The bondage number of graphs on topological surfaces and Teschner's conjecture

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Abstract

The bondage number of a graph is the smallest number of its edges whose removal results in a graph having a larger domination number. We provide constant upper bounds for the bondage number of graphs on topological surfaces, and improve upper bounds for the bondage number in terms of the maximum vertex degree and the orientable and non-orientable genera of graphs. Also, we present stronger upper bounds for graphs with no triangles and graphs with the number of vertices larger than a certain threshold in terms of graph genera. This settles Teschner's Conjecture in affirmative for almost all graphs. As an auxiliary result, we show tight lower bounds for the number of vertices of graphs 2-cell embeddable on topological surfaces of a given genus.

Key words: Bondage number, Domination number, Topological surface, Embedding on a surface, Euler's formula, Triangle-free graphs

1. Introduction

We consider simple finite non-empty graphs. For a graph G, its vertex and edge sets are denoted, respectively, by $V(G)$ and $E(G)$, $|V(G)| = n$ and $|E(G)| = m$. A graph is *trivial* if it has only one vertex. We also use the following standard notation: $d(v)$ for the degree of a vertex v in G, $\Delta = \Delta(G)$ for the maximum vertex degree of G, $\delta = \delta(G)$ for the minimum vertex degree of G, and $N(v)$ for the neighbourhood of a vertex v in G.

A set $D \subseteq V(G)$ is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$. Clearly, for any spanning subgraph H of G, $\gamma(H) \geq \gamma(G)$. The *bondage number* of G, denoted by $b(G)$, is the minimum cardinality of a set of edges $B \subseteq E(G)$ such that $\gamma(G - B) > \gamma(G)$, where $V(G - B) = V(G)$ and $E(G - B) = E(G)\backslash B$. In a sense, the bondage number $b(G)$ measures integrity and

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reliability of the domination number $\gamma(G)$ with respect to the edge removal from G, which may correspond, for example, to link failures in communication networks.

The bondage number was introduced by Bauer et al. [1] (see also Fink et al. [5]). Recently, it has been shown by Hu and Xu [11] that the decision problem for the bondage number is NP-hard. Also, they have conjectured that determining an actual set of edges corresponding to the bondage number is not even an NP-problem, which implies it is important to have any reasonable estimations and bounds on the bondage number in terms of other graph parameters and properties. Two unsolved classical conjectures for the bondage number of arbitrary and planar graphs are given below.

Conjecture 1 (Teschner [20]). *For any graph G,*

$$b(G) \leq \frac{3}{2}\Delta(G).$$

Hartnell and Rall [9] and Teschner [21] showed that for the cartesian product $G_n = K_n \times K_n$, $n \geq 2$, the bound of Conjecture 1 is sharp, i.e. $b(G_n) = \frac{3}{2}\Delta(G_n)$. Teschner [20] also proved that Conjecture 1 holds when $\gamma(G) \leq 3$.

Conjecture 2 (Dunbar et al. [4]). *If G is a planar graph, then*

$$b(G) \leq \Delta(G) + 1.$$

Trying to prove Conjecture 2, Kang and Yuan [15] have shown the following, with a simpler topological proof later discovered by Carlson and Develin [3].

Theorem 3 ([15, 3]). *For any connected planar graph G,*

$$b(G) \leq \min\{8,\ \Delta(G) + 2\}.$$

This solves Conjecture 2 when $\Delta(G) \geq 7$, and Conjecture 1 for planar graphs with $\Delta(G) \geq 4$. Also, it is shown in [3] that $b(G) \leq \Delta(G) + 3$ for any connected toroidal graph G, which solves Conjecture 1 for toroidal graphs with $\Delta(G) \geq 6$. In [8], we generalized this for any topological surface as follows.

Theorem 4 ([8]). *For a connected graph G of orientable genus h and non-orientable genus k,*

$$b(G) \leq \min\{\Delta(G) + h + 2,\ \Delta(G) + k + 1\}. \tag{1}$$

Also, in [8], we indicated that the upper bound (1) can be improved for larger values of the genera h and k by adjusting the proofs and stated the following general conjecture.

Conjecture 5 ([8]). *For a connected graph G of orientable genus h and non-orientable genus k,*

$$b(G) \leq \min\{c_h,\ c'_k,\ \Delta(G) + o(h),\ \Delta(G) + o(k)\},$$

where c_h and c'_k are constants depending, respectively, on the orientable and non-orientable genera of G.

Notice that it is sufficient to consider connected graphs because the bondage number of a disconnected graph G is the minimum of the bondage numbers of its components.

In this paper, we provide constant upper bounds for the bondage number of graphs on topological surfaces, which can be used as the first estimation for the constants c_h and c'_k of Conjecture 5. We also explicitly improve upper bounds of Theorem 4, and give tight lower bounds for the number of vertices of graphs 2-cell embeddable on topological surfaces of a given genus. Moreover, we present stronger upper bounds for graphs with no triangles and graphs with the number of vertices larger than a certain threshold in terms of genera h and k. This shows that, for almost all graphs, the bondage number is at most eleven and also settles Conjecture 1 in affirmative for almost all graphs.

2. Graphs on the topological surfaces

The planar graphs are precisely the graphs that can be drawn with no crossing edges on the sphere S_0. A topological surface S can be obtained from the sphere S_0 by adding a number of handles or crosscaps. If we add h, $h \geq 1$, handles to S_0, we obtain an orientable surface S_h, which is often referred to as the *h-holed torus*. The number h is called the *orientable genus* of S_h. If we add k, $k \geq 1$, crosscaps to the sphere S_0, we obtain a non-orientable surface N_k. The number k is called the *non-orientable genus* of N_k. Any topological surface is homeomorphically equivalent either to S_h ($h \geq 0$), or to N_k ($k \geq 1$). For example, S_1, N_1, N_2 are the *torus*, the *projective plane*, and the *Klein bottle*, respectively.

