On the stability and uniqueness of the flow of a fluid through porous media

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Abstract

In this short note we study the stability of flows of a fluid through porous media that satisfies a generalization of Brinkman's equation to include inertial effects. Such flows could have relevance to enhanced oil recovery and also to the ow of dense liquids through porous media. In any event, one cannot ignore the fact that flows through porous media are inherently unsteady and thus at least a part of the inertial term needs to be retained in many situations. We study the stability of the rest state and find it to be asymptotically stable. Next, we study the stability of a base flow and find that the flow is asymptotically stable, provided the base flow is sufficiently slow. Finally, we establish results concerning the uniqueness of the flow under appropriate conditions, and present some corresponding numerical results.

1 Introduction

In this note we shall study the stability of flows of a fluid that is governed by a generalization to Brinkman's equation that takes into account the effect of inertia. Brinkman (see Brinkman [1, 2]) developed an equation for the flow of a fluid through a porous solid which reduces to the equation developed by Darcy [3] for the flow through a porous medium when one ignores the frictional effects within the fluid and to the equations governing Stokes flow, when the effect of the friction at the pores are ignored. Forchheimer [4] suggested a modified "drag" due to the friction at the pore as he found the predictions of Darcy's equation to be not in keeping with experimental effects. The interaction term that he introduced leads to the equation becoming non-linear. However, in the models proposed by Darcy, Brinkman and Forchheimer the non-linearity of the inertial effect is ignored, and the equations proposed by Darcy and Brinkman are linear. The justification offered by Darcy and Brinkman to ignore the effects of inertia is that the flow in a porous media is expected to be slow. However, as shown by Munaf, et al. [5], inertial effects can become important in the flow of fluids through porous media under certain circumstances. In fact, in problems such as enhanced oil recovery where the oil is driven by steam at high pressure, when the pressure gradients are high or when the flow of dense fluids is considered, inertial effects can become important, or at least significant enough to be not ignored. It might be necessary, in flows involving high pressures and high pressure gradients, to include the effect of the pressure on the viscosity as well as the "Drag" term that arises due to frictional effects at the pore. Recently, Subramaniam and Rajagopal [6] investigated flow of fluids at high pressures while the gradients of pressure is also high and allowed for the viscosity and the "Drag coefficient" to depend on the pressure. They found the results to be markedly different from the results for the constant viscosity and constant "Drag coefficient" in that the flow rates are very different and they also found the development of boundary layers (regions where the vorticity is much larger than the rest of the flow domain) wherein the high pressures are confined. Later, Kannan and Rajagopal [7] also studied the flow of fluids through an inclined porous media at high pressures and pressure gradients due to the effects of gravity and they also found results that show the development of boundary layers wherein the vorticity is concentrated. The flows considered by Subramaniam and Rajagopal [6] and Kannan and Rajagopal [7] are steady flows and due to the special form assumed for the flow field, the inertial term is identically zero. However, the flow field assumed in these and several other studies can only be viewed as approximations to flows that take place in a porous medium as they assume that the flow is unidirectional. It is important to recognize that flows through porous media are inherently unsteady and thus one has to include at the very least the unsteady part of the inertial term. Moreover, flow through porous media is never truly one-dimensional and thus one cannot neglect the non-linear term in the inertia on that basis. In fact, when very high pressure gradients are involved the flow will be turbulent. Here, we shall not consider turbulent flows. We shall however modify Brinkman's equation to take into account the effects of inertia. A detailed discussion of the various assumptions that go into the development of Brinkman's equation can be found in the recent article on a hierarchy for approximations for the flow of fluids through porous media by Rajagopal [8][1]. Brinkman very astutely observed that "*Equation* (2)[2], *however, cannot be used as such. A first objection is that no viscous stress*

[1]There are several obvious typographical errors which appear in the paper indicating poor proof reading on the part of the author. The sign in front of in equations (3.4), (3.7), (3.11), (3.14) and (4.8) should be a negative sign instead of a positive sign.

