

Randomized algorithms and upper bounds for multiple domination in graphs and networks

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Abstract

We consider four different types of multiple domination and provide new improved upper bounds for the k - and k -tuple domination numbers. They generalise two classical bounds for the domination number and are better than a number of known upper bounds for these two multiple domination parameters. Also, we explicitly present and systematize randomized algorithms for finding multiple dominating sets, whose expected orders satisfy new and recent upper bounds. The algorithms for k - and k -tuple dominating sets are of linear time in terms of the number of edges of the input graph, and they can be implemented as local distributed algorithms. Note that the corresponding multiple domination problems are known to be *NP*-complete.

Keywords: Randomized algorithm, k -Domination, k -Tuple domination, α -Domination, α -Rate domination

1. Introduction

Domination is one of the fundamental concepts in graph theory with various applications to wireless and ad hoc networks, biological networks, distributed computing, social networks and web graphs [1, 5, 6, 14]. Dominating sets are also used as models for facility location problems in operational research. An important role is played by multiple domination. For example,

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k - and k -tuple dominating sets are used for balancing efficiency and fault tolerance in wireless sensor networks [6, 21, 22].

Wireless sensor networks and ad hoc mobile networks can be considered as natural examples of applications of multiple domination. A *wireless sensor network* (WSN) usually consists of up to several hundred small autonomous devices to measure some physical parameters. Each device contains a processing unit and a limited memory as well as a radio transmitter and a receiver to be able to communicate with its neighbours. Also, it contains a limited power battery and is constrained in energy consumption. There is a *base station*, which is a special sensor node used as a sink to collect information gathered by other sensor nodes and to provide a connection between the WSN and a usual network.

A routing algorithm allows the sensor nodes to self-organize into a WSN. As stated in [18], an important goal in WSN design is to maximize the functional lifetime of a sensor network by using energy efficient distributed algorithms, networking and routing techniques. To maximize the functional lifetime, it is important to select some sensor nodes to behave as a backbone set to support routing communications in an efficient and fault tolerant way. The backbone set can be considered as a dominating set in the corresponding underlying graph of the network.

Dominating sets of several different kinds have proved to be useful and effective for modelling backbone sets. In the recent literature (e.g., see [6, 21, 22]), particular attention has been paid to construction of m -connected k - and k -tuple dominating sets in WSNs. Several probabilistic, approximating and deterministic approaches have been proposed and analyzed. The backbone set of sensor nodes should be selected as small as possible and, on the other hand, it should guarantee high efficiency and reliability of networking and communications. This trade-off requires construction of multiple dominating sets providing energy efficient and reliable data dissemination and communications.

In this paper, we provide new upper bounds for the k - and k -tuple domination numbers and explicitly describe effective and efficient randomized algorithms to construct multiple dominating sets, whose expected orders satisfy the new and recently discovered upper bounds. The algorithms arise from probabilistic constructions used to prove the corresponding bounds. All the presented algorithms can be implemented in parallel or as local distributed algorithms in the spirit of [18]. The new upper bounds generalise two classical bounds for the domination number and improve a number of

known upper bounds for the multiple domination parameters presented in [4, 8, 9, 12, 20, 23].

2. Randomized algorithms for multiple domination

2.1. Basic notions, notation, classical results and related work

We consider networks represented by undirected simple finite graphs. If G is a graph of order n , then $V(G) = \{v_1, v_2, \dots, v_n\}$ is the set of vertices of G and d_i denotes the degree of v_i , $i = 1, \dots, n$. Denote by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ the minimum and maximum vertex degrees of G , respectively. Let $N(v)$ denote the neighbourhood of a vertex v in G , and $N[v] = N(v) \cup \{v\}$ be the closed neighbourhood of v . A set $X \subseteq V(G)$ is called a *dominating set* if every vertex not in X is adjacent to at least one vertex in X . The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$.

A set X is called a *k-dominating set* if every vertex not in X has at least k neighbours in X . The minimum cardinality of a *k-dominating set* of G is the *k-domination number* $\gamma_k(G)$. A set X is called a *k-tuple dominating set* of G if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. It is obvious that a *k-tuple dominating set* is also a *k-dominating set*. The minimum cardinality of a *k-tuple dominating set* of G is the *k-tuple domination number* $\gamma_{\times k}(G)$. The *k-tuple domination number* is only defined for graphs with $\delta \geq k - 1$. Clearly, $\gamma_{\times k}(G) \geq \gamma_k(G)$.