A graph G is *embeddable* on a topological surface S if it admits a drawing on the surface with no crossing edges. Such a drawing of G on the surface S is called an *embedding* of G on S. Notice that there can be many different embeddings of the same graph G on a particular surface S. The embeddings can be distinguished and classified by different properties. The set of faces of a particular embedding of G on S is denoted by $F(G)$, $|F(G)| = f$.

An embedding of G on the surface S is a *2-cell embedding* if each face of the embedding is homeomorphic to an open disk. In other words, a 2-cell embedding is an embedding on S that "fits" the surface. This is expressed in Euler's formula (2) of Theorem 6 below. For example, a cycle C_n ($n \geq 3$) does not have a 2-cell embedding on the torus, but it has 2-cell embeddings on the sphere and the projective plane. Similarly, a planar graph may have 2-cell and non-2-cell embeddings on the torus. An algorithm to transform a planar 2-cell embedding into a toroidal 2-cell embedding, whenever possible, can be found in Gagarin et al. [7], pp. 358–360. Similar algorithms to transform a 2-cell embedding of genus h, $h \geq 1$, (resp., k, $k \geq 1$) into a 2-cell embedding of genus $h+1$ (resp., $k+1$), whenever possible, can be devised for orientable (resp., non-orientable) surfaces by analogy, with more cases to consider. See also how to transform a planar 2-cell embedding of a graph containing a cycle into a projective-planar 2-cell embedding in Kocay and Kreher [16], p. 364.

The following result is usually known as (generalized) *Euler's formula*. We state it here in a form similar to Thomassen [22].

Theorem 6 (Euler's Formula [22]). *Given a connected graph G with n vertices and m edges 2-cell embedded on a topological surface S,*

$$n - m + f = \chi(S), \tag{2}$$

where either $\chi(S) = 2 - 2h$ and $S = S_h$, or $\chi(S) = 2 - k$ and $S = N_k$, and f is the number of faces of the 2-cell embedding on S.

Equation (2) is usually referred to as *Euler's formula* for an orientable surface S_h of genus h, $h \geq 0$, or a non-orientable surface N_k of genus k, $k \geq 1$, and the invariant $\chi(S)$ is the *Euler characteristic* of an orientable surface $S = S_h$ or a non-orientable surface $S = N_k$, respectively. Notice that $\chi(S) \leq 2$.

The *orientable genus* of a graph G is the smallest integer $h = h(G)$ such that G admits an embedding on an orientable topological surface S of genus h. The *non-orientable genus* of G is the smallest integer $k = k(G)$ such that G can be embedded on a non-orientable topological surface S of genus k. In general, $h(G) \neq k(G)$ (e.g., see [16], pp. 367–368), and the embeddings on $S_{h(G)}$ and, in most cases, on $N_{k(G)}$ must be 2-cell embeddings. If G is not a tree and $k(G) = 2h(G) + 1$, then, in addition to 2-cell embeddings on $N_{k(G)}$, there are non-2-cell embeddings on $N_{k(G)}$ (see Parsons et al. [17]). Note that a tree has no 2-cell embeddings on non-orientable surfaces. In general, it is an NP-complete problem to determine the orientable genus $h(G)$ (see Thomassen [23]).

Lemma 7. *Given a non-trivial graph G 2-cell embedded on a topological surface S of the Euler characteristic $\chi(S)$, the number of vertices of G is*

$$n \geq \frac{3 + \sqrt{17 - 8\chi(S)}}{2} > \sqrt{4 - 2\chi(S)} + 1. \tag{3}$$

Proof. From Euler's formula (2),

$$n - m + f = \chi(S).$$

Since $f \geq 1$ and $m \leq n(n-1)/2$, we have

$$\chi(S) = n - m + f \geq n - \frac{n(n-1)}{2} + 1,$$

which gives

$$n^2 - 3n + 2(\chi(S) - 1) \geq 0.$$

Solving the corresponding quadratic equation for n, we obtain

$$n = \frac{3 \pm \sqrt{17 - 8\chi(S)}}{2}. \tag{4}$$

Since n is a positive integer at least two, the statement of Lemma 7 follows. □

Plugging in $\chi(S) = 2 - 2h$ and $\chi(S) = 2 - k$ into (3) yields

$$n \geq \frac{3 + \sqrt{16h+1}}{2} > 2\sqrt{h} + 1, \quad h \geq 0,$$

and

$$n \geq \frac{3 + \sqrt{8k+1}}{2} > \sqrt{2k} + 1, \quad k \geq 1,$$

respectively.

The *maximum orientable* (resp., *non-orientable*) *genus* $h_M(G)$ (resp., $k_M(G)$) of a graph G is the largest integer h (resp., k) such that G has a 2-cell embedding on S_h (resp., N_k). The maximum genera of graphs are well-studied parameters (for example, see Huang [13] and Ringel [18]). Notice that, if $h(G)$ is the orientable genus of G, then G has 2-cell embeddings on the orientable surfaces of genus h, $h(G) \leq h \leq h_M(G)$. Similarly, G has 2-cell embeddings on the non-orientable surfaces of genus k, $k(G) \leq k \leq k_M(G)$.

