Morphing Hexagonal Frameworks

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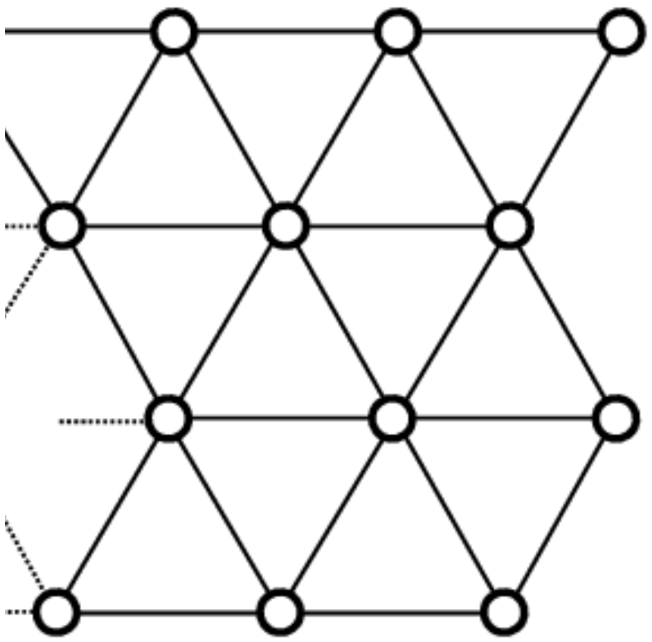
Vision

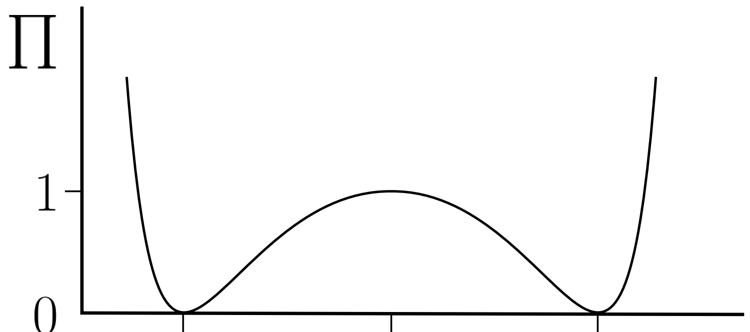
- To develop mathematical tools to analyse and design morphing structures.
- In particular, we want to reduce dependence on brute-force computations.
- In this talk I'll introduce our approach in a simple context:
  - Planar Triangular Bistable-Edge Frameworks

Planar Triangular Bistable-Edge Frameworks

- Framework:
  - Edges undergo only axial deformation.
  - Edges are hinged at the nodes.
- Bistable edges:
  - All edges are of length either 1 or $L \in (\frac{1}{2}, 1)$
- Planar
  - The framework remains planar at all times.

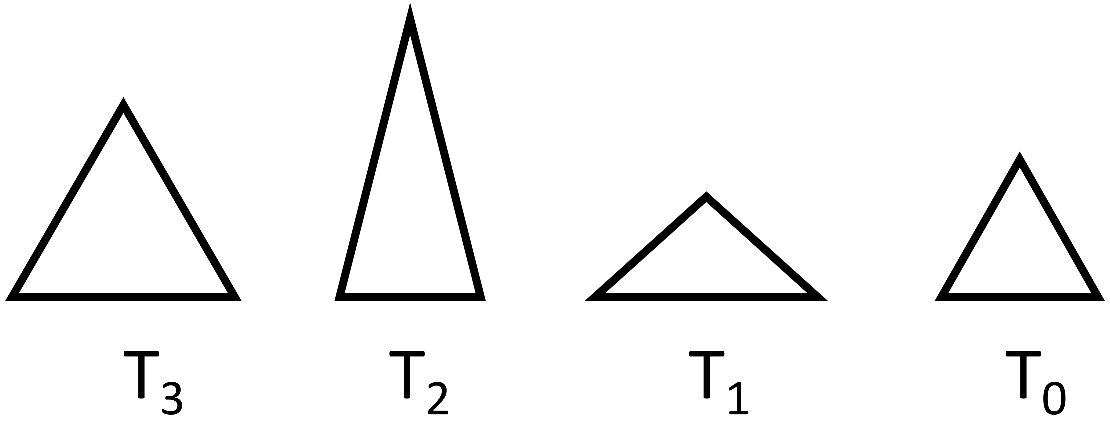
Question: What are the zero-energy states?