Let α be a real number satisfying $0 < \alpha \leq 1$. A set $X \subseteq V(G)$ is called an *α -dominating set* of G if for every vertex $v \in V(G) - X$, $|N(v) \cap X| \geq \alpha d_v$, i.e. v is adjacent to at least $\lceil \alpha d_v \rceil$ vertices of X . The minimum cardinality of an *α -dominating set* of G is called the *α -domination number* $\gamma_\alpha(G)$. The α -domination was introduced by Dunbar et al. [7]. It is easy to see that $\gamma(G) \leq \gamma_\alpha(G)$, and $\gamma_{\alpha_1}(G) \leq \gamma_{\alpha_2}(G)$ for $\alpha_1 < \alpha_2$. Also, $\gamma(G) = \gamma_\alpha(G)$ if α is sufficiently close to 0. In [10], we define a set $X \subseteq V(G)$ to be an *α -rate dominating set* of G if $|N[v] \cap X| \geq \alpha d_v$ for any vertex $v \in V(G)$. The concept of α -rate domination is similar to the concept of *k-tuple domination*, and an α -rate dominating set can be considered as a particular case of an α -dominating set in the same graph. We call the minimum cardinality of an α -rate dominating set of G the *α -rate domination number* $\gamma_{\times \alpha}(G)$. It is easy to see that $\gamma_\alpha(G) \leq \gamma_{\times \alpha}(G)$.

The following fundamental result was independently proved by Alon and Spencer [2], Arnaoutov [3], Lovász [16] and Payan [19]. Notice that a simple

deterministic algorithm to construct a dominating set satisfying bound (1) can be found in [2].

Theorem 1 ([2, 3, 16, 19]). *For any graph G ,*

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n. \quad (1)$$

Similar upper bounds for the double and triple domination numbers are known (see [12] and [20]). For $t \leq \delta$, the *closed t -degree of a graph G* is defined as follows:

$$\tilde{d}_t = \tilde{d}_t(G) = \frac{1}{n} \sum_{i=1}^n \binom{d_i + 1}{t}.$$

Note that \tilde{d}_1 is the average degree $d = d(G)$ of G plus 1. Zverovich [23] and Chang [4] have recently proved the following upper bound for the k -tuple domination number, which originally has been stated as a conjecture by Rautenbach and Volkmann in [20]. Both proofs independently exploit the idea of randomly generating a k -tuple dominating set from [9].

Theorem 2 ([4, 23]). *For any graph G with $\delta \geq k - 1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln \tilde{d}_{k-1} + 1}{\delta - k + 2} n. \quad (2)$$

Theorems 4, 6, 8 and 9 below generalise bound (1) and also the following Caro–Roditty bound (3), which is one of the strongest known upper bounds for the domination number:

Theorem 3 ([14], p. 48). *For any graph G with $\delta \geq 1$,*

$$\gamma(G) \leq \left(1 - \frac{\delta}{(1 + \delta)^{1+1/\delta}} \right) n. \quad (3)$$

2.2. k -Tuple domination

The following theorem improves the upper bound of Theorem 2. Also, the probabilistic construction used in the proof of Theorem 4 implies randomized Algorithm 1 to find a k -tuple dominating set, whose order satisfies the bound of Theorem 4 with a positive probability (Algorithm 1 is written on the same lines with the algorithm to find an α -rate dominating set). In other words,

the expectation of the order of the set D returned by Algorithm 1 satisfies the upper bound of Theorem 4. For $t \leq \delta$, we define

$$\delta' = \delta - k + 1, \quad b_t = b_t(G) = \binom{\delta}{t}, \quad \text{and} \quad \tilde{b}_t = \tilde{b}_t(G) = \binom{\delta + 1}{t}.$$

Theorem 4. *For any graph G with $\delta \geq k$,*

$$\gamma_{\times k}(G) \leq \left(1 - \frac{\delta'}{\tilde{b}_{k-1}^{1/\delta'}(1 + \delta')^{1+1/\delta'}}\right) n. \quad (4)$$