The bounds of Lemma 7 are tight. Euler's formula (2) implies $h_M(G) \leq \lfloor \frac{m-n+1}{2} \rfloor$ and $k_M(G) \leq m - n + 1$, and 4-edge connected graphs are known to be *upper-embeddable*, i.e. to have $h_M(G) = \lfloor \frac{m-n+1}{2} \rfloor$ (e.g. see Jungerman [14]). Notice that complete graphs K_n, $n \geq 5$, are 4-edge connected, and $h(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$ (e.g. see [2], p. 281). Then,
for $h = 3$, Lemma 7 gives $n \geq 5$, and K_5 has $h(K_5) = 1$, $h_M(K_5) = 3$;
for $h = 5$, Lemma 7 gives $n \geq 6$, and K_6 has $h(K_6) = 1$, $h_M(K_6) = 5$;
for $h = 14$, Lemma 7 gives $n \geq 9$, and K_9 has $h(K_9) = 3$, $h_M(K_9) = 14$;
for $h = 18$, Lemma 7 gives $n \geq 10$, and K_{10} has $h(K_{10}) = 4$, $h_M(K_{10}) = 18$;
etc. In general, for $h = 4y^2 \pm y$, $y \in \mathbb{Z}^+$, the bound of Lemma 7 is tight. Notice that the proof of Lemma 7 implies that there is only one face in the 2-cell embedding of K_{4y+2} or K_{4y+1} on the surface of its maximum genus $h_M(K_{4y+2}) = 4y^2 + y$ or $h_M(K_{4y+1}) = 4y^2 - y$, respectively. Similar observations can be easily obtained for the non-orientable surfaces: a connected graph G which is not a tree has $k_M(G) = m - n + 1$ (see [18]).

A *triangle-free graph* G is a graph having no cycles of length 3. The lower bounds of Lemma 7 can be easily improved for graphs with no triangles as follows.

Corollary 8. *Given a non-trivial triangle-free graph G 2-cell embedded on a topological surface S of the Euler characteristic $\chi(S)$, the number of vertices of G is*

$$n \geq 2\sqrt{2 - \chi(S)} + 2. \tag{5}$$

Proof. The number of edges of a triangle-free graph G is $m \leq n^2/4$ (e.g. see [2], p. 45). The rest of the proof is similar to that of Lemma 7. □

Notice that bipartite graphs are triangle-free, and all the results for the triangle-free graphs in this paper hold for the bipartite graphs as well.

3. Constant upper bounds for general graphs on the topological surfaces

Hartnell and Rall [10] proved the following.

Theorem 9 (Hartnell and Rall [10]). *The number of edges of a connected graph G having n vertices and the bondage number $b(G)$ is $m \geq \frac{n}{4}(b(G)+1)$, and the bound is sharp.*

We use Theorem 9, Lemma 7, and Euler's formula (2) to establish the following upper bounds for the bondage number of graphs.

Theorem 10. *Let G be a connected graph embedded on the surface of its orientable or non-orientable genus S of the Euler characteristic $\chi(S)$ and having $n = |V(G)|$ vertices. Then*

(i) $\chi(S) \geq 1$ (G is planar or projective-planar) implies $b(G) \leq 10$;

(ii) $\chi(S) \leq 0$ and $n > -12\chi(S)$ imply $b(G) \leq 11$;

(iii) $\chi(S) \leq -1$ and $n \leq -12\chi(S)$ imply

$$b(G) \leq 11 + \frac{3\chi(S)(\sqrt{17 - 8\chi(S)} - 3)}{\chi(S) - 1} = 11 + O(\sqrt{-\chi(S)}).$$

Proof. As a corollary of Euler's formula (2), since $3f \leq 2m$ for $n \geq 3$, in general,

$$m \leq 3(n - \chi(S)). \tag{6}$$

Then, (6) and Theorem 9 give

$$\frac{n(b(G)+1)}{4} \leq m \leq 3(n - \chi(S)),$$

which implies

$$b(G) \leq 11 - \frac{12\chi(S)}{n}. \tag{7}$$

The statements of Theorem 10 follow directly, applying the bound (3) of Lemma 7 to obtain the statement of Theorem 10(iii). □

Plugging in $\chi(S) = 2 - 2h$ and $\chi(S) = 2 - k$ into (7) gives

$$b(G) \leq 11 + \frac{24(h-1)}{n} \qquad \text{and} \qquad b(G) \leq 11 + \frac{12(k-2)}{n},$$

respectively.

Clearly, in the case of planar graphs, Theorem 3 provides a better upper bound, $b(G) \leq c_0 \leq 8$, than Theorem 10(i). Since there are no restrictions on the number of vertices in the cases of toroidal ($h = 1$, $\chi(S) = 0$), projective-planar ($k = 1$, $\chi(S) = 1$), and Klein bottle ($k = 2$, $\chi(S) = 0$) graphs in Theorem 10, we have the following general constant upper bounds.

Corollary 11. *For any connected projective-planar graph G, $b(G) \le c'_1 \le 10$, and for any connected toroidal or Klein bottle graph G, $b(G) \le 11$, i.e. $c_1 \le 11$ and $c'_2 \le 11$.*

The formulae of Theorem 10(iii) provide constant upper bounds for the surfaces of higher genera as follows.

Corollary 12. *For a connected graph G of orientable genus $h = h(G) \ge 2$ and non-orientable genus $k = k(G) \ge 3$, we have*

Orientable genus, h	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$b(G) \le c_h \le$	16	20	24	27	29	32	34	36	38	40	42	43	45	47

Non-orientable genus, k	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$b(G) \le c'_k \le$	14	16	18	20	22	24	25	27	28	29	30	32	33	34

Table 1: Constant upper bounds for the bondage number of graphs on topological surfaces of higher genera ($h \le 15$ and $k \le 16$).

Since only the direct arguments with Euler's formulae have been used in Theorem 10 and Lemma 7, in the case $k = 2h$, the bounds for N_k coincide with the corresponding upper bounds for S_h, $h \ge 1$. The Euler characteristics and corresponding Euler's formulae are the same in this case. However, surfaces S_h and N_{2h}, $h \ge 1$, are not equivalent, i.e. *non-homeomorphic* (e.g. see [22], pp. 129–130), and the classes of graphs of orientable genus h and non-orientable genus $k = 2h$, $h \ge 1$, are quite different. Therefore, we conjecture that refinements of the results of Theorem 10 are going to provide different constant upper bounds for the bondage number of graphs embeddable on S_h and N_{2h}, $h \ge 1$.

