t-s)v(s) ds$$

$$= \frac{1}{2} w(x,t) + w(x,t) \frac{[S(t)\Phi_\alpha(\mu+L)](x)}{w(x,t)} \int_0^t \left\| \frac{F(w(s))}{v(s)} \right\|_{\infty} ds$$

$$\le w(x,t) \left[\frac{1}{2} + C\varepsilon^{\frac{\theta}{N}-\eta}\right] \le w(x,t)$$

in Q_T, where we have used the fact that

$$w(x,t) \geq 2\Phi_\alpha^{-1}\left(S(t)\Phi_\alpha(\mu)\right) \geq 2S(t)\mu$$

by Jensen's inequality. Hence, w is a supersolution in Q_T and Theorem 4.2 now follows from Lemma 2.2 and (4.19). $\qquad\square$

Corollary 4.2. *Let $\mu \in \mathcal{L}_0$ and assume conditions* (F1) *and* (F2) *hold with $p = p_\theta$ and $q \geq -1$. Let $\alpha > 0$ and set*

$$h(\tau) \quad := \begin{cases} (\log(e+\tau))^{q+1} & \text{if } q > -1, \\ \log(e + \log(e+\tau)) & \text{if } q = -1, \end{cases}$$
$$\psi_\alpha^\pm(\tau) := \tau h(\tau)^{\pm\alpha},$$

for $\tau \in (0, \infty)$. Then, there exists $\varepsilon > 0$ such that if μ satisfies

$$\sup_{x \in \mathbb{R}^N} \psi_\alpha^-\left[\fint_{B(x,\sigma)} \psi_\alpha^+(\mu)\,dy\right] \leq \varepsilon \sigma^{-N} h(\sigma^{-1})^{-\frac{N}{\theta}} \tag{4.26}$$

for small enough $\sigma > 0$, then problem (P) *possesses a solution u in Q_T for some $T > 0$, with u satisfying*

$$0 \leq u(x,t) \leq \begin{cases} Ct^{-\frac{N}{\theta}}|\log t|^{-\frac{N(q+1)}{\theta}} & \text{if } q > -1, \\ Ct^{-\frac{N}{\theta}}[\log|\log t|]^{-\frac{N}{\theta}} & \text{if } q = -1 \end{cases}$$

in Q_T for some $C > 0$.

Proof. Let $\alpha > 1$. Set

$$G(\tau) := (\log \tau)^q, \quad H(\tau) := \begin{cases} (\log \tau)^{q+1} & \text{if } q > -1, \\ \log(\log \tau) & \text{if } q = -1, \end{cases} \quad \Psi_\alpha(\tau) := \tau H(\tau)^{-\alpha}.$$

Then, for any $a \geq 1$ and $b > 0$, we have

$$G(a\tau^b) = (\log a + b\log\tau)^q = b^q(\log\tau)^q(1 + o(1))$$
$$\asymp (\log\tau)^q = G(\tau) \quad \text{for any } a \geq 1 \text{ and } b > 0,$$
$$G(\tau) = o(\tau^\delta) \quad \text{for any } \delta > 0,$$

as $\tau \to \infty$. Thus, condition (B2) holds. Let Φ_α be the inverse function of Ψ_α, and we show that conditions (B3)–(B5) in Theorem 4.2 hold.

Consider the case of $q > -1$. Then,

$$H'(\tau) = (q+1)\tau^{-1}(\log\tau)^q,$$
$$\Psi_\alpha'(\tau) = (\log\tau)^{-\alpha(q+1)} - \alpha(q+1)(\log\tau)^{-\alpha(q+1)-1} = (\log\tau)^{-\alpha(q+1)}(1 + o(1)) > 0,$$
$$\Psi_\alpha''(\tau) = -\alpha(q+1)\tau^{-1}(\log\tau)^{-\alpha(q+1)-1} + \alpha(q+1)(\alpha(q+1)+1)\tau^{-1}(\log\tau)^{-\alpha(q+1)-2}$$
$$= -\alpha(q+1)\tau^{-1}(\log\tau)^{-\alpha(q+1)-1}(1 + o(1)) < 0,$$

as $\tau \to \infty$. We see that

$$\tau^{-1}G(\tau) = \tau^{-1}(\log \tau)^q \asymp H'(\tau) \quad \text{as } \tau \to \infty,$$

$$G(\tau H(\tau)^{-1}) = \left[\log(\tau(\log \tau)^{-(q+1)})\right]^q \asymp (\log \tau)^q = G(\tau) \quad \text{as } \tau \to \infty,$$

$$\lim_{\tau \to \infty} H(\tau) = \infty, \qquad H(\tau) = o(\tau^\delta) \quad \text{as } \tau \to \infty \text{ for any } \delta > 0.$$