PROOF. For each vertex $v \in V(G)$, we select δ vertices from $N(v)$ and denote the resulting set by $N'(v)$. Let $p = 1 - 1/\left(\tilde{b}_{k-1}(1 + \delta')\right)^{1/\delta'}$ and let A be a set formed by an independent choice of vertices of G , where each vertex is selected with the probability p . For $m = 0, 1, \dots, k-1$, we denote $B_m = \{v_i \in V(G) - A : |N'(v_i) \cap A| = m\}$. Also, for $m = 0, 1, \dots, k-2$, we denote $A_m = \{v_i \in A : |N'(v_i) \cap A| = m\}$. For each set A_m , we form a set A'_m in the following way. For every vertex $v \in A_m$, we take $k - m - 1$ neighbours from $N'(v) - A$ and add them to A'_m . Such neighbours always exist because $\delta \geq k$. It is obvious that $|A'_m| \leq (k - m - 1)|A_m|$. For each set B_m , we form a set B'_m by taking $k - m - 1$ neighbours from $N'(v) - A$ for every vertex $v \in B_m$. We have $|B'_m| \leq (k - m - 1)|B_m|$.

We construct the set D as follows: $D = A \cup \left(\bigcup_{m=0}^{k-2} A'_m\right) \cup \left(\bigcup_{m=0}^{k-1} B_m \cup B'_m\right)$. It is easy to see that D is a k -tuple dominating set. The expectation of $|D|$ is

$$\begin{aligned} \mathbb{E}(|D|) &\leq \mathbb{E}\left(|A| + \sum_{m=0}^{k-2} |A'_m| + \sum_{m=0}^{k-1} |B_m| + \sum_{m=0}^{k-1} |B'_m|\right) \\ &\leq \mathbb{E}(|A|) + \sum_{m=0}^{k-2} (k - m - 1)\mathbb{E}(|A_m|) + \sum_{m=0}^{k-1} (k - m)\mathbb{E}(|B_m|). \end{aligned}$$

We have

$$\begin{aligned} \mathbb{E}(|A_m|) &= \sum_{i=1}^n \mathbb{P}(v_i \in A_m) = \sum_{i=1}^n p \binom{\delta}{m} p^m (1-p)^{\delta-m} = p^{m+1} (1-p)^{\delta-m} b_m n \quad \text{and} \\ \mathbb{E}(|B_m|) &= \sum_{i=1}^n \mathbb{P}(v_i \in B_m) = \sum_{i=1}^n (1-p) \binom{\delta}{m} p^m (1-p)^{\delta-m} = p^m (1-p)^{\delta-m+1} b_m n. \end{aligned}$$

Taking into account that $b_{-1} = 0$, we obtain

$$\begin{aligned}
\mathbb{E}(|D|) &\leq pn + \sum_{m=0}^{k-2} (k-m-1)p^{m+1}(1-p)^{\delta-m}b_m n + \sum_{m=0}^{k-1} (k-m)p^m(1-p)^{\delta-m+1}b_m n \\
&= pn + \sum_{m=1}^{k-1} (k-m)p^m(1-p)^{\delta-m+1}b_{m-1} n + \sum_{m=0}^{k-1} (k-m)p^m(1-p)^{\delta-m+1}b_m n \\
&= pn + (1-p)^{\delta-k+2}n \sum_{m=0}^{k-1} (k-m)p^m(1-p)^{k-m-1}(b_{m-1} + b_m).
\end{aligned}$$

Furthermore, for $0 \leq m \leq k-1$,

$$\begin{aligned}
(k-m)(b_{m-1} + b_m) &= (k-m) \binom{\delta+1}{m} \leq \prod_{j=1}^{\delta-k+2} \frac{(k-m+j-1)}{j} \binom{\delta+1}{m} \\
&= \binom{\delta-m+1}{\delta-k+2} \binom{\delta+1}{m} = \binom{k-1}{m} \binom{\delta+1}{k-1} = \binom{k-1}{m} \tilde{b}_{k-1}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\mathbb{E}(|D|) &\leq pn + (1-p)^{\delta'+1}n\tilde{b}_{k-1} \sum_{m=0}^{k-1} \binom{k-1}{m} p^m (1-p)^{k-m-1} \\
&= pn + (1-p)^{\delta'+1}n\tilde{b}_{k-1} \leq \left(1 - \frac{\delta'}{\tilde{b}_{k-1}^{1/\delta'}(1+\delta')^{1+1/\delta'}}\right)n,
\end{aligned}$$

as required. The proof of the theorem is complete. \square

The proof of Theorem 4 implies the following result, which improves the bound of Theorem 2 and generalises the classical bound (1).