Zero-Energy States for Triangles

- Consider a triangle with bistable edges
  - The edges can be of length either 1 or L
- Any combination of side lengths is possible because $L \in (\frac{1}{2}, 1)$

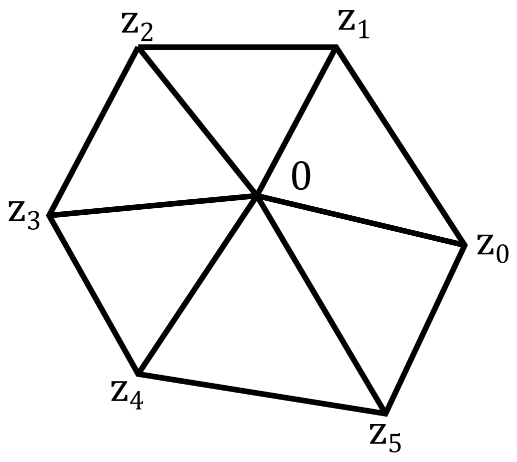


Note: T_0 and T_3 are similar triangles.

- Nomenclature:
  - Triangle T_n has n edges of side 1

Zero-Energy States for Hexagons

- Question: Which combinations of un/transformed edges form hexagons?
- Planarity: The six central angles should add to 2π.
- There are 2^{12} = 4096 possibilities to search.
  - Computationally feasible.
  - But not a good strategy for realistic frameworks.
  - See O'Donnell et al, Frontiers in Materials, 2020 for complete cl energy hexagons based on semi-analytic method.

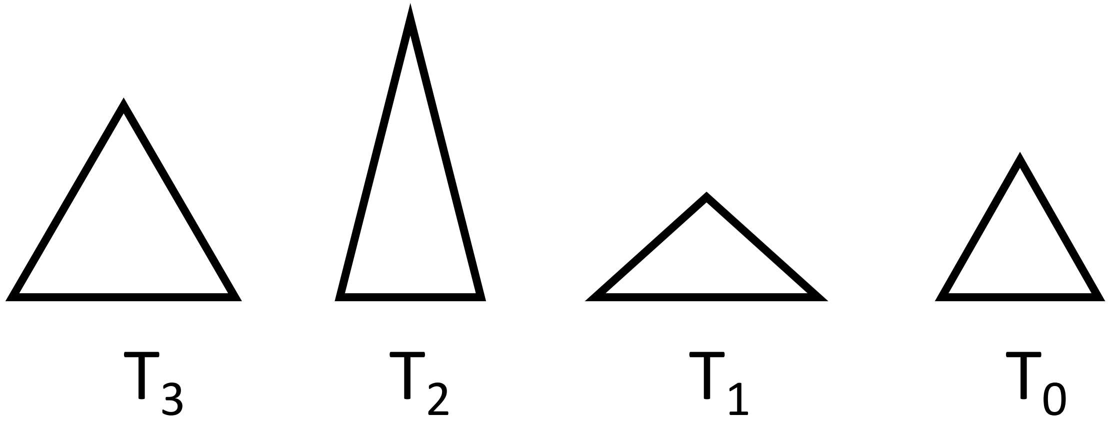


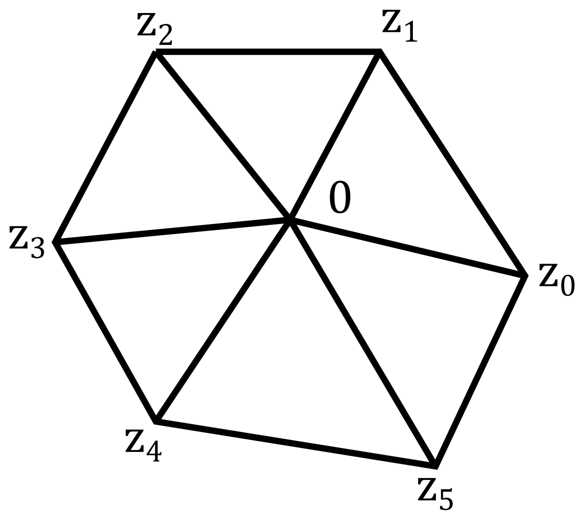
Goal: To develop tools to study such frameworks and not rely only on brute-force computation.

Lemma 1 (IC-MO'D, 2022)

For generic L, a zero-energy hexagon contains at most two elements from

$\{ \{T_0, T_3\}, T_1, T_2 \}$.