Thus, condition (B3) holds. Furthermore, we observe that Ψ_α is strictly increasing and concave for large enough τ, that is, the inverse function Φ_α of Ψ_α^{-1} exists and it is strictly increasing and convex for large enough τ. Then, condition (B4) holds. In addition, for any $\eta > 0$, setting

$$P(\tau) = \tau^\eta H(\tau)^{-\alpha} G(\tau) = \tau^\eta (\log \tau)^{-\alpha(q+1)+q},$$

by Lemma 2.6, we see that $P'(\tau) > 0$ for large enough τ. This implies that condition (B5) also holds. Thus, conditions (B3)–(B5) hold in the case of $q > -1$.

Consider the case of $q = -1$. It follows that

$$H'(\tau) = \tau^{-1}(\log \tau)^{-1},$$

$$\Psi_\alpha'(\tau) = (\log(\log \tau))^{-\alpha} - \alpha(\log \tau)^{-1}(\log(\log \tau))^{-\alpha-1} = (\log(\log \tau))^{-\alpha}(1 + o(1)) > 0,$$

$$\Psi_\alpha''(\tau) = -\alpha\tau^{-1}(\log \tau)^{-1}(\log(\log \tau))^{-\alpha-1} + \alpha\tau^{-1}(\log \tau)^{-2}(\log(\log \tau))^{-\alpha-1}$$

$$\qquad + \alpha(\alpha+1)\tau^{-1}(\log \tau)^{-2}(\log(\log \tau))^{-\alpha-2}$$

$$= -\alpha\tau^{-1}(\log \tau)^{-1}(\log(\log \tau))^{-\alpha-1}(1 + o(1)) < 0,$$

as $\tau \to \infty$. Similarly to the case of $q > -1$, we have

$$\tau^{-1}G(\tau) = \tau^{-1}(\log \tau)^{-1} = H'(\tau) \quad \text{as } \tau \to \infty,$$

$$G(\tau H(\tau)^{-1}) = \left[\log(\tau(\log(\log \tau))^{-1})\right]^{-1} \asymp (\log \tau)^{-1} = G(\tau) \quad \text{as } \tau \to \infty,$$

$$\lim_{\tau \to \infty} H(\tau) = \infty, \qquad H(\tau) = o(\tau^\delta) \quad \text{as } \tau \to \infty \text{ for any } \delta > 0.$$

Thus, condition (B3) holds. Furthermore, we see that Ψ_α is strictly increasing and concave for large enough τ, that is, the inverse function Ψ_α^{-1} exists and it is strictly increasing and convex for large enough τ. Then, condition (B4) holds. In addition, for any $\eta > 0$, setting

$$P(\tau) = \tau^\eta H(\tau)^{-\alpha} G(\tau) = \tau^\eta (\log(\log \tau))^{-\alpha}(\log \tau)^{-1},$$

by Lemma 2.6 we see that $P'(\tau) > 0$ for large enough τ. This implies that condition (B5) also holds. Thus, conditions (B3)–(B5) hold in the case of $q = -1$.