[2]By equation (2) Brinkman is referring to Darcy's equation.

has been defined with relation to it. The viscous shearing stresses acting on a volume element of a fluid have been neglected; only the damping force of the porous mass η^ν/k has been retained. This is a good approximation for small permeabilities." When the permeability is large, it is necessary to include the effect of the viscous dissipation within the fluid has to be taken into account in the modeling. Brinkman's equation can be derived systematically from the theory of mixtures (see Truesdell [9, 10], Bowen [11], Atkin and Craine [12], Samohyl [13], Rajagopal and Tao [14] for a detailed discussion of the mechanics of mixtures) by making the following assumptions (see [8]):

(1) The solid is a rigid porous solid and thus the balance of linear momentum of the solid can be ignored, the stresses in the solid are whatever they need to be to meet the balance of linear momentum of the solid.

(2) Frictional effects between the fluid and the pore as well as frictional effects in the fluid due to the viscosity of the fluid are included[3]. The fluid will be assumed to be a linearly viscous fluid.

(3) The flow is sufficiently slow that inertial effects in the fluid can be neglected.

(4) The fluid density is assumed to be a constant.

(5) The flow is steady.

We shall not enforce the requirement that inertial effects be neglected or that the flow be steady. Based on this generalization of the model due to Brinkman, we shall consider the stability of the base flow to finite disturbances and conditions under which we can establish its uniqueness. The seminal works of Reynolds [16] and Orr [17], followed by the work of Synge [18], Kampe de Feriet [19], Berker [20], Thomas [21], Hopf [22] laid the foundation to the stability of the flows of the Navier-Stokes fluids to finite disturbances and Serrin [23] built upon this work and was able to use it to obtain numerical results concerning the stability of flows and extended the work of Hopf and Thomas under which one could establish the uniqueness of flows of the Navier-Stokes fluid. We shall follow such a procedure to establish the asymptotic stability of the base flow of a fluid that satisfies the equations developed by Brinkman and establish conditions under which the solution is unique. We show that the base flow is asymptotically stable, i.e., the disturbances decay to the basic flow, provided the base flow is sufficiently

[3]A detailed discussion of the various interaction mechanisms between constituents of fluids can be found in the article by Johnson, Massoudi and Rajagopal [15].

slow in the sense that the eigenvalue associated with the symmetric part of the velocity gradient is small with respect to the viscosity of the fluid. We are also able to establish that the base flow is unique under the same conditions. Several mathematical studies (Qin and Kaloni [24, 25], Qin, Gao and Kaloni [26], Guo and Kaloni [27], Franchi and Straughan [28], Payne and Straughan [29]) have been carried out concerning the convection of flows in porous media that couples Brinkman's equation with the energy equation, with the coupling between the two equations due to a term due to the effect of buoyancy due to a Oberbeck-Boussinesq approximation (see Oberbeck [30, 31], Boussinesq [32]), but this classic approximation that is widely used is not an approximation that retains terms of like order in a perturbation (see the paper by Rajagopal, et al. [33] for a detailed discussion of the issues). An up to date discussion of the literature pertinent to the stability of flows in porous media can be found in the recent book by Straughan [34]. In the next section we document the governing equations and in Section 3 we study the asymptotic stability of the rest state. In the final section we carry out the asymptotic stability analysis, and provide some corresponding numerical results.

2 Governing equations

The equation developed by Brinkman [1, 2] is

$$-\nabla p + \mu \Delta \boldsymbol{v} - \alpha \boldsymbol{v} + \rho \boldsymbol{b} = \mathbf{0}. \tag{2.1}$$

In the above equation, μ denotes the fluid viscosity, α the drag coefficient due to the frictional resistance offered by the pore to the flow of the fluid, p the pressure, $\boldsymbol{v}$ the velocity and $\boldsymbol{b}$ the body force. We shall assume that both the viscosity and drag coefficient are positive. Since it is assumed that the fluid density ρ is constant, the fluid can only undergo isochoric motions and thus we have

$$\mathrm{div}\boldsymbol{v} = \mathbf{0}. \tag{2.2}$$

Equations (2.1) and (2.2) provide four scalar equations for the three components of the velocity and pressure. The above model due to Brinkman assumes that the flow is sufficiently slow that inertial effects in the fluid can be ignored. We shall consider a generalization that takes into account inertial effects due to the flow, namely

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right] = -\nabla p + \mu \Delta \boldsymbol{v} - \alpha \boldsymbol{v} + \rho \boldsymbol{b}. \tag{2.3}$$

We shall henceforth assume that the body force field is conservative with potential ϕ, *i.e.*, $\boldsymbol{b} = -\nabla\phi$. Then, equation (2.3) can be rewritten as

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla P + \mu\Delta\boldsymbol{v} - \alpha\boldsymbol{v}, \tag{2.4}$$

where $P = p + \rho\phi$.