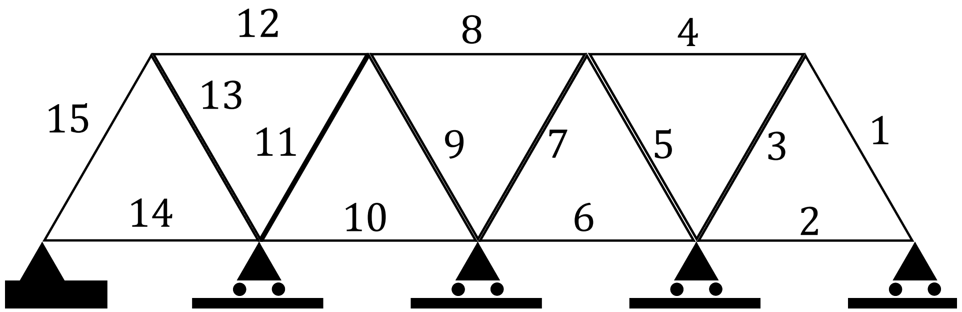


For generic L, a zero-energy hexagon contains at most two elements from $\{ \{T_0, T_3\}, T_1, T_2 \}$.

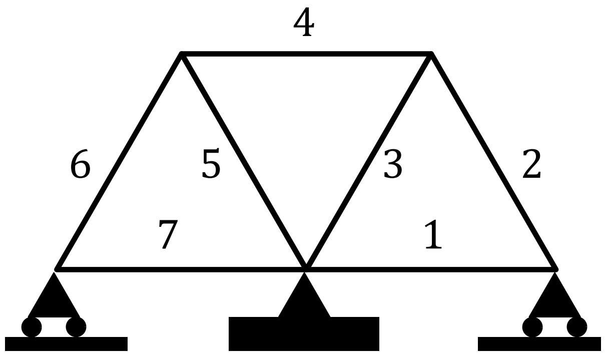
- The interior angles of a triangle add to π.
- So three copies each of two triangles would form a hexagon, provided the sides match.
- It turns out that this is also the only way to form a hexagon, with two exceptions:
  - For two values of L ($\sqrt{2-\sqrt{3}} = 0.51762$ and 0.6180) other hexagons are possible.
  - Because T_0 and T_3 are similar they are interchangeable as far as angles are concerned, so we can have a hexagon with T_0, T_3 and T_1 or T_2.

Boundary Effects

- What happens at a boundary that is constrained to remain a straight line?
  - For example:



- The smallest non-trivial 'unit cell' is a half-hexagon:
  - With the constraint that the bottom edges form a straight line.



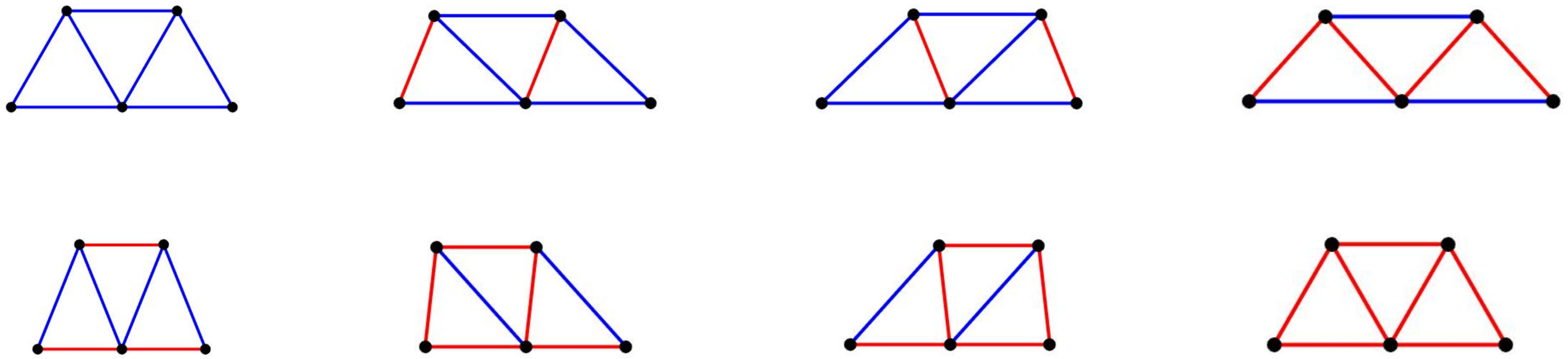
- Once again we can computationally investigate the $2^7 = 128$ possibilities…

- But we can do better!

Lemma 2 (IC-MO'D, 2022)

For generic L, a zero-energy half-hexagon contains (3 copies of) precisely one of the triangles T_0, T_1, T_2, T_3.