Assume (4.26). By Lemma 2.6 (with $a = 1$, $b = -\alpha(q+1)$, and $c = 0$ for $q > -1$ and with $a = 1$, $b = 0$, and $c = -\alpha$ for $q = -1$), we have

$$\Phi_\alpha(\tau) = \Psi_\alpha^{-1}(\tau) \asymp \begin{cases} \tau(\log \tau)^{\alpha(q+1)} & \text{for } q > -1, \\ \tau(\log(\log \tau))^\alpha & \text{for } q = -1, \end{cases}$$

as $\tau \to \infty$. Since

$$\Phi_\alpha^{-1}(\tau) = \Psi_\alpha(\tau) \le C\psi_\alpha^-(\tau), \qquad \Phi_\alpha(\tau) = \Psi_\alpha^{-1}(\tau) \le C\psi_\alpha^+(\tau),$$

for large enough τ, taking large enough $R > 0$ if necessary, we see that

$$\Phi_\alpha^{-1}\left[\fint_{B(x,\sigma)} \Phi_\alpha(\mu(y) + R) \, dy\right] \le C\psi_\alpha^-\left[\fint_{B(x,\sigma)} C\psi_\alpha^+(\mu(y) + R) \, dy\right]. \quad (4.27)$$

Furthermore, we see that

$$\psi_\alpha^+(\tau + R) \le C\psi_\alpha^+(\tau) + C, \quad \psi_\alpha^-(C\tau + C) \le C\psi_\alpha^-(\tau) + C$$

for $\tau > 0$. Then, by (4.26) and (4.27) we see that

$$\Phi_\alpha^{-1}\left[\fint_{B(x,\sigma)} \Phi_\alpha(\mu(y) + R) \, dy\right] \le C\psi_\alpha^-\left[\fint_{B(x,\sigma)} C\psi_\alpha^+(\mu(y)) \, dy + C\right]$$

$$\le C\psi_\alpha^-\left[\fint_{B(x,\sigma)} \psi_\alpha^+(\mu(y)) \, dy\right] + C$$

$$\le C\varepsilon\sigma^{-N}h(\sigma^{-1})^{-\frac{N}{\theta}} + C \le C\varepsilon\sigma^{-N}H(\sigma^{-1})^{-\frac{N}{\theta}}$$

for all small enough $\sigma > 0$.

Let f be as in (4.5). Since $f(\tau) \asymp \tau^{p_\theta}(\log \tau)^q$ as $\tau \to \infty$, condition (B1) holds with F replaced by f. We deduce from Theorem 4.2 that problem (P) with F replaced by f possesses a solution v in Q_T for some $T > 0$ such that

$$0 \le v(x, t) \le \begin{cases} Ct^{-\frac{N}{\theta}} |\log t|^{-\frac{N(q+1)}{\theta}} & \text{if } q > -1, \\ Ct^{-\frac{N}{\theta}} [\log |\log t|]^{-\frac{N}{\theta}} & \text{if } q = -1, \end{cases}$$

for all $(x, t) \in Q_T$. This together with $f(\tau) \ge F(\tau)$ (by (4.6)) and Lemma 2.3 implies that problem (P) possesses a solution u in Q_T such that

$$0 \le u(x, t) \le v(x, t) \le \begin{cases} Ct^{-\frac{N}{\theta}} |\log t|^{-\frac{N(q+1)}{\theta}} & \text{if } q > -1, \\ Ct^{-\frac{N}{\theta}} [\log |\log t|]^{-\frac{N}{\theta}} & \text{if } q = -1, \end{cases}$$

for all $(x, t) \in Q_T$. Thus, Corollary 4.2 follows. $\qquad \square$

4.3. Sufficiency: case (C)

In this section, we consider nonlinearities F which generalize case (C).

Theorem 4.3. *Let $\mu \in \mathcal{L}_0$ and let F be an increasing, nonnegative continuous function in $[0, \infty)$ such that*

(C1) *there exist $R \ge 0$ and $d > 1$ such that the function $(R, \infty) \ni \tau \mapsto \tau^{-d}F(\tau) \in (0, \infty)$ is increasing.*

Furthermore, assume that there exists a continuous function G in $[R, \infty)$ satisfying the following conditions:

(C2) *there exists $p \in [d, d+1)$ such that $G(\tau) \geq \tau^{-p} F(\tau) > 0$ as $\tau \to \infty$;*

(C3) *for any $a \geq 1$, $b > 0$, and $c \in \mathbb{R}$, $G(a\tau^b G(\tau)^c) \asymp G(\tau)$ as $\tau \to \infty$;*

(C4) *there exists $\delta \in (0, 1)$ such that the function $(R, \infty) \ni \tau \mapsto \tau^{-\delta} G(\tau)$ is decreasing.*