4. Graphs with no triangles

The triangle-free graphs are exactly the graphs of girth at least 4. Fischermann et al. [6] have shown the following.

Theorem 13 (Fischermann et al. [6])**.** *A connected planar triangle-free graph G has $b(G) \le 6$.*

We provide a simple proof of Theorem 13 and generalize it to all the other topological surfaces as follows. The proof uses the bound of Theorem 9.

Theorem 14. *Let G be a connected triangle-free graph embedded on the surface of its orientable or non-orientable genus S of the Euler characteristic $\chi(S)$ and having $n = |V(G)|$ vertices. Then*

(i) $\chi(S) \ge 1$ (G is planar or projective-planar) implies $b(G) \le 6$;

(ii) $\chi(S) \le 0$ and $n > -8\chi(S)$ imply $b(G) \le 7$;

(iii) $\chi(S) \le -1$ *and* $n \le -8\chi(S)$ *imply*

$$b(G) \le 7 - \frac{4\chi(S)}{1 + \sqrt{2 - \chi(S)}}.$$

PROOF. In the case of triangle-free graphs, $4f \le 2m$ and $f \le m/2$. Then, similarly to (6), as a corollary to Euler's formula (2) for $n \ge 3$, we obtain

$$m \le 2(n - \chi(S)). \tag{8}$$

Further, (8) and Theorem 9 give

$$\frac{n(b(G) + 1)}{4} \le m \le 2(n - \chi(S)),$$

which implies

$$b(G) \le 7 - \frac{8\chi(S)}{n}. \tag{9}$$

The statements of Theorem 14 follow directly, applying the bound (5) of Corollary 8 to obtain the statement of Theorem 14(iii). □

Plugging in $\chi(S) = 2 - 2h$ and $\chi(S) = 2 - k$ into (9) gives

$$b(G) \le 7 + \frac{16(h-1)}{n} \qquad \text{and} \qquad b(G) \le 7 + \frac{8(k-2)}{n},$$

respectively.

Notice that, in the case of planar graphs, Theorem 14(i) provides the same upper bound, $b(G) \le 6$, as the previously known upper bound of Theorem 13. Conclusions similar to Corollaries 11 and 12 with constant upper bounds for the bondage number of triangle-free graphs on topological surfaces can be drawn from Theorem 14 as well. In general, Theorem 14 provides stronger bounds than Theorem 10 in the case of triangle-free graphs.

5. Improved upper bounds in terms of the maximum vertex degree and the genera

One of the classical upper bounds on the bondage number can be stated as follows.

Lemma 15 (Hartnell and Rall [9]). *For any edge uv in a graph G, we have $b(G) \le d(u) + d(v) - 1 - d_{uv}$, where $d_{uv} = |N(u) \cap N(v)|$. In particular, this implies that $b(G) \le \Delta(G) + \delta(G) - 1$* (see also [1, 5]).

Having a graph G embedded on a surface S, each edge $e_i = uv \in E(G)$, $i = 1, \dots, m$, is assigned two weights, $w_i = \frac{1}{d(u)} + \frac{1}{d(v)}$ and $f_i = \frac{1}{m'} + \frac{1}{m''}$, where m' is the number of edges on the boundary of a face on one side of e_i, and m'' is the number of edges on the boundary of the face on the other side of e_i. Notice that, in an embedding on a surface, an edge e_i may be not separating two distinct faces, but instead can appear twice on the boundary of the same face, and, in this case, $f_i = \frac{2}{m'} = \frac{2}{m''}$.

We have

$$\sum_{i=1}^{m} w_i = n, \qquad \sum_{i=1}^{m} f_i = f,$$

and, by Euler's formula (2),

$$\sum_{i=1}^{m}(w_i + f_i - 1) = n + f - m = \chi(S),$$

or, in other words,

$$\sum_{i=1}^{m}\left(w_i + f_i - 1 - \frac{\chi(S)}{m}\right) = 0. \tag{10}$$

Now, each edge $e_i = uv \in E(G)$, $i = 1, \ldots, m$, can be associated with the weight $Q(e_i) = w_i + f_i - 1 - \frac{\chi(S)}{m}$ called, depending on S, the *oriented* or *non-oriented curvature* of the edge e_i, respectively.

Theorem 16. *Let G be a connected graph 2-cell embeddable on an orientable surface of genus $h \geq 0$. Then*

$$b(G) \leq \begin{cases} \Delta(G) + \lceil h^{0.7} \rceil + 2 & \text{for } h \leq 5, \\ \Delta(G) + \lceil h^{0.7} \rceil + 3 & \text{for } h \geq 6. \end{cases} \tag{11}$$

Proof. Suppose G is 2-cell embedded on the h-holed torus S_h, and denote by τ the function given by

$$\begin{cases} \lceil h^{0.7} \rceil - 1 & \text{for } h \leq 5, \\ \lceil h^{0.7} \rceil & \text{for } h \geq 6. \end{cases}$$

Then we have to prove the following:

$$b(G) \leq \Delta(G) + \tau + 3.$$

If $\delta(G) \leq \tau + 4$, then, by Lemma 15,

$$b(G) \leq \Delta(G) + \delta(G) - 1 \leq \Delta(G) + \tau + 3,$$

as required, and inequality (11) holds. Therefore, we can assume that $\delta(G) \geq \tau + 5$.