Corollary 5. *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2) + \ln \tilde{b}_{k-1} + 1}{\delta-k+2} n.$$

PROOF. Using the inequality $1-p \leq e^{-p}$, the proof of Theorem 4 implies a weaker upper bound for $\mathbb{E}(|D|)$:

$$\mathbb{E}(|D|) \leq pn + e^{-p(\delta'+1)}n\tilde{b}_{k-1}.$$

The result easily follows if we put $p = \min\{1, \frac{\ln(\delta'+1) + \ln \tilde{b}_{k-1}}{\delta'+1}\}$. Note that if $p = 1$, then $\frac{\ln(\delta'+1) + \ln \tilde{b}_{k-1}}{\delta'+1} \geq 1$ and the upper bound is obviously true. \square

Algorithm 1: Randomized k -tuple dominating set (resp., α -rate dominating set)

Input: A graph G and an integer k , $k \leq \delta$ (resp., a real number α , $0 < \alpha \leq 1$).

Output: A k -tuple (resp., α -rate) dominating set D of G .

begin

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    Compute  $p = 1 - 1/\left((1 + \delta')\tilde{b}_{k-1}\right)^{1/\delta'}$ 
    (resp.,  $p' = 1 - 1/\left((1 + \widehat{\delta})\tilde{d}_\alpha\right)^{1/\widehat{\delta}}$ );
    Initialize  $A = \emptyset$ ;                                /* Form a set  $A \subseteq V(G)$  */
    foreach vertex  $v \in V(G)$  do
    | with the probability  $p$  (resp.,  $p'$ ), decide if  $v \in A$  or  $v \notin A$ ;
    end
    Initialize  $B = \emptyset$ ;                                /* Form a set  $B \subseteq V(G) - A$  */
    foreach vertex  $v \in V(G)$  do
    | Compute  $r = |N[v] \cap A|$ ;
    | if  $r < k$  (resp.,  $r < \alpha d_v$ ) then
    | | if  $v \in A$  then
    | | | add any  $k - r$  (resp.,  $\lceil \alpha d_v \rceil - r$ ) vertices from  $N(v) - A$ 
    | | | into  $B$ ;
    | | else
    | | | /*  $v \notin A$  */
    | | | add  $v$  and any  $k - r - 1$  (resp.,  $\lceil \alpha d_v \rceil - r - 1$ ) vertices
    | | | from  $N(v) - A$  into  $B$ ;
    | | end
    | end
    end
    Put  $D = A \cup B$ ;                                /*  $D$  is a  $k$ -tuple (resp.,  $\alpha$ -rate)
    dominating set */
    return  $D$ ;

```

end

In some cases, Theorem 4 provides a much better upper bound than the bound of Corollary 5, and hence the bound of Theorem 2. For example, let G be a 20-regular graph. Then, according to Corollary 5, $\gamma_{\times 5}(G) < 0.738n$, while Theorem 4 yields $\gamma_{\times 5}(G) < 0.543n$. Thus, a k -tuple dominating set returned by Algorithm 1 in this case is expected to be much smaller than the upper bound of Theorem 2.

2.3. k -Domination

Algorithm 2 presented below is a randomized algorithm to find a k -dominating set whose order satisfies the upper bound of Theorem 6 with a positive probability (Algorithm 2 is written on the same lines with the algorithm to find an α -dominating set). The algorithm is based on the probabilistic construction used in the proof of Theorem 6, and the expectation of the order of the set D returned by Algorithm 2 satisfies the upper bound of Theorem 6.

Theorem 6. *For any graph G with $\delta \geq k$,*

$$\gamma_k(G) \leq \left(1 - \frac{\delta'}{b_{k-1}^{1/\delta'} (1 + \delta')^{1+1/\delta'}} \right) n.$$

PROOF. For each vertex $v \in V(G)$, we select δ vertices from $N(v)$ and denote the resulting set by $N'(v)$. Let $p = 1 - 1/(b_{k-1}(1 + \delta')^{1/\delta'})$ and let A be a set formed by an independent choice of vertices of G , where each vertex is selected with the probability p . For $m = 0, 1, \dots, k - 1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N'(v_i) \cap A| = m\}.$$

We construct the set D as follows:

$$D = A \cup \left(\bigcup_{m=0}^{k-1} B_m \right).$$

It is easy to see that D is a k -dominating set. The expectation of $|D|$ is

$$\mathbb{E}(|D|) \leq \mathbb{E} \left(|A| + \sum_{m=0}^{k-1} |B_m| \right) = \mathbb{E}(|A|) + \sum_{m=0}^{k-1} \mathbb{E}(|B_m|).$$

We have