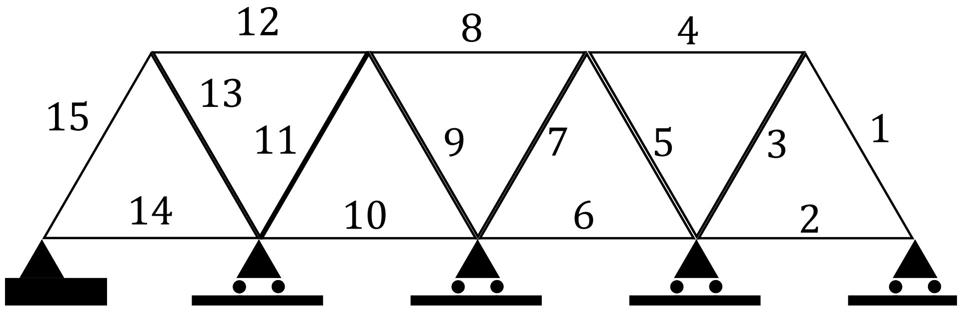
There are 8 possibilities:

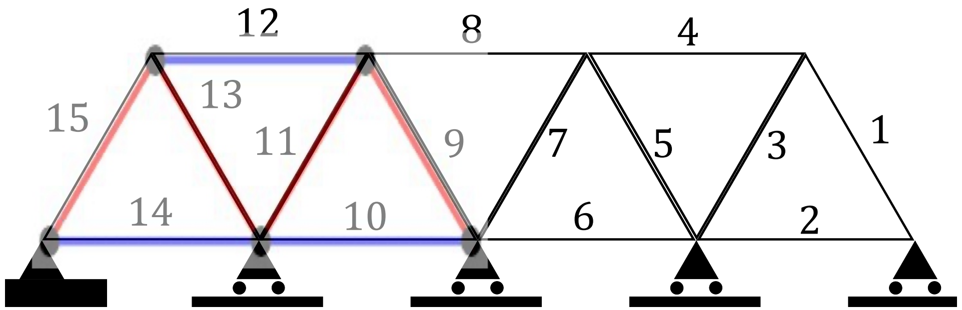


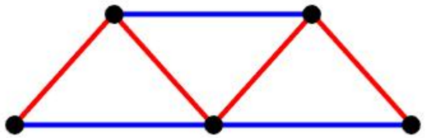
Note that the bottom edges are both un/transformed.

Corollary: Straight boundaries are hard constraints!

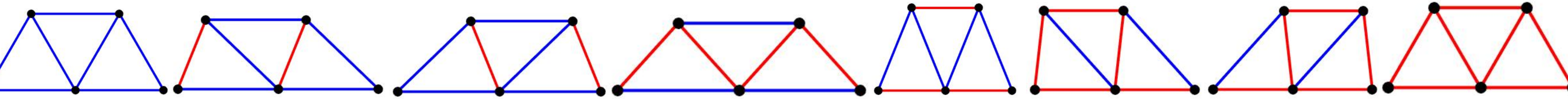
- All triangles attached to a straight boundary are identical:





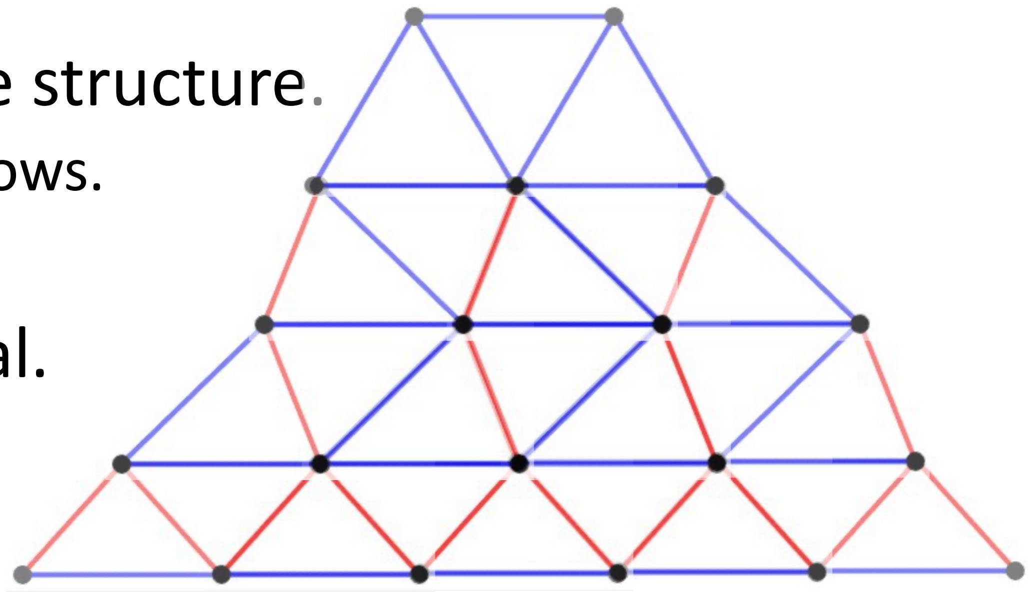


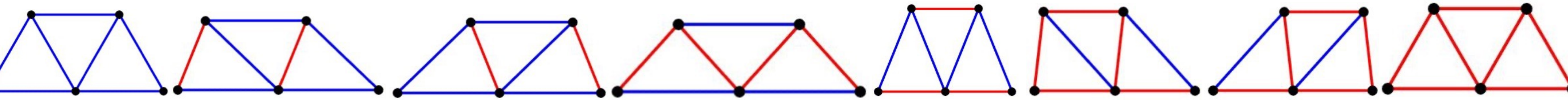
- The entire boundary is un/transformed.



Corollary: Straight boundaries are a hard constraint!

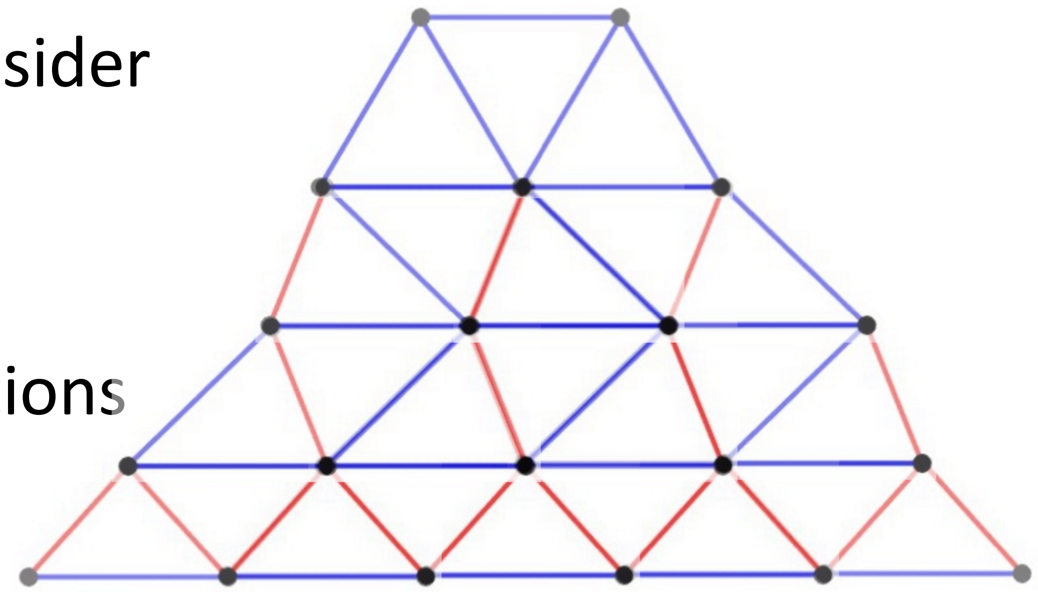
- The un/transformation is propagated into the structure.
  - An n-edge straight boundary propagates to n-1 rows.
- The triangles in each interior row are identical.
  - And there are at most 4 kinds of triangles.
- Can easily calculate the possible homogenised strains in such a region.