3 Uniqueness and stability in bounded domains

Let Ω be a bounded domain and let d denote its diameter. Let us non-dimensionalize eqs. (2.4) and (2.2) according to

$$\boldsymbol{x}^* = \frac{\boldsymbol{x}}{d}, \quad \boldsymbol{v}^* = \frac{\boldsymbol{v}}{V}, \quad t^* = \frac{V}{d}t, \quad P^* = \frac{P}{\rho V^2}, \tag{3.1}$$

V being a reference velocity (here, the maximum modulus of the velocity field will henceforth be taken as a reference value). By dropping the asterisks for simplicity of notation, equations (2.4) and (2.2) become

$$\begin{cases} \mathrm{DaRe}\left[\dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\mathrm{DaRe}\nabla P + \mathrm{Da}\Delta\boldsymbol{v} - \boldsymbol{v}, \\ \mathrm{div}\boldsymbol{v} = 0, \end{cases} \tag{3.2}$$

where $\mathrm{Re} = \rho V d/\mu$ and $\mathrm{Da} = \mu/(\alpha d^2)$ are the Reynolds and Darcy numbers, respectively. Let $m_0 = (\bar{\boldsymbol{v}}, \bar{P})$ be a solution to (3.2) in Ω satisfying a Dirichlet-type boundary condition on $\partial\Omega$ and let us study its uniqueness and stability. We first introduce the perturbations $(\boldsymbol{u}, \Pi)$ to the basic solution m_0, i.e.,

$$\bar{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{u}, \quad P = \bar{P} + \Pi, \tag{3.3}$$

and then we write down the evolution equations of the perturbations

$$\begin{cases} \mathrm{DaRe}\left[\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\bar{\boldsymbol{v}} + (\bar{\boldsymbol{v}}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] & \\ \qquad\qquad = -\mathrm{DaRe}\nabla\Pi + \mathrm{Da}\Delta\boldsymbol{u} - \boldsymbol{u} & \text{in } \Omega\times]0,+\infty[, \\ \mathrm{div}\boldsymbol{u} = 0 & \text{in } \Omega\times]0,+\infty[, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega\times]0,+\infty[. \end{cases} \tag{3.4}$$

On forming the scalar product of $(3.4)_1$ with $\boldsymbol{u}$, integrating over the domain Ω and taking into account $(3.4)_2$, $(3.4)_3$ and that $\mathrm{div}\bar{\boldsymbol{v}} = 0$, we obtain

$$\mathrm{DaRe}\frac{\mathrm{d}E}{\mathrm{d}t} = -2\mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t)E(t), \tag{3.5}$$

where

$$E(t) = \|\boldsymbol{u}(\cdot, t)\|_2^2 = \int_\Omega |\boldsymbol{u}(\boldsymbol{x}, t)|^2 \mathrm{d}V \tag{3.6}$$

is the kinetic energy associated with the perturbations, the functional $\mathcal{G}$ is defined as

$$\mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t) = \frac{\|\boldsymbol{u}\|_2^2 + \mathrm{Da}\left(\|\nabla\boldsymbol{u}\|_2^2 + \mathrm{Re}\displaystyle\int_\Omega \boldsymbol{u}\cdot\bar{\mathbf{D}}\boldsymbol{u}\mathrm{d}V\right)}{\|\boldsymbol{u}\|_2^2}, \tag{3.7}$$

and

$$\bar{\mathbf{D}} = \frac{1}{2}\left[\nabla\bar{\boldsymbol{v}} + (\nabla\bar{\boldsymbol{v}})^T\right]. \tag{3.8}$$