$$\mathbb{E}(|B_m|) = \sum_{i=1}^n \mathbb{P}(v_i \in B_m) = \sum_{i=1}^n (1-p) \binom{\delta}{m} p^m (1-p)^{\delta-m} = p^m (1-p)^{\delta-m+1} b_m n.$$

Therefore,

$$\begin{aligned} \mathbb{E}(|D|) &\leq pn + \sum_{m=0}^{k-1} p^m (1-p)^{\delta-m+1} b_m n \\ &= pn + (1-p)^{\delta-k+2} n \sum_{m=0}^{k-1} p^m (1-p)^{k-m-1} b_m. \end{aligned}$$

Furthermore, for $0 \leq m \leq k-1$,

$$b_m = \binom{\delta}{m} \leq \binom{\delta-m}{\delta-k+1} \binom{\delta}{m} = \binom{k-1}{m} \binom{\delta}{k-1} = \binom{k-1}{m} b_{k-1}.$$

We obtain

$$\begin{aligned} \mathbb{E}(|D|) &\leq pn + (1-p)^{\delta'+1} n b_{k-1} \sum_{m=0}^{k-1} \binom{k-1}{m} p^m (1-p)^{k-m-1} \\ &= pn + (1-p)^{\delta'+1} n b_{k-1} \\ &\leq \left(1 - \frac{\delta'}{b_{k-1}^{1/\delta'} (1+\delta')^{1+1/\delta'}} \right) n, \end{aligned}$$

as required. The proof of Theorem 6 is complete. \square

An analogue of Theorem 2 and Corollary 5 for the k -domination number easily follows from Theorem 6:

Corollary 7. *For any graph G with $\delta \geq k$,*

$$\gamma_k(G) \leq \frac{\ln(\delta - k + 2) + \ln b_{k-1} + 1}{\delta - k + 2} n.$$

PROOF. The proof is similar to that of Corollary 5. \square

It may be pointed out that Corollary 7 generalises the classical bound (1).

2.4. Related results for α -domination and α -rate domination

The concept of α -domination is different from k -domination in that a vertex must be dominated by a percentage of the vertices in its neighbourhood instead of a fixed number of its neighbours. However, the above randomized algorithms for α -domination and k -domination are very similar. Intuitively, in a homogeneous WSN, since sensor nodes may fail or consume all of their energy resources in an unbalanced and poorly predictable way, it might be more effective and reasonable to dominate a sensor node by a certain percentage of its neighbourhood nodes instead of a fixed number of neighbours.

The problem of deciding whether $\gamma_\alpha(G) \leq q$ for a positive integer q is known to be *NP*-complete [7]. Therefore, it is important to have good upper bounds for the α -domination number and efficient algorithms to find ‘small’ α -dominating sets. The following bounds for the α -domination number are proved in [7], where m is the number of edges in G (note that $m = \frac{1}{2} \sum_{i=1}^n d_i = (\tilde{d}_1 - 1)n/2 = dn/2$):

$$\frac{\alpha\delta n}{\Delta + \alpha\delta} \leq \gamma_\alpha(G) \leq \frac{\Delta n}{\Delta + (1 - \alpha)\delta} \quad (5)$$

and

$$\frac{2\alpha m}{(1 + \alpha)\Delta} \leq \gamma_\alpha(G) \leq \frac{(2 - \alpha)\Delta n - (2 - 2\alpha)m}{(2 - \alpha)\Delta}. \quad (6)$$

For $0 < \alpha \leq 1$, the α -degree of a graph G is defined as follows:

$$\hat{d}_\alpha = \hat{d}_\alpha(G) = \frac{1}{n} \sum_{i=1}^n \left(\left\lceil \alpha d_i \right\rceil - 1 \right).$$

Also, we put

$$\hat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1.$$

The following theorem proved in [10] generalises the upper bound (3) for the α -domination number. Indeed, if $d_i \geq 1$ are fixed for all $i = 1, \dots, n$, and α is sufficiently close to 0, then $\hat{\delta} = \delta$ and $\hat{d}_\alpha = 1$. Notice that in some cases Theorem 8 provides a much better bound than the upper bounds in (5) and (6). For example, if G is a 1000-regular graph, then Theorem 8 gives $\gamma_{0.1}(G) < 0.305n$, while (5) and (6) yield only $\gamma_{0.1}(G) < 0.527n$.

Theorem 8 ([10]). *For any graph G ,*

$$\gamma_\alpha(G) \leq \left(1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1+1/\hat{\delta}} \hat{d}_\alpha^{1/\hat{\delta}}} \right) n. \quad (7)$$

Algorithm 2, written on the same lines with the algorithm to find a k -dominating set, is a randomized algorithm to find an α -dominating set D , whose order satisfies the upper bound of Theorem 8 with a positive probability. In other words, the expectation of the order of set D returned by Algorithm 2 satisfies the upper bound of Theorem 8 (see [10] for details).