Let us suppose that $b(G) \geq \Delta(G) + \tau + 4$. Then, by Lemma 15, for any edge $e_i = uv$ we have

$$d(u) + d(v) - 1 - d_{uv} \geq b(G) \geq \Delta(G) + \tau + 4,$$

which implies

$$d(u) + d(v) \geq \Delta(G) + \tau + 5 + d_{uv}. \tag{12}$$

Without loss of generality, assume $d(u) \leq d(v)$. There are now three cases to consider.

Case 1: $d(u) = \tau + 5$. By (12), $d(v) \geq \Delta(G) + d_{uv}$, which implies $d(v) = \Delta(G)$ and $d_{uv} = 0$. Therefore, $m' \geq 4$ and $m'' \geq 4$ in this case, and

$$Q(e_i) = w_i + f_i - 1 + \frac{2h-2}{m} \leq \frac{2}{\tau+5} - \frac{1}{2} + \frac{2h-2}{m}.$$

If $h = 0$, then $\tau = -1$ and

$$Q(e_i) \leq \frac{2}{4} - \frac{1}{2} - \frac{2}{m} < 0.$$

Now suppose that $h \geq 1$. Since $d_{uv} = 0$, we obtain $n \geq d(u) + d(v) \geq 2\tau + 10$, and hence

$$m \geq \frac{n\delta(G)}{2} \geq \frac{(2\tau+10)(\tau+5)}{2} = (\tau+5)^2.$$

Thus, for $h \geq 1$,

$$Q(e_i) \leq \frac{2}{\tau+5} - \frac{1}{2} + \frac{2h-2}{(\tau+5)^2} < 0.$$

Case 2: $d(u) = \tau + 6$. Then $d(v) \geq \tau + 6$. By (12), $d(v) \geq \Delta(G) - 1 + d_{uv}$. If $d_{uv} \geq 2$, then $d(v) \geq \Delta(G) + 1$, a contradiction. Therefore, $d_{uv} \leq 1$ and, without loss of generality, $m' \geq 3$ and $m'' \geq 4$ in this case. We have

$$Q(e_i) \leq \frac{2}{\tau+6} + \frac{1}{3} + \frac{1}{4} - 1 + \frac{2h-2}{m} = \frac{2}{\tau+6} - \frac{5}{12} + \frac{2h-2}{m}.$$

If $h = 0$, then $\tau = -1$ and

$$Q(e_i) \leq \frac{2}{5} - \frac{5}{12} - \frac{2}{m} = -\frac{1}{60} - \frac{2}{m} < 0.$$

Let us consider the sub-case when $h \geq 1$. Because $d_{uv} \leq 1$, we have $n \geq d(u) + d(v) - 1 \geq 2\tau + 11$, and therefore

$$m \geq \frac{d(u) + d(v) + (n-2)\delta(G)}{2} \geq \frac{2\tau + 12 + (2\tau+9)(\tau+5)}{2} = \frac{2\tau^2 + 21\tau + 57}{2}.$$

Thus, if $h \geq 1$, then

$$Q(e_i) \leq \frac{2}{\tau+6} - \frac{5}{12} + \frac{2(2h-2)}{2\tau^2 + 21\tau + 57} < 0.$$

Case 3: $d(u) \geq \tau + 7$. Then $d(v) \geq \tau + 7$, and $m' \geq 3$, $m'' \geq 3$. We obtain

$$Q(e_i) \leq \frac{2}{\tau+7} - \frac{1}{3} + \frac{2h-2}{m}.$$

If $h = 0$, then $\tau = -1$ and

$$Q(e_i) \leq \frac{2}{6} - \frac{1}{3} - \frac{2}{m} < 0.$$

Suppose that $h \geq 1$. The inequality (12) implies

$$n \geq d(u) + d(v) - d_{uv} \geq \Delta(G) + \tau + 5 \geq 2\tau + 12.$$

Hence

$$m \geq \frac{d(u) + d(v) + (n-2)\delta(G)}{2} \geq \frac{2\tau + 14 + (2\tau + 10)(\tau + 5)}{2} = \tau^2 + 11\tau + 32,$$

and, for $h \geq 1$, $h \in \mathbb{Z}^+$,

$$Q(e_i) \leq \frac{2}{\tau + 7} - \frac{1}{3} + \frac{2h - 2}{\tau^2 + 11\tau + 32} < 0.$$

Thus, $Q(e_i) < 0$ for any edge e_i ($i = 1, \ldots, m$) and for any $h \geq 0$, $h \in \mathbb{Z}$. We obtain $\sum_{i=1}^{m} Q(e_i) < 0$, which contradicts to (10). Therefore, $b(G) \leq \Delta(G) + \tau + 3$, as required. □

If the genus h is fixed, then by excluding a finite number of graphs we can improve the bound of Theorem 16 as shown below.

Corollary 17. *For a connected graph G 2-cell embeddable on an orientable surface of genus $h \geq 1$, we have:*
(a) $b(G) \leq \Delta(G) + \lceil \ln^2 h \rceil + 3$ *if* $n \geq h$;
(b) $b(G) \leq \Delta(G) + \lceil \ln h \rceil + 3$ *if* $n \geq h^{1.9}$;
(c) $b(G) \leq \Delta(G) + 4$ *if* $n \geq h^{2.5}$.

Proof. Let us introduce the following notation:
(a) $\tau = \lceil \ln^2 h \rceil$;
(b) $\tau = \lceil \ln h \rceil$;
(c) $\tau = 1$.

Then the proof is very similar to that of Theorem 16, with some differences in the three cases which are considered below.