Let $\alpha > 1$. Then, there exists $\varepsilon > 0$ such that if μ satisfies

$$\sup_{x \in \mathbb{R}^N} \left[\fint_{B(x,\sigma)} \mu(y)^\alpha \, dy \right]^{\frac{1}{\alpha}} \leq \varepsilon \sigma^{-\frac{\theta}{p-1}} G(\sigma^{-1})^{-\frac{1}{p-1}} \tag{4.28}$$

for small enough $\sigma > 0$, then problem (P) possesses a solution u in Q_T for some $T > 0$, with u satisfying

$$0 \leq u(x, t) \leq 2[S(t)\mu^\alpha](x)^{\frac{1}{\alpha}} + R \leq Ct^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}}$$

in Q_T for some $C > 0$.

Proof. Let $\varepsilon \in (0, 1)$ be chosen later, and assume (4.28). Without loss of generality, we may assume that $\alpha \in (1, d)$. Indeed, if $\alpha \geq d$ and (4.28) holds, then for any $\alpha' \in (1, d)$ we can write $\mu^\alpha = (\mu^{\alpha'})^{\frac{\alpha}{\alpha'}}$ and apply Jensen's inequality to give

$$\sup_{x \in \mathbb{R}^N} \left[\fint_{B(x,\sigma)} \mu(y)^{\alpha'} \, dy \right]^{\frac{1}{\alpha'}} \leq \sup_{x \in \mathbb{R}^N} \left[\fint_{B(x,\sigma)} \mu(y)^\alpha \, dy \right]^{\frac{1}{\alpha}} \leq \varepsilon \sigma^{-\frac{\theta}{p-1}} G(\sigma^{-1})^{-\frac{1}{p-1}}$$

for small enough $\sigma > 0$. Consequently, (4.28) also holds for $\alpha' \in (1, d)$.

Set

$$w(x, t) := 2[S(t)\mu^\alpha](x)^{\frac{1}{\alpha}} + R. \tag{4.29}$$

It follows from (C3), Lemma 2.1, and (4.28) that

$$0 \leq [S(t)\mu^\alpha](x) \leq Ct^{-\frac{N}{\theta}} \sup_{z \in \mathbb{R}^N} \int_{B(z,t^{\frac{1}{\theta}})} \mu(y)^\alpha \, dy$$

$$\leq C\varepsilon^\alpha t^{-\frac{\alpha}{p-1}} G(t^{-\frac{1}{\theta}})^{-\frac{\alpha}{p-1}} \leq C\varepsilon^\alpha t^{-\frac{\alpha}{p-1}} G(t^{-1})^{-\frac{\alpha}{p-1}}$$

in Q_T for small enough T. On the other hand, by (C1) and (C2) we see that

$$\lim_{\tau \to \infty} \tau G(\tau) \geq C \lim_{\tau \to \infty} \tau^{1-p} F(\tau) = C \lim_{\tau \to \infty} \tau^{d+1-p} \tau^{-d} F(\tau) = \infty,$$

since $p < d + 1$. These imply that

$$R \leq w(x, t) \leq R + C\varepsilon t^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}} \leq C\varepsilon t^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}} \tag{4.30}$$

$$\leq Ct^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}} \tag{4.31}$$

in Q_T. Since $1 < \alpha < d$, by (C1)–(C3), (4.30), and (4.31) we obtain

$$\frac{F(w(x,t))}{w(x,t)^\alpha} = w(x,t)^{d-\alpha} \frac{F(w(x,t))}{w(x,t)^d}$$

$$\leq C\left[C\varepsilon t^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}}\right]^{d-\alpha}\left[Ct^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}}\right]^{-d} F\left(Ct^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}}\right)$$

$$\leq C\left[C\varepsilon t^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}}\right]^{d-\alpha}\left[Ct^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}}\right]^{p-d} G\left(Ct^{-\frac{1}{p-1}} G(t^{-1})^{-\frac{1}{p-1}}\right)$$

$$\leq C\varepsilon^{d-\alpha} t^{-\frac{p-\alpha}{p-1}} G(t^{-1})^{\frac{\alpha-1}{p-1}} \tag{4.32}$$