Theorem 8 easily implies the following generalisation of the well-known bound of Theorem 1:

$$\gamma_\alpha(G) \leq \frac{\ln(\widehat{\delta} + 1) + \ln \widehat{d}_\alpha + 1}{\widehat{\delta} + 1} n.$$

The concept of α -rate domination combines the concepts of α -domination and k -tuple domination. For $0 < \alpha \leq 1$, the *closed α -degree* of a graph G is defined as follows:

$$\widetilde{d}_\alpha = \widetilde{d}_\alpha(G) = \frac{1}{n} \sum_{i=1}^n \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1}.$$

In fact, the only difference between the α -degree and the closed α -degree is that to compute the latter, we choose from $d_i + 1$ vertices instead of d_i , i.e. from the *closed* neighbourhood $N[v_i]$ of v_i instead of $N(v_i)$, $i = 1, \dots, n$.

Algorithm 1 above, written on the same lines with the algorithm to find a k -tuple dominating set, is a randomized algorithm to find an α -rate dominating set D . The expectation of the order of the α -rate dominating set D returned by Algorithm 1 satisfies the upper bound of Theorem 9 below. This can be easily deduced from the detailed proof of Theorem 9 in [10]. Also, Theorem 9 provides an analogue of the Caro–Roditty bound (Theorem 3) for the α -rate domination number.

Theorem 9 ([10]). *For any graph G and $0 < \alpha \leq 1$,*

$$\gamma_{\times\alpha}(G) \leq \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \widetilde{d}_\alpha^{1/\widehat{\delta}}} \right) n. \quad (8)$$

Note that, similar to Theorem 8, Theorem 9 also implies the following generalisation of the classical upper bound (1):

$$\gamma_{\times\alpha}(G) \leq \frac{\ln(\widehat{\delta} + 1) + \ln \widetilde{d}_\alpha + 1}{\widehat{\delta} + 1} n.$$

3. Complexity and implementation

For complexity analysis, we only consider sequential implementation of the presented algorithms for k - and k -tuple dominating sets. The complexity analysis of the algorithms for α - and α -rate dominating sets is similar – the only difference is in the calculation of p' .

An essential part of the algorithms is to compute the binomial coefficients $\binom{a}{b}$. By definition,

$$\binom{a}{b} = \frac{a!}{b!(a-b)!} = \frac{a(a-1)\dots(a-b+1)}{b!} = \frac{a(a-1)\dots(b+1)}{(a-b)!},$$

and it can be computed in $O(a)$ time in terms of elementary operations of multiplication, division, addition and subtraction. However, since in the worst case scenario the required memory usage to store the products is $O(a \log a)$, writing to (reading from) the memory would require $O(a \log a)$ time. In practice, to overcome the memory and reading (writing) operations requirements (e.g., see pages 93–96 in [17]), the commonly used approach is to compute the binomial coefficient by using dynamic programming and Pascal's triangle. In this case, the time complexity to compute the binomial coefficient would be $O(ab) = O(a^2)$, and the memory usage is $O(b) = O(a)$.

We assume that computing the binomial coefficient is done in $O(a^2)$ time and that an input graph G has no isolated vertices. It is easy to see that the minimum vertex degree δ of G can be computed in $O(m)$ time, where m is the number of edges in G . We will show that Algorithm 1 can take up to $O(m) = O(m+n)$ time, where $n = |V(G)|$. More precisely, in reference to Algorithm 1, a worst case scenario when k is close to $\delta/2$ may require $O(\delta^2)$ steps to compute \tilde{b}_{k-1} , and δ' can be computed in $O(1)$. Therefore, in total, it takes $O(\delta^2)$ steps to compute the probability p . Note that $O(\delta^2)$ does not exceed $O(m)$. Clearly, it takes $O(n)$ time to find the set A . The numbers $r = |N[v] \cap A|$ for each $v \in V(G)$ can be computed separately or when finding the set A . In any case, we need to keep track of them only up to $r = k$. Since we may need to browse through all the neighbours of vertices in A , in total it can take $O(m)$ steps to calculate all the necessary r 's for each vertex $v \in V(G)$. Then the set B can be also found in $O(m)$ steps. Thus, in total, Algorithm 1 runs in $O(m)$ time. For Algorithm 2, a complexity analysis similar to that of Algorithm 1 shows that it can take up to $O(m)$ steps to find a k -dominating set.