Case 1: $d(u) = \tau + 5$. We have already shown that

$$Q(e_i) \leq \frac{2}{\tau + 5} - \frac{1}{2} + \frac{2h - 2}{m}.$$

Now,

$$m \geq \frac{n\delta(G)}{2} \geq \frac{h^x(\tau + 5)}{2}$$

and, similarly to the proof of Theorem 16,

$$m \geq \frac{n\delta(G)}{2} \geq \frac{(2\tau + 10)(\tau + 5)}{2} = (\tau + 5)^2.$$

This implies, respectively,

$$Q(e_i) \leq Q'(e_i) = \frac{2}{\tau + 5} - \frac{1}{2} + \frac{2(2h - 2)}{h^x(\tau + 5)}$$

and

$$Q(e_i) \le Q''(e_i) = \frac{2}{\tau+5} - \frac{1}{2} + \frac{2h-2}{(\tau+5)^2}.$$

We obtain:
(a) For $x = 1$, $Q'(e_i) < 0$ if $h \ge 12$, and $Q''(e_i) < 0$ if $h \le 11$;
(b) For $x = 1.9$, $Q'(e_i) < 0$ if $h \ne 2$, and $Q''(e_i) < 0$ if $h = 2$;
(c) For $x = 2.5$, $Q'(e_i) < 0$ if $h \ge 1$.

Case 2: $d(u) = \tau + 6$. Then, similarly to the proof of Theorem 16,

$$Q(e_i) \le \frac{2}{\tau+6} - \frac{5}{12} + \frac{2h-2}{m}.$$

Now,

$$m \ge \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \ge \frac{2\tau+12+(n-2)(\tau+5)}{2} \ge \frac{2+h^x(\tau+5)}{2}$$

and also

$$m \ge \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \ge \frac{2\tau+12+(2\tau+9)(\tau+5)}{2} = \frac{2\tau^2+21\tau+57}{2}.$$

This implies, respectively,

$$Q(e_i) \le Q'(e_i) = \frac{2}{\tau+6} - \frac{5}{12} + \frac{2(2h-2)}{2+h^x(\tau+5)}$$

and

$$Q(e_i) \le Q''(e_i) = \frac{2}{\tau+6} - \frac{5}{12} + \frac{2(2h-2)}{2\tau^2+21\tau+57}.$$

We have:
(a) For $x = 1$, $Q'(e_i) < 0$ if $h \ge 17$, and $Q''(e_i) < 0$ if $h \le 16$;
(b) For $x = 1.9$, $Q'(e_i) < 0$ if $h \ne 2$, and $Q''(e_i) < 0$ if $h = 2$;
(c) For $x = 2.5$, $Q'(e_i) < 0$ if $h \ge 1$.

Case 3: $d(u) \ge \tau + 7$. Then

$$Q(e_i) \le \frac{2}{\tau+7} - \frac{1}{3} + \frac{2h-2}{m}.$$

We obtain

$$m \ge \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \ge \frac{2\tau+14+(n-2)(\tau+5)}{2} \ge \frac{4+h^x(\tau+5)}{2}$$

and, similarly to the proof of Theorem 16,

$$m \ge \frac{d(u)+d(v)+(n-2)\delta(G)}{2} = \tau^2+11\tau+32.$$

This implies, respectively,

$$Q(e_i) \le Q'(e_i) = \frac{2}{\tau+7} - \frac{1}{3} + \frac{2(2h-2)}{4+h^x(\tau+5)}$$

and

$$Q(e_i) \le Q''(e_i) = \frac{2}{\tau+7} - \frac{1}{3} + \frac{2h-2}{\tau^2+11\tau+32}.$$

We have:
(a) For $x = 1$, $Q'(e_i) < 0$ if $h \ge 28$, and $Q''(e_i) < 0$ if $h \le 27$;
(b) For $x = 1.9$, $Q'(e_i) < 0$ if $h \ge 5$, and $Q''(e_i) < 0$ if $h \le 4$;
(c) For $x = 2.5$, $Q'(e_i) < 0$ if $h \ne 2$, and $Q''(e_i) < 0$ if $h = 2$.

Thus, $Q(e_i) < 0$ for any edge e_i $(i = 1, \dots, m)$ and for any $h \ge 1$, $h \in \mathbb{Z}$. We obtain $\sum_{i=1}^{m} Q(e_i) < 0$, which contradicts to (10). Therefore, $b(G) \le \Delta(G) + \tau + 3$, as required. □

Theorem 18. *Let G be a connected graph 2-cell embeddable on a non-orientable surface of genus $k \ge 1$. Then*

$$b(G) \le \begin{cases} \Delta(G) + \lceil k^{0.6} \rceil + 1 & \text{for } k \le 5, \\ \Delta(G) + \lceil k^{0.6} \rceil + 2 & \text{for } k \ge 6. \end{cases} \tag{13}$$

Proof. The proof is similar to that of Theorem 16 above and goes as follows. Let G be 2-cell embedded on N_k. Denote by τ the function given by

$$\begin{cases} \lceil k^{0.6} \rceil - 1 & \text{for } k \le 5, \\ \lceil k^{0.6} \rceil & \text{for } k \ge 6. \end{cases}$$

Then we have to prove that

$$b(G) \le \Delta(G) + \tau + 2.$$

If $\delta(G) \le \tau + 3$, then, by Lemma 15,

$$b(G) \le \Delta(G) + \delta(G) - 1 \le \Delta(G) + \tau + 2,$$

as required, and inequality (13) holds.

Therefore, assume $\delta(G) \ge \tau + 4$. Suppose that $b(G) \ge \Delta(G) + \tau + 3$. Then, by Lemma 15, for any edge $e_i = uv$,

$$d(u) + d(v) - 1 - d_{uv} \ge b(G) \ge \Delta(G) + \tau + 3,$$

i.e.

$$d(u) + d(v) \ge \Delta(G) + \tau + 4 + d_{uv}. \tag{14}$$

Without loss of generality, $d(u) \le d(v)$. There are three cases to consider.