in Q_T for small enough T. Similarly, by (4.31) we have

$$w(x,t)^{\alpha-1} \leq Ct^{-\frac{\alpha-1}{p-1}} G(t^{-1})^{-\frac{\alpha-1}{p-1}} \tag{4.33}$$

in Q_T. On the other hand, by (C4) we see that

$$\int_0^t s^{-\frac{p-\alpha}{p-1}} G(s^{-1})^{\frac{\alpha-1}{p-1}}\, ds = \int_{t^{-1}}^\infty \tau^{\frac{p-\alpha}{p-1}-2} G(\tau)^{\frac{\alpha-1}{p-1}}\, d\tau$$

$$= \int_{t^{-1}}^\infty \tau^{-1-(1-\delta)\frac{\alpha-1}{p-1}} [\tau^{-\delta} G(\tau)]^{\frac{\alpha-1}{p-1}}\, d\tau \tag{4.34}$$

$$\leq C[t^\delta G(t^{-1})]^{\frac{\alpha-1}{p-1}} t^{(1-\delta)\frac{\alpha-1}{p-1}} = Ct^{\frac{\alpha-1}{p-1}} G(t^{-1})^{\frac{\alpha-1}{p-1}}$$

for all $t \in (0, T)$ and small enough T. Therefore, taking small enough ε, by Jensen's inequalities, (4.32), (4.33), and (4.34) we obtain

$$[S(t)\mu](x) + \int_0^t S(t-s)F(w(s))\, ds$$

$$\leq [S(t)\mu^\alpha](x)^{\frac{1}{\alpha}} + C\int_0^t \left\|\frac{F(w(s))}{w(s)^\alpha}\right\|_{L^\infty(\mathbb{R}^N)} S(t-s)[S(s)\mu^\alpha + R^\alpha]\, ds$$

$$\leq \frac{1}{2}w(x,t) + C\varepsilon^{d-\alpha}[S(t)\mu^\alpha + R^\alpha]\int_0^t s^{-\frac{p-\alpha}{p-1}} G(s^{-1})^{\frac{\alpha-1}{p-1}}\, ds$$

$$\leq \frac{1}{2}w(x,t) + C\varepsilon^{d-\alpha} w(x,t)^\alpha \int_0^t s^{-\frac{p-\alpha}{p-1}} G(s^{-1})^{\frac{\alpha-1}{p-1}}\, ds$$

$$\leq \frac{1}{2}w(x,t) + C\varepsilon^{d-\alpha} \left\|w(t)^{\alpha-1}\right\|_{L^\infty(\mathbb{R}^N)} t^{\frac{\alpha-1}{p-1}} G(t^{-1})^{\frac{\alpha-1}{p-1}} w(x,t)$$

$$\leq w(x,t)\left[\frac{1}{2} + C\varepsilon^{d-\alpha}\right] \leq w(x,t)$$

in Q_T. Hence, w is a supersolution in Q_T and Theorem 4.3 now follows from Lemma 2.2, (4.29), and (4.30). $\qquad\square$

Corollary 4.3. *Let $\mu \in \mathcal{L}_0$ and assume conditions* (F1) *and* (F2) *hold. For any $\alpha > 1$, there exists $\varepsilon > 0$ such that if μ satisfies*

$$\sup_{x\in\mathbb{R}^N} \left[\fint_{B(x,\sigma)} \mu(y)^\alpha\, dy\right]^{\frac{1}{\alpha}} \leq \varepsilon\sigma^{-\frac{\theta}{p-1}} |\log\sigma|^{-\frac{q}{p-1}} \tag{4.35}$$

for all small enough $\sigma > 0$, then problem (P) *possesses a solution u in Q_T for some $T > 0$, with u satisfying*

$$0 \leq u(x, t) \leq 2[S(t)\mu^\alpha](x)^{\frac{1}{\alpha}} + R \leq Ct^{-\frac{1}{p-1}}|\log t|^{-\frac{q}{p-1}}$$

in Q_T for some $R, C > 0$.