Let $\lambda_i(\boldsymbol{x}, t)$ $(i = 1, 2, 3)$ be the eigenvalues of the symmetric second-order tensor $\bar{\mathbf{D}}(\boldsymbol{x}, t)$ and assume that

$$\lambda_{\min} = \inf_{t\geq 0}\min_{\boldsymbol{x}\in\Omega}\min\{\lambda_1(\boldsymbol{x}, t), \lambda_2(\boldsymbol{x}, t), \lambda_3(\boldsymbol{x}, t)\} > -\infty. \tag{3.9}$$

(It is worth noting that, since $\mathrm{div}\bar{\boldsymbol{v}} = \mathrm{tr}\bar{\mathbf{D}} = 0$, $\lambda_{\min}$ is non-positive and $\lambda_{\min}$ vanishes if and only if the velocity field $\bar{\boldsymbol{v}}$ is constant in $\Omega \times [0, +\infty[$.) Then, the functional G defined through (3.7) is bounded from below in $\mathcal{I} \times [0, +\infty[$, $\mathcal{I}$ being the space of the kinematically admissible perturbations, that is the space of divergence-free vector fields defined in Ω and vanishing on $\partial\Omega$. In fact, assumption (3.9) and the Poincaré inequality [35, 36],

$$\|\nabla\boldsymbol{u}\|_2^2 \geq C(\Omega)\|\boldsymbol{u}\|_2^2 \quad \forall\boldsymbol{u} \in \mathcal{I}, \tag{3.10}$$

yield

$$\begin{aligned}\mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t) &\geq \frac{\|\boldsymbol{u}\|_2^2 + \mathrm{Da}\left(\|\nabla\boldsymbol{u}\|_2^2 - \mathrm{Re}|\lambda_{\min}|\|\boldsymbol{u}\|_2^2\right)}{\|\boldsymbol{u}\|_2^2} \\ &\geq 1 + \mathrm{Da}\left[C(\Omega) - \mathrm{Re}|\lambda_{\min}|\right] \quad \forall(\boldsymbol{u}, t) \in \mathcal{I} \times [0, +\infty[.\end{aligned} \tag{3.11}$$

Moreover, by following [37] one can prove that for all $t \in [0, +\infty[$ the functional $\mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t)$ admits minimum in I. Then, by virtue of (3.11)

$$\gamma \equiv \inf_{t\geq 0}\min_{\boldsymbol{u}\in\mathcal{I}}\mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t) \geq 1 + \mathrm{Da}\left[C(\Omega) - \mathrm{Re}|\lambda_{\min}|\right]. \tag{3.12}$$

We are now in position to prove the following theorem.

Theorem 1. *Let* $m_0 = (\bar{\boldsymbol{v}}, P)$ *be a solution to* (3.2) *satisfying Dirichlet-type boundary conditions such that*

$$\gamma = \inf_{t \geq 0} \min_{\boldsymbol{u} \in \mathcal{I}} \mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t) > 0, \tag{3.13}$$

with $\mathcal{G}$ *as in* (3.7). *Then,* m_0 *is globally exponentially stable.*

Proof. From (3.5) and (3.13) we deduce that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -\frac{2\gamma}{\mathrm{DaRe}} E(t). \tag{3.14}$$

Integrating (3.14) yields

$$E(t) \leq E(0) \exp\left(-\frac{2\gamma t}{\mathrm{DaRe}}\right), \tag{3.15}$$

and hence the kinetic energy associated with the perturbations decay exponentially in time.

□

Another sufficient condition for the stability of the basic motion m_0 is given by the following corollary.

Corollary 1. *Let* $m_0 = (\bar{\boldsymbol{v}}, P)$ *be a solution to* (3.2) *satisfying Dirichlet-type boundary conditions on* $\partial\Omega \times [0, +\infty[$ *such that* (3.9) *holds. Assume that*

$$\mathrm{Re} < \frac{1 + \mathrm{Da}C(\Omega)}{\mathrm{Da}|\lambda_{\min}|}. \tag{3.16}$$

Then, m_0 *is globally exponentially stable.*

Proof. The proof follows immediately from Theorem 1 and (3.12). □

It is worth noting that the stability condition (3.16) implies (3.13) but the vice-versa does not hold. In addition, the stability condition (3.16) is easier to apply as it does not require to solve any variational problem.