Case 1: $d(u) = \tau + 4$. By (14), $d(v) \geq \Delta(G) + d_{uv}$, which implies $d(v) = \Delta(G)$, $d_{uv} = 0$, and $m' \geq 4$, $m'' \geq 4$. In this case, we have

$$Q(e_i) = w_i + f_i - 1 + \frac{k-2}{m} \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{k-2}{m}.$$

If $k = 1$, then

$$Q(e_i) \leq \frac{2}{4} - \frac{1}{2} - \frac{1}{m} < 0.$$

Suppose that $k \geq 2$. Now, $d_{uv} = 0$ implies $n \geq d(u) + d(v) \geq 2\tau + 8$, and hence

$$m \geq \frac{n\delta(G)}{2} \geq \frac{(2\tau+8)(\tau+4)}{2} = (\tau+4)^2.$$

Thus, if $k \geq 2$, then

$$Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{k-2}{(\tau+4)^2} < 0.$$

Case 2: $d(u) = \tau + 5$. Then $d(v) \geq \tau + 5$. By (14), $d(v) \geq \Delta(G) - 1 + d_{uv}$, which implies $d_{uv} \leq 1$, and, without loss of generality, $m' \geq 3$ and $m'' \geq 4$ in this case. We have

$$Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{k-2}{m}.$$

If $k = 1$, then

$$Q(e_i) \leq \frac{2}{5} - \frac{5}{12} - \frac{1}{m} = -\frac{1}{60} - \frac{1}{m} < 0.$$

Now consider the sub-case $k \geq 2$. We have $n \geq d(u) + d(v) - 1 \geq 2\tau + 9$, and

$$m \geq \frac{d(u) + d(v) + (n-2)\delta(G)}{2} \geq \frac{2\tau + 10 + (2\tau+7)(\tau+4)}{2} = \tau^2 + 8.5\tau + 19.$$

Thus, for $k \geq 2$,

$$Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{k-2}{\tau^2 + 8.5\tau + 19} < 0.$$

Case 3: $d(u) \geq \tau + 6$. Then $d(v) \geq \tau + 6$, and $m' \geq 3$, $m'' \geq 3$. We have

$$Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{k-2}{m}.$$

If $k = 1$, then

$$Q(e_i) \leq -\frac{1}{m} < 0.$$

Suppose that $k \geq 2$. The inequality (14) implies

$$n \geq d(u) + d(v) - d_{uv} \geq \Delta(G) + \tau + 4 \geq 2\tau + 10.$$

Hence

$$m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+12+(2\tau+8)(\tau+4)}{2} = \tau^2+9\tau+22,$$

and, for $k \geq 2$, $k \in \mathbb{Z}$,

$$Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{k-2}{\tau^2+9\tau+22} < 0.$$

Thus, $Q(e_i) < 0$ for any edge e_i $(i = 1, \ldots, m)$ and for any $k \geq 1$, $k \in \mathbb{Z}$. We obtain $\sum_{i=1}^{m} Q(e_i) < 0$, which contradicts to (10). Therefore, $b(G) \leq \Delta(G)+\tau+2$, as required. □

The bounds of Theorem 18 can be improved if the number of vertices n is restricted from below by some function of the non-orientable genus k. If k is fixed, then only a finite number of graphs is excluded by such a function.

Corollary 19. *Let G be a connected graph 2-cell embeddable on a non-orientable surface of genus $k \geq 1$. Then*
(a) $b(G) \leq \Delta(G) + \lceil \ln^2 k \rceil + 2$ *if* $n \geq k/6$;
(b) $b(G) \leq \Delta(G) + \lceil \ln k \rceil + 2$ *if* $n \geq k^{1.6}$;
(c) $b(G) \leq \Delta(G) + 3$ *if* $n > k^2$.

Proof. For $k = 1$, the result follows from Theorem 18. Let us consider the case $k \geq 2$ and introduce the following notation:
(a) $\tau = \lceil \ln^2 k \rceil$;
(b) $\tau = \lceil \ln k \rceil$;
(c) $\tau = 1$.

Then the proof is very similar to that of Theorem 18, with some differences in the three cases which are considered below.

Case 1: $d(u) = \tau + 4$ and

$$Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{k-2}{m}.$$

(a) We obtain $m \geq \frac{n\delta(G)}{2} \geq \frac{k(\tau+4)}{12}$, and hence

$$Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{12(k-2)}{k(\tau+4)} < 0 \quad \text{if} \quad k \geq 122.$$

Similarly to the proof of Theorem 18,

$$Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{k-2}{(\tau+4)^2} < 0 \quad \text{if} \quad k \leq 121.$$

(b) Since $m \geq \frac{n\delta(G)}{2} \geq \frac{k^{1.6}(\tau+4)}{2}$, we have

$$Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{2(k-2)}{k^{1.6}(\tau+4)} < 0 \quad \text{for} \quad k \geq 2.$$

(c) In this sub-case, $m \geq \frac{n\delta(G)}{2} > \frac{k^2(\tau+4)}{2}$, and so

$$Q(e_i) < \frac{2}{\tau+4} - \frac{1}{2} + \frac{2(k-2)}{k^2(\tau+4)} < 0 \quad \text{for} \quad k \geq 2.$$

Case 2: $d(u) = \tau + 5$ and

$$Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{k-2}{m}.$$

(a) $m \geq \frac{k(\tau+4)}{12}$ implies

$$Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{12(k-2)}{k(\tau+4)} < 0 \quad \text{if} \quad k \geq 219.$$

Similarly to the proof of Theorem 18,

$$Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{k-2}{\tau^2+8.5\tau+19} < 0 \quad \text{if} \quad k \leq 218.$$

(b) $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+10+(k^{1.6}-2)(\tau+4)}{2} = \frac{k^{1.6}(\tau+4)+2}{2}$ implies

$$Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{2(k-2)}{k^{1.6}(\tau+4)+2} < 0 \quad \text{for} \quad k \geq 2.$$