Proof. Let $d \in (1, p)$ with $d > p - 1$. Let $\kappa, L > 0$, and set

$$f(\tau) := \kappa\tau^d \int_0^\tau s^{-d} \left(\int_0^s \xi^{p-2}[\log(e + \xi)]^q \, d\xi \right) ds + L \qquad (4.36)$$

for $\tau \in (0, \infty)$. It follows from Lemma 2.8 (iii) that

$$\tau^d \int_0^\tau s^{-d} \left(\int_0^s \xi^{p-2}[\log(e + \xi)]^q \, d\xi \right) ds \asymp \tau^p[\log \tau]^q \qquad (4.37)$$

as $\tau \to \infty$. We take large enough κ and L so that $F(\tau) \leq f(\tau)$ in $[0, \infty)$. On the other hand, since

$$\left(\frac{f(\tau)}{\tau^d} \right)' = \tau^{-d} \left[\kappa \left(\int_0^\tau \xi^{p-2}[\log(e + \xi)]^q \, d\xi \right) ds - L\tau^{-1}d \right] > 0$$

for large enough τ, condition (C1) in Theorem 4.3 holds in (R, ∞) with F replaced by f for some $R > 0$.

Taking large enough R if necessary and setting $G(\tau) := (\log \tau)^q$ for $\tau \in (R, \infty)$, we see that the function $(R, \infty) \ni \tau \mapsto \tau^{-\frac{1}{2}}G(\tau)$ is decreasing (i.e., $\delta = 1/2$ in (C4)). By (4.36) and (4.37), we find $C > 0$ such that

$$\tau^{-p}f(\tau) \leq CG(\tau)$$

for all $\tau \in (R, \infty)$. Then, conditions (C2)–(C4) in Theorem 4.3 hold with F replaced by f. Therefore, by Theorem 4.3 there exists $\varepsilon > 0$ such that if μ satisfies (4.35), then problem (P) with F replaced by f possesses a solution v in Q_T for some $T > 0$ such that

$$0 \leq v(x, t) \leq 2[S(t)\mu^\alpha](x)^{\frac{1}{\alpha}} + R \leq Ct^{-\frac{1}{p-1}}|\log t|^{-\frac{q}{p-1}}$$

in Q_T, for some $C > 0$. This together with Lemma 2.3 implies that problem (P) possesses a solution u in Q_T such that

$$0 \leq u(x, t) \leq v(x, t) \leq 2[S(t)\mu^\alpha](x)^{\frac{1}{\alpha}} + R \leq Ct^{-\frac{1}{p-1}}|\log t|^{-\frac{q}{p-1}}$$

in Q_T. Thus, Corollary 4.3 follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.4. A special case: Dirac measure as initial data

Here, we provide a necessary and sufficient condition on the nonlinearity F for the solvability of problem (P) in the special case when $\mu = \delta_y$, the Dirac measure in $\mathbb{R}^N$ based at point y. This problem was considered in [7] for the opposite sign pure power law case $F(u) = -u^p$, i.e., dissipative F.

Corollary 4.4. *Suppose F satisfies*

(D1) *F is nonnegative and locally Lipschitz continuous in $[0, \infty)$;*
(D2) *there exist $R > 0$ and $d > 1$ such that*
   (i) *the function $(R, \infty) \ni \tau \mapsto \tau^{-d} F(\tau) \in (0, \infty)$ is increasing;*
   (ii) *F is convex in (R, ∞).*

Let $y \in \mathbb{R}^N$. Then, problem (P) possesses a local-in-time solution with $\mu = \delta_y$ if and only if

$$\int_1^\infty \tau^{-p_\theta - 1} F(\tau) \, d\tau < \infty. \tag{4.38}$$

Proof. Assume that problem (P) possesses a solution with $\mu = \delta_y$ in Q_T for some $T > 0$. Set

$$f(\tau) := 0 \quad \text{for} \quad 0 \le \tau \le R, \quad f(\tau) := F(\tau) - \tau^d R^{-d} F(R) \quad \text{for} \quad \tau > R \tag{4.39}$$

Then, by (D2)-(i) we see that f is increasing and $F \ge f$ in $[0, \infty)$. Applying Theorem 3.1 with $z = y$, so that $m_\sigma(z) = \delta_y(B(y, \sigma)) \equiv 1$, we find $\gamma \ge 1$ such that

$$\int_{\gamma^{-1} T^{-\frac{N}{\theta}}}^{\gamma^{-1} \sigma^{-N}} s^{-p_\theta - 1} f(s) \, ds \le \gamma^{p_\theta + 1}, \qquad 0 < \sigma < T^{\frac{1}{\theta}}.$$

Letting $\sigma \to 0$, we have

$$\int_{\gamma^{-1} T^{-\frac{N}{\theta}}}^\infty s^{-p_\theta - 1} f(s) \, ds \le \gamma^{p_\theta + 1}.$$

This together with (4.39) implies (4.38).