We conclude this Section by remarking that if a solution to (3.2) under a prescribed initial condition on the velocity field meets the hypotheses of Theorem 1 or Corollary 1, then it is unique.

4 Laminar solutions

In this Section we are interested in the stability of laminar flows trough a porous medium that is bounded in only one direction. Then, once introduced a Cartesian frame of reference $Oxyz$ with fundamental unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, the porous layer may be represented by the domain $\Omega_d = \mathbb{R}^2 \times [0, d]$ and the laminar flows whose stability we shall investigate are of the form

$$\boldsymbol{v} = U(z)\boldsymbol{i}. \tag{4.1}$$

As done in the previous Section, we non-dimensionalize equations (2.4) and (2.2) according to (3.1) (in which d is now the thickness of the porous layer and $V = \max_{z\in[0,d]} |U(z)|$ to obtain (3.2) again. It is easy to check that the following solutions to (3.2) represent all the possible laminar flows of the form (4.1):

$$\begin{cases} U(z) = c_1 \exp(\tau z) + c_2 \exp(-\tau z) + A_0, \\ P = \bar{P}(x) = -\dfrac{A_0}{\mathrm{DaRe}}x + P_0, \end{cases} \tag{4.2}$$

where c_1, c_2, A_0 and P_0 are integration constants and $\tau = 1/\sqrt{\mathrm{Da}}$.

As special cases of (4.2), for

- $U(0) = 0$, $U(1) = 1$ and $A_0 = 0$ one obtains the Couette flow

$$\begin{cases} U(z) = \dfrac{\sinh(\tau z)}{\sinh \tau}, \\ P = \bar{P}(x) = P_0, \end{cases} \tag{4.3}$$

- $U(0) = U(1) = 0$ and $A_0 \neq 0$ we get the Poiseuille flow

$$\begin{cases} U(z) = \mathrm{sign}(A_0)\dfrac{\cosh\left(\dfrac{\tau}{2}\right) - \cosh\left[\tau\left(z - \dfrac{1}{2}\right)\right]}{\cosh\dfrac{\tau}{2} - 1}, \\ P = \bar{P}(x) = -\dfrac{A_0}{\mathrm{DaRe}}x + P_0. \end{cases} \tag{4.4}$$

5 Stability of laminar flows

Let $\boldsymbol{u} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$ and Π be the perturbations to the velocity and pressure fields given by (4.2), *i.e.*,

$$\boldsymbol{v} = U(z)\boldsymbol{i} + \boldsymbol{u}, \quad P = \bar{P}(x) + \Pi. \tag{5.1}$$

From (3.2) we deduce that the perturbations satisfy the following equations

$$\begin{cases} \mathrm{DaRe}\left[\dfrac{\partial \boldsymbol{u}}{\partial t} + U\dfrac{\partial \boldsymbol{u}}{\partial x} + U'w\boldsymbol{i} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] = -\mathrm{DaRe}\nabla\Pi - \boldsymbol{u} + \mathrm{Da}\Delta\boldsymbol{u}, \\ \mathrm{div}\boldsymbol{u} = 0, \end{cases} \tag{5.2}$$

the prime denoting differentiation with respect to z, and the boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad z = 0, 1. \tag{5.3}$$

From here on we shall assume that the perturbations $\boldsymbol{u}$ and Π are periodic with periods $2\pi/a_x$ and $2\pi/a_y$ in the x and y directions ($a_x > 0$, $a_y > 0$). Let us denote by Ω_p the periodicity cell

$$\Omega_p = \left[-\frac{\pi}{a_x}, \frac{\pi}{a_x}\right] \times \left[-\frac{\pi}{a_y}, \frac{\pi}{a_y}\right] \times [0, 1], \tag{5.4}$$

and let $a = \sqrt{a_x^2 + a_y^2}$ be the wave number.