Algorithms 1 and 2 are presented here in a form consistent with the proofs of the corresponding theorems. However, when implementing these algorithms, the output sets D can be constructed more efficiently and effectively by a recursive extension of the corresponding initial set A . In other words, instead of adding missing vertices into the sets B , we can add them directly into A . This can result in a smaller k -tuple, k -, α - or α -rate dominating set D , respectively.

It is easy to see that, as soon as probability p (resp., p') is known to all the vertices (sensor nodes in a WSN), Algorithms 1 and 2 can be easily and efficiently implemented in parallel or as local distributed algorithms. This is particularly important in case of WSNs (see [18] for details). To compute the probability p (resp., p') and to distribute its value to all the network nodes (graph vertices) in a WSN, one needs to use a data gathering round and a data distribution round coordinated from a base station or a selected super-node (vertex). When this is done, to construct the corresponding multiple dominating set for the whole network (graph), each network node (graph vertex) only needs to gather and communicate information locally in its own neighbourhood. It would be also interesting to obtain reasonable online versions of these algorithms for a realistic case scenario when the network changes dynamically and network nodes obtain information about the whole network and local neighbourhoods gradually in time.

4. Final remarks

Some bounds for the connected k -domination number can be found in [11]. To the best of our knowledge, the concept of α -domination is still to be explored in WSNs. Intuitively, since sensor nodes may fail or consume their energies in an unbalanced and poorly predictable way, it might be more effective and reasonable to dominate a sensor node by a certain percentage of its neighbourhood nodes instead of a fixed number of neighbours. Construction and analysis of multiple dominating sets should lead to a better balance between efficiency and fault tolerance in WSNs and help to extend the functional lifetime of a network. It seems reasonable to do simulations with random data by analogy with the models and results presented in [6].

Another direction in this research could be to use concentration bounds to show that the probability of significant deviation of the algorithmic outputs from their expected values are sufficiently small. However, proving some of

the concentration bounds has shown up to be quite challenging and seems to be a marvelous extension of this research.

We wonder if it is possible to derandomize any of the presented algorithms or to obtain independent deterministic algorithms to find corresponding dominating sets satisfying the upper bounds of Theorems 4, 6, 8 and 9. Harant and Henning [13] have recently found a realization algorithm to find a double dominating set satisfying the upper bound of Theorem 2. The algorithm can be considered as a derandomization of the corresponding probabilistic construction used in [12] to prove the upper bound for the double domination number. Algorithms approximating the α - and α -rate domination numbers up to a certain degree of precision would be interesting as well. For the k -tuple domination number, an interesting approximation algorithm was found by Klasing and Laforest [15].

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