Conversely, under condition (4.38), we apply Theorem 4.1 to obtain a local-in-time solution of problem (P) with $\mu = \delta_y$. Thus, Corollary 4.4 follows. $\square$

We mention that the integral condition (4.38) also appears in [30, Theorem 5.1] as a necessary and sufficient condition for existence with L^1 initial data. See also the informal argument preceding the proof of Theorem 4.1 of that work, where a Dirac delta function is considered as initial data.

5. Proof of the main theorem

Proof of Theorem 1.1. Assertion (i) is proved by Corollary 3.1 (ii), Remark 2.1, and Corollary 4.1.

We now prove the nonexistence parts of statements (1) and (2) in assertion (ii). Suppose first that (1.3) holds and there exists a local solution of problem (P). Then,

$$\sup_{z\in\mathbb{R}^N}\mu(B(z,\sigma)) \geq \gamma_1 \int_{B(0,\sigma)} |x|^{-N}|\log|x||^{-1}[\log|\log|x||]^{-\frac{N}{\theta}-1}\,dx \geq C_1\gamma_1[\log|\log\sigma|]^{-\frac{N}{\theta}}$$

for small enough $\sigma > 0$. For large enough γ_1, we then obtain a contradiction to Corollary 3.1. Hence, no local solution can exist for such γ_1. Now suppose that (1.5) holds. Then, there exists $C_2 > 0$ such that

$$\sup_{z\in\mathbb{R}^N}\mu(B(z,\sigma)) \geq \gamma_2 \int_{B(0,\sigma)} |x|^{-N}|\log|x||^{-\frac{N(q+1)}{\theta}-1}\,dx \geq C_2\gamma_2|\log\sigma|^{-\frac{N(q+1)}{\theta}}$$

for small enough $\sigma > 0$. Again, we can obtain a contradiction to Corollary 3.1 for large enough γ_2 and deduce that problem (P) possesses no local-in-time solution for such γ_2.

Next, we prove the existence parts of statements (1) and (2) in assertion (ii). Assume therefore that either (1.4) with $\varepsilon_1 \in (0,1)$ or (1.6) with $\varepsilon_2 \in (0,1)$ hold. Let $\alpha > 0$ and set

$$h(\tau) := \begin{cases} \log(e+\log(e+\tau)) & \text{if } 1.4 \text{ holds,} \\ (\log(e+\tau))^{q+1} & \text{if } 1.6 \text{ holds,} \end{cases} \qquad \psi_\alpha^\pm(\tau) := \tau h(\tau)^{\pm\alpha}.$$

If (1.4) holds, then

$$\psi_\alpha^+(\mu(x)) \leq C\mu(x)\log(e+\log(e+\mu(x)))^\alpha$$
$$\leq C\varepsilon_1|x|^{-N}|\log|x||^{-1}[\log|\log|x||]^{-\frac{N}{\theta}-1+\alpha}\chi_{B(0,R)}(x) + C, \quad x \in \mathbb{R}^N.$$

This implies that

$$\sup_{x\in\mathbb{R}^N} \psi_\alpha^-\left[\fint_{B(x,\sigma)} \psi_\alpha^+(\mu)\,dy\right] \leq \psi_\alpha^-\left(C\varepsilon_1\sigma^{-N}[\log|\log\sigma|]^{-\frac{N}{\theta}+\alpha}\right)$$
$$\leq C\varepsilon_1\sigma^{-N}[\log|\log\sigma|]^{-\frac{N}{\theta}} \leq C\varepsilon_1\sigma^{-N}h(\sigma^{-1})^{-\frac{N}{\theta}}$$

for small enough $\sigma > 0$.