5.1 Linear stability

On linearizing equations (5.2) we obtain

$$\begin{cases} \mathrm{DaRe}\left[\dfrac{\partial \boldsymbol{u}}{\partial t} + U\dfrac{\partial \boldsymbol{u}}{\partial x} + U'w\boldsymbol{i}\right] = -\mathrm{DaRe}\nabla\Pi - \boldsymbol{u} + \mathrm{Da}\Delta\boldsymbol{u}, \\ \mathrm{div}\boldsymbol{u} = 0. \end{cases} \tag{5.5}$$

By taking the third components of curl and curlcurl of $(5.5)_1$, and taking into account $(5.5)_2$, we deduce that the components of the perturbation to the velocity field may be found by solving the following boundary value problem

$$\begin{cases} \mathrm{DaRe}\left(-\dfrac{\partial \Delta w}{\partial t} - U\dfrac{\partial \Delta w}{\partial x} + U''\dfrac{\partial w}{\partial x}\right) = \Delta w - \mathrm{Da}\Delta\Delta w, \\ \mathrm{DaRe}\left(\dfrac{\partial \zeta}{\partial t} + U\dfrac{\partial \zeta}{\partial x} - U'\dfrac{\partial w}{\partial y}\right) = -\zeta + \mathrm{Da}\Delta\zeta, \\ \Delta_* u = -\dfrac{\partial^2 w}{\partial x \partial z} - \dfrac{\partial \zeta}{\partial y}, \\ \Delta_* v = -\dfrac{\partial^2 w}{\partial y \partial z} + \dfrac{\partial \zeta}{\partial x}, \\ w = \dfrac{\partial w}{\partial z} = 0 \quad \text{on } z = 0, 1, \\ \zeta = 0, \end{cases} \tag{5.6}$$

where $\zeta = \text{curl}\boldsymbol{u} \cdot \boldsymbol{k}$ and

$$\Delta_* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5.7}$$

is the two-dimensional Laplacian. Finally, once the components of $\boldsymbol{u}$ are determined, the perturbation to the pressure field may be found by means of $(5.5)_1$. From (5.6) we deduce that the unique independent component of $\boldsymbol{u}$ is w as, once it is determined by solving equation $(5.6)_1$ under the boundary conditions $(5.6)_4$, all the other unknown scalar fields may be determined from the remaining equations. Since the coefficients in $(5.6)_1$ depend only on z, equation $(5.6)_1$ admits solutions which depend on x, y and t exponentially. We consider therefore solutions of the form

$$w(x, y, z, t) = W(z) \exp[i(a_x x + a_y y - a_x ct)], \tag{5.8}$$

in which it is understood that the real parts of these expressions must be taken into consideration to obtain physically meaningful quantities. The wave speed c may be complex, *i.e.*, $c = c_r + ic_i$, and the expression (5.8) thus represent waves which travel in the x and y co-ordinate directions with phase speed $a_x c_r / a$ and which grow or decay in time given by $\exp(-a_x c_i t)$. Such a wave is stable if $c_i > 0$, unstable if $c_i < 0$, and neutrally stable if $c_i = 0$. If we now let $D = \mathrm{d}/\mathrm{d}z$ and $R = a_x \mathrm{Re}$, then on substituting the expression (5.8) into equation $(5.6)_1$ and boundary conditions $(5.6)_4$ we obtain the following boundary value problem[4]

$$\begin{cases} [\mathrm{Da}(D^2 - a^2) - 1](D^2 - a^2)W = i\mathrm{Da}R[(U - c)(D^2 - a^2) - U'']W, \\ W = DW = 0 \quad \text{at } z = 0, 1. \end{cases} \tag{5.9}$$

The fourth-order system (5.9) was solved using the Chebyshev-tau method [38], which is a spectral technique coupled with the QZ algorithm.

[4] Equation $(5.9)_1$ represents the generalization of the Orr-Sommerfeld equation to laminar flows in a porous medium.