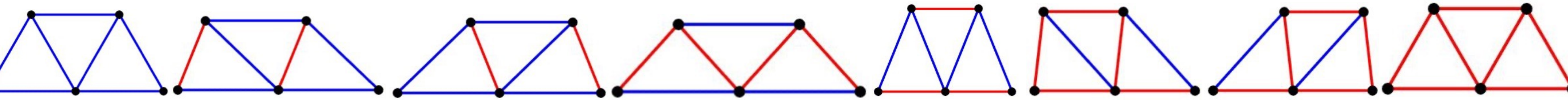




Straight boundaries: Summary

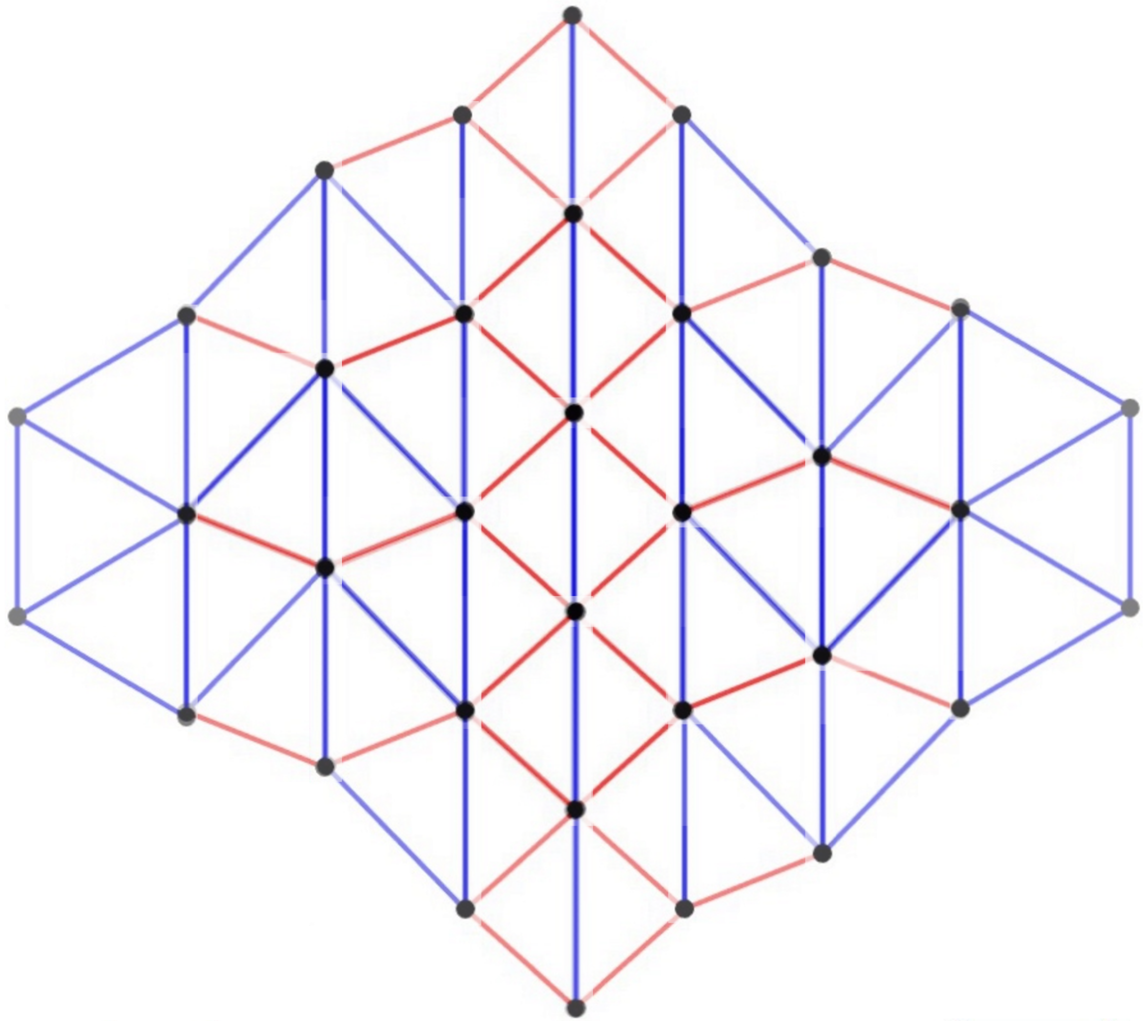
- Our analysis completely characterises the $2^{2n-1} = 2^9 = 512$ zero-energy states of this region.
- A brute-force approach would have had to consider $2^{3(n-1)(n+2)/2+1} = 2^{43} > 8*10^{12}$ possibilities.
- A database can be developed for straight line boundaries of various lengths so the computations need not be repeated.

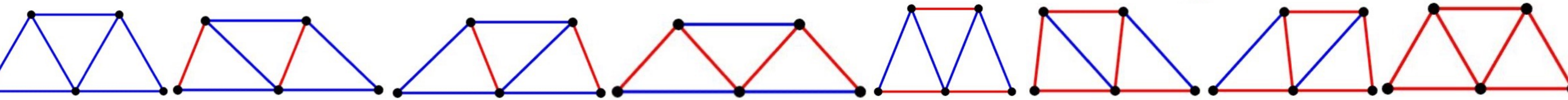




Similarly for straight lines within a structure

- The un/transformation is propagated into the structure.
  - An n-edge straight line propagates to 2(n-1) rows.
  - As before, can characterise all zero-energy states.
  - The efficiency in contrast to brute-force computation is even more pronounced in this case.