Similarly, if (1.6) holds, then

$$\psi_\alpha^+(\mu(x)) \leq C\mu(x)[\log(e+\mu(x))]^{\alpha(q+1)}$$
$$\leq C\varepsilon_2|x|^{-N}|\log|x||^{-\frac{N(q+1)}{\theta}-1+\alpha(q+1)}\chi_{B(0,R)}(x) + C, \quad x \in \mathbb{R}^N.$$

This implies that

$$\sup_{x\in\mathbb{R}^N} \psi_\alpha^-\left[\fint_{B(x,\sigma)} \psi_\alpha^+(\mu)\,dy\right] \leq \psi_\alpha^-(C\varepsilon_2\sigma^{-N}|\log\sigma|^{-\frac{N(q+1)}{\theta}+\alpha(q+1)})$$
$$\leq C\varepsilon_2\sigma^{-N}|\log\sigma|^{-\frac{N(q+1)}{\theta}} \leq C\varepsilon_2\sigma^{-N}h(\sigma^{-1})^{-\frac{N}{\theta}}$$

for small enough $\sigma > 0$. Therefore, by Corollary 4.2 we see that, if $\varepsilon_1 > 0$ (respectively $\varepsilon_2 > 0$) is small enough, then problem (P) possesses a local-in-time solution. Thus, statements (1) and (2) in assertion (ii) follow.

Finally, we prove statement (3) in assertion (ii). Assume that (1.7) holds. Then,

$$\sup_{z \in \mathbb{R}^N} \mu(B(z, \sigma)) \geq \gamma_3 \int_{B(0,\sigma)} |x|^{-\frac{\theta}{p-1}} |\log |x||^{-\frac{q}{p-1}} \, dx \geq C_1 \gamma_3 \sigma^{N - \frac{\theta}{p-1}} |\log \sigma|^{-\frac{q}{p-1}}$$

for small enough $\sigma > 0$. This together with Corollary 3.1 implies that problem (P) possesses no local-in-time solution for large enough γ_3. Conversely, suppose that (1.8) holds. Since $p > p_\theta$, we find $\alpha > 1$ such that $\alpha\theta/(p-1) < N$. Then, we have

$$\sup_{x \in \mathbb{R}^N} \left[\fint_{B(x,\sigma)} \mu(y)^\alpha \, dy \right]^{\frac{1}{\alpha}} \leq \left[C\varepsilon_3^\alpha \sigma^{-\frac{\alpha\theta}{p-1}} |\log \sigma|^{-\frac{\alpha q}{p-1}} + C K_3^\alpha \right]^{\frac{1}{\alpha}}$$

$$\leq C\varepsilon_3 \sigma^{-\frac{\theta}{p-1}} |\log \sigma|^{-\frac{q}{p-1}}$$

for small enough $\sigma > 0$. By Corollary 4.3, we see that, if $\varepsilon_3 > 0$ is small enough, then problem (P) possesses a local-in-time solution. Thus, statement (3) in assertion (ii) follows. The proof is complete. $\qquad\square$

Remark 5.1. The arguments in the proof of Theorem 1.1 are readily adapted to further log-refinements. For example, suppose that (F2) is replaced by

(F2′) $F(\tau) \asymp \tau^p [\log \tau]^q [\log(\log \tau)]^r$ as $\tau \to \infty$ for some $p > 1$ and $q, r \in \mathbb{R}$.

Then, we can show that problem (P) possesses a local-in-time solution if and only if

$$\sup_{z \in \mathbb{R}^N} \mu(B(z, 1)) < \infty$$

in the cases when (i) $p < p_\theta$, (ii) $p = p_\theta$ and $q < -1$, and (iii) $p = p_\theta$, $q = -1$, and $r < -1$. In the other cases, we divide condition (F2′) into four cases:

(1) $p = p_\theta$ and $q = r = -1$.
(2) $p = p_\theta$, $q = -1$, and $r > -1$.
(3) $p = p_\theta$, $q > -1$, and $r \in \mathbb{R}$.
(4) $p > p_\theta$ and $q, r \in \mathbb{R}$,

and we can identify the optimal singularities of the initial data for solvability of problem (P). Since inclusion of the proofs here would make the paper unduly long, we leave the details to the interested reader.

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