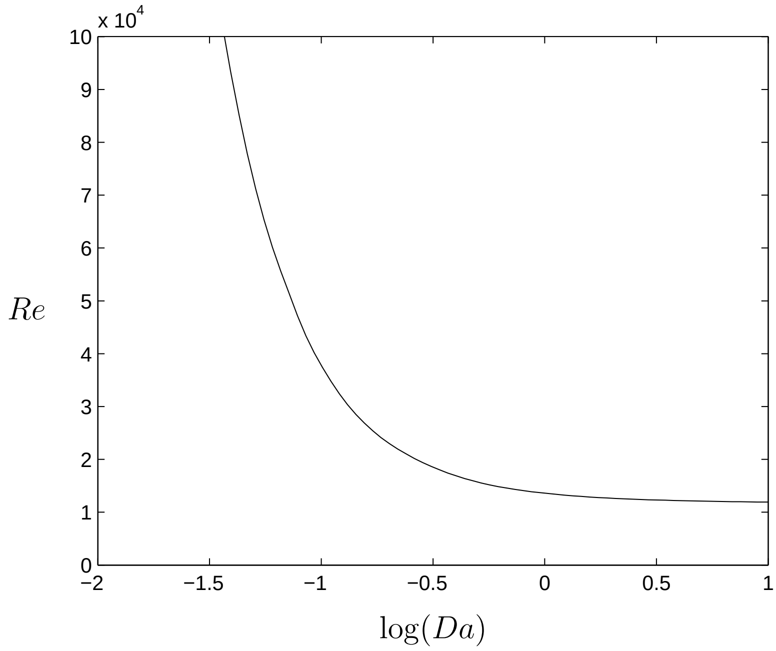

Figure 1: Visual representation of the Poiseuille flow linear instability thresholds with critical Reynolds number Re plotted against $\log(Da)$.

For Poiseuille flow, the numerical results correspond to comparable studies on Brinkman flow [39]. Couette flow does not yield instability thresholds utilising linear theory.

5.2 Nonlinear stability

In order to study the nonlinear stability of the laminar flows (4.2) we follow the same arguments as in Section 3 but modifying the notations slightly. More precisely, we introduce the functional

$$\mathcal{F}(U, \boldsymbol{u}) \equiv \frac{\|\boldsymbol{u}\|_2^2 + \mathrm{Da}\left(\|\nabla \boldsymbol{u}\|_2^2 + \mathrm{Re}\displaystyle\int_{\Omega_p} U' u w \,\mathrm{d}V\right)}{\|\boldsymbol{u}\|_2^2}, \tag{5.10}$$

and set

$$\gamma(a_x, a_y) \equiv \min_{\boldsymbol{u} \in \mathcal{I}_p} \mathcal{F}(U, \boldsymbol{u}), \tag{5.11}$$

where the space of the kinematically admissible perturbations $\mathcal{I}_p$ is the space of the divergence-free vector fields $\boldsymbol{u}$ defined in Ω_p such that

$$\begin{cases} \boldsymbol{u}\left(-\dfrac{\pi}{a_x}, y, z\right) = \boldsymbol{u}\left(\dfrac{\pi}{a_x}, y, z\right) & \forall (y,z) \in \left[-\dfrac{\pi}{a_y}, \dfrac{\pi}{a_y}\right] \times [0,1], \\ \boldsymbol{u}\left(x, -\dfrac{\pi}{a_y}, z\right) = \boldsymbol{u}\left(x, \dfrac{\pi}{a_y}, z\right) & \forall (x,z) \in \left[-\dfrac{\pi}{a_x}, \dfrac{\pi}{a_x}\right] \times [0,1], \\ \boldsymbol{u}\,(x,y,0) = \boldsymbol{u}\,(x,y,1) = \boldsymbol{0} & \forall (x,y) \in \left[-\dfrac{\pi}{a_x}, \dfrac{\pi}{a_x}\right] \times \left[-\dfrac{\pi}{a_y}, \dfrac{\pi}{a_y}\right]. \end{cases} \tag{5.12}$$

In this way, we may state that if $\gamma_p(a_x, a_y) > 0$ then the laminar flow (4.2) is nonlinearly exponentially stable with respect to all perturbations periodic along x and y direction with periods $2\pi/a_x$ and $2\pi/a_y$ as

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 \leq \|\boldsymbol{u}(\cdot,0)\|_2^2 \exp\left[-\frac{2\gamma_p(a_x,a_y)}{\mathrm{DaRe}}t\right] \quad \forall \boldsymbol{u} \in \mathcal{I}_p. \tag{5.13}$$

The Euler-Lagrange equations corresponding to the variational problem (5.11) are

$$\begin{cases} \nabla\chi + (1-\sigma)\boldsymbol{u} - \mathrm{Da}\Delta\boldsymbol{u} + \dfrac{1}{2}\mathrm{DaRe}U'(w\boldsymbol{i} + u\boldsymbol{k}) = \boldsymbol{0}, \\ \mathrm{div}\boldsymbol{u} = 0, \end{cases} \tag{5.14}$$

where χ is a Lagrange multiplier associated with the incompressibility constraint. Then, the number $\gamma_p(a_x, a_y)$ is the least eigenvalue σ of the characteristic-value problem (5.14)and (5.12).