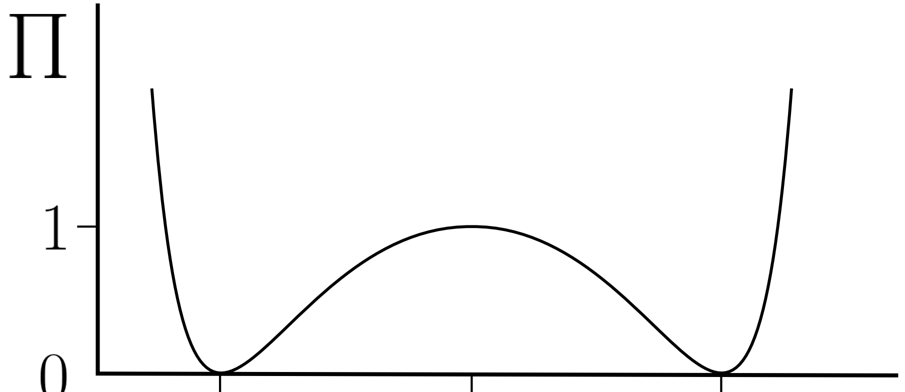
Actuation

- Changing the length of an edge requires that an energy barrier be overcome.

- In practise, we would not do this for every edge whose length we want to change.

- Rather we would do this for some edges – by using actuators – and allow other edges to passively change length.

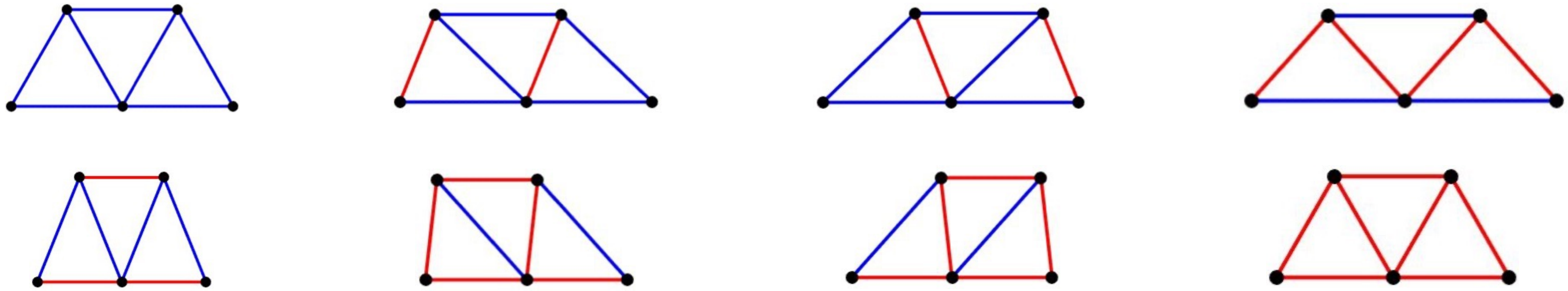
- Questions:
  - How many actuators do we need?
  - Where should they be located?



Observation 3 (IC-MO'D, 2022)

All 8 zero-energy half-hexagon can be accessed via $\log_2 8 = 3$ actuators.

(Not accounting for transmission losses.)



Need one actuator each for:

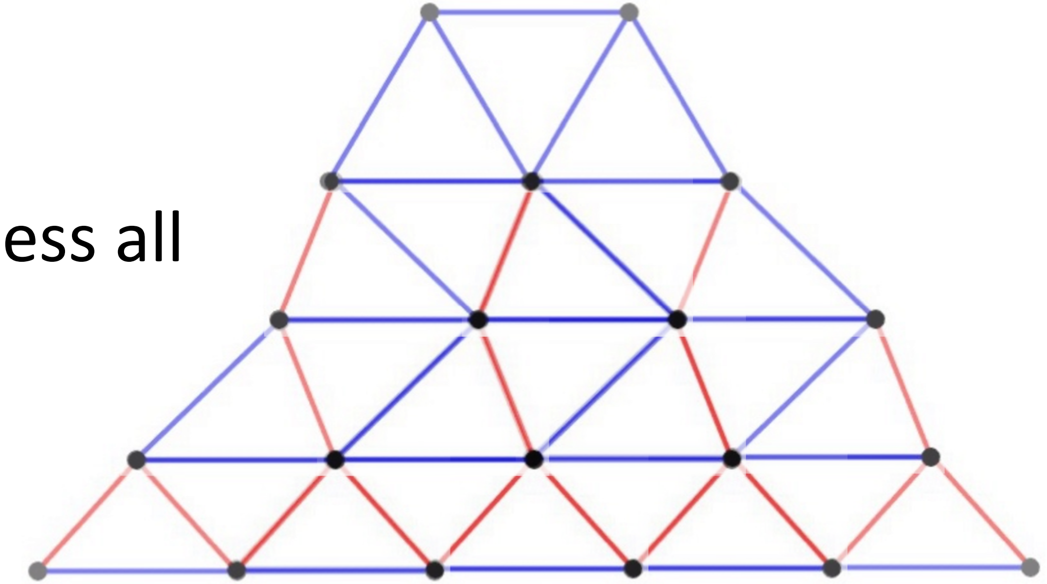
- All horizontal edges
- All right-leaning edges
- All left-leaning edges