(c) $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} > \frac{2\tau+10+(k^2-2)(\tau+4)}{2} = \frac{k^2(\tau+4)+2}{2}$ implies

$$Q(e_i) < \frac{2}{\tau+5} - \frac{5}{12} + \frac{2(k-2)}{k^2(\tau+4)+2} < 0 \quad \text{for} \quad k \geq 2.$$

Case 3: $d(u) \geq \tau + 6$ and

$$Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{k-2}{m}.$$

(a) $m \geq \frac{k(\tau+4)}{12}$ implies

$$Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{12(k-2)}{k(\tau+4)} < 0 \quad \text{if} \quad k \geq 439.$$

Similarly to the proof of Theorem 18,

$$Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{k-2}{\tau^2+9\tau+22} < 0 \quad \text{if} \quad k \leq 438.$$

(b) $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+12+(k^{1.6}-2)(\tau+4)}{2} = \frac{k^{1.6}(\tau+4)+4}{2}$ implies

$$Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{2(k-2)}{k^{1.6}(\tau+4)+4} < 0 \quad \text{for} \quad k \geq 2.$$

(c) $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} > \frac{2\tau+12+(k^2-2)(\tau+4)}{2} = \frac{k^2(\tau+4)+4}{2}$ implies

$$Q(e_i) < \frac{2}{\tau+6} - \frac{1}{3} + \frac{2(k-2)}{k^2(\tau+4)+4} \leq 0 \quad \text{for} \quad k \geq 2.$$

Thus, $Q(e_i) < 0$ for any e_i ($i = 1, \dots, m$) and any $k \geq 2$, $k \in \mathbb{Z}$, and so $\sum_{i=1}^{m} Q(e_i) < 0$, contrary to (10). Therefore, $b(G) \leq \Delta(G) + \tau + 2$, as required. $\square$

Note that asymptotically the bounds of Theorems 16 and 18 are not best possible, and for large values of genera h and k it is better to use the following bound (or a similar slightly stronger result of Huang [12]):

$$b(G) \leq \min\left\{\Delta(G) + \left\lfloor \frac{3+\sqrt{1+48h}}{2} \right\rfloor,\ \Delta(G) + \left\lfloor \frac{3+\sqrt{1+24k}}{2} \right\rfloor\right\}, \qquad (15)$$

$h \geq 1$ and $k \geq 1$. The bound (15) can be seen as a simple corollary to Lemma 15 and the upper bounds on the minimum vertex degree in terms of graph genera: $\delta(G) \leq \lfloor \frac{5+\sqrt{1+48h}}{2} \rfloor$ for $h \geq 1$, $\delta(G) \leq \lfloor \frac{5+\sqrt{1+24k}}{2} \rfloor$ for $k \geq 1$ (e.g., see Sachs [19]).

However, when the number of vertices is restricted from below by simple linear functions of genera h and k, Corollaries 17(a) and 19(a) provide much better bounds.

6. Teschner's conjecture and final remarks

Combining Corollary 11 with the results of [8] and Theorem 6 in [3], we obtain a statement similar to Theorem 3.

Theorem 20. *For any connected projective-planar graph G,*

$$b(G) \leq \min\{10,\ \Delta(G) + 2\},$$

and for any connected toroidal or Klein bottle graph G,

$$b(G) \leq \min\{11,\ \Delta(G) + 3\}.$$

As mentioned previously, the bondage number of a disconnected graph G is the minimum of the bondage numbers of its components. Theorem 20 and Theorem 3 settle Teschner's conjecture (Conjecture 1) in affirmative for all planar and projective-planar graphs G with $\Delta(G) \geq 4$, and all toroidal and Klein bottle graphs G with $\Delta(G) \geq 6$, respectively. For a fixed non-orientable surface, Theorem 10(ii) and Corollary 19(c) settle Teschner's conjecture in affirmative for almost all graphs G with $\Delta(G) \geq 6$. For a fixed orientable surface, Theorem 10(ii) and Corollary 17(c) imply a similar statement for almost all graphs with $\Delta(G) \geq 8$.

For each particular surface S of orientable genus h or non-orientable genus k the number of graphs embeddable on S is infinite, and the number of graphs not satisfying the conditions of Theorem 10(ii) and Corollaries 17(c) and 19(c) is finite. Also, almost all graphs have the maximum degree greater than a given constant. Therefore, we can informally conclude that Teschner's conjecture holds for almost all graphs in general. The above comments provide particular cases where Teschner's conjecture must be verified to be settled for all graphs. We hope that the hierarchies of upper bounds by graph genera are going to be useful to solve Conjecture 1 for all graphs.

Computations in the proofs of Theorems 16 and 18 and Corollaries 17 and 19 have been done by using software Maple 9.5 and MS Excel. As presented, these four proofs rely on assumptions of appropriate asymptotic behaviour of some functions. It would be interesting to obtain similar proofs without using computational tools or the assumptions. However, since asymptotically the upper bound (15) is stronger, one needs to do the computations only for a finite number of cases, and there is no need to justify the asymptotic behaviour of those functions.

For the constant upper bounds of Theorems 10 and 3 (see also Corollaries 11 and 12), we would suggest to refine them to obtain tight constant upper bounds. For example, Fischermann et al. [6] asked whether there exist planar graphs with bondage numbers 6, 7 or 8. A class of planar graphs with the bondage number equal to 6 is shown in [3], and hence $6 \le c_0 \le 8$ in the case of planar graphs. The next surfaces to consider should be the torus S_1, the projective plane N_1 and the Klein bottle N_2, and it would be interesting to improve for them, if possible, the results of Corollary 11 (see also Theorem 20).

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