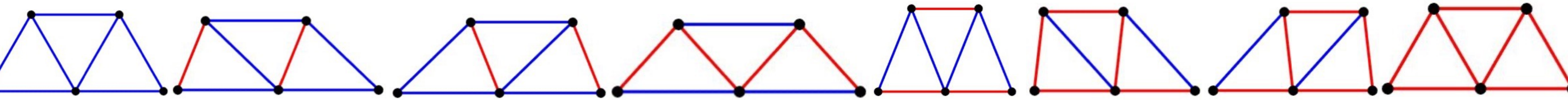
Activation at straight boundaries

- The ‘region of influence’ of an n-edge straight boundary
  - Needs at most 2n+1 actuators
  - To access 2^{2n+1} zero-energy states.

- (Far) Fewer actuators would be needed to access all possible zero-energy homogenised strains.

- Similarly for straight lines within a structure.



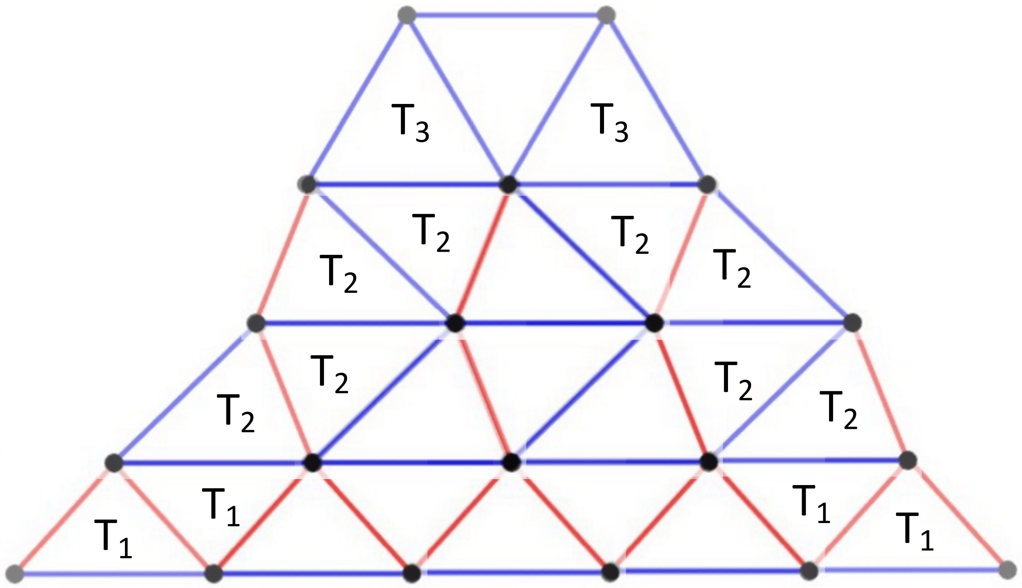


Idea being investigated: Skeleton of actuators

- Consider this 'region of influence' of a straight boundary.
  - One actuator controls all horizontal edges.
  - Each side edge on the boundary of this region needs a separate actuator.

- From Lemma 1 these actuators influence the rest of the structure.

- From Lemma 2 the influence on the two sides is the same.



Lemma 1: For generic L, a zero-energy hexagon contains at most two elements from $\{ \{T_0, T_3\}, T_1, T_2 \}$

Conclusions

- Geometric analysis of morphing frameworks appears to be a promising endeavour.
- Planes and straight lines are useful constraints.
- Yet to be done: Morphing surfaces
  - 3D deformations of such 2D frameworks