Since the Euler-Lagrange equations (5.14) are linear we may follow the same arguments as those employed for the linear stability analysis. More specifically, we take the third components of curl and curlcurl of $(5.14)_1$, use $(5.6)_3$ and $(5.14)_2$ and look for solutions of the form

$$\begin{cases} w(x,y,z) = W(z)\exp[i(a_x x + a_y y)], \\ \zeta(x,y,z) = \mathrm{curl}\boldsymbol{u}\cdot\boldsymbol{k} = \Psi(z)\exp[i(a_x x + a_y y)] \end{cases} \tag{5.15}$$

to reduce the eigenvalue problem (5.14) and (5.12) to

$$\begin{cases} \mathrm{Da}(D^2-a^2)^2W + (\sigma-1)(D^2-a^2)W \\ \qquad\qquad + \dfrac{\mathrm{DaRe}}{2}\left(2ia_xU'DW + ia_yU'\Psi + ia_xU''W\right) = 0, \\ \mathrm{Da}(D^2-a^2)\Psi + (\sigma-1)\Psi + \dfrac{\mathrm{DaRe}}{2}ia_yU'W = 0, \\ W = DW = \Psi = 0 \quad \text{at } z = 0,1. \end{cases} \tag{5.16}$$

Finally, from (28) we may state the following theorem.

Theorem 2. *Assume that*

$$\gamma_{\mathrm{cr}} \equiv \min_{a_x, a, y > 0} \gamma_p(a_x, a_y) > 0. \tag{5.17}$$

Then the laminar flow (4.2) *is globally exponentially stable.*

The sixth-order system (5.16) was solved using the Chebyshev-tau method [38]. We let $a_x = a\sqrt{\gamma}$ and $a_y = a\sqrt{1-\gamma}$, such that $\gamma \in [0,\, 1]$ for the range of a_x and a_y values which comprise wavenumber a.

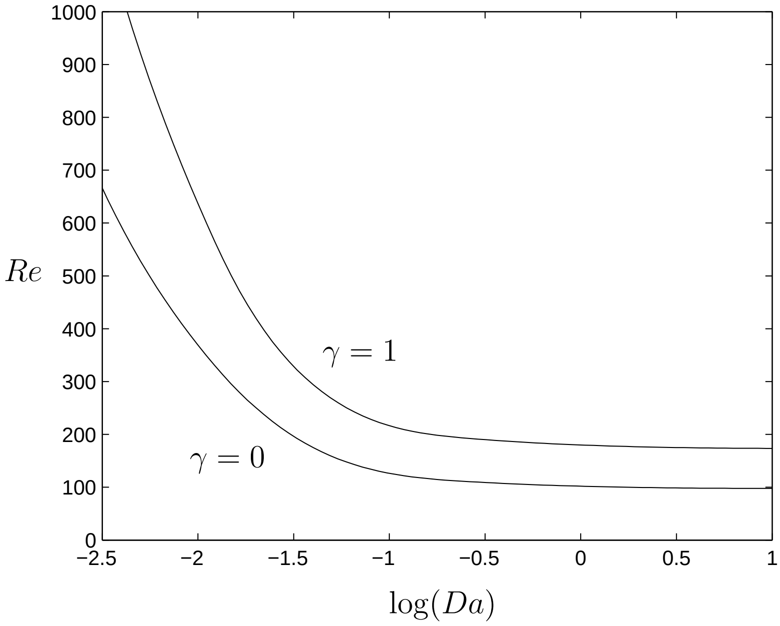

Figure 2: Visual representation of the Poiseuille flow nonlinear stability thresholds with critical Reynolds number Re plotted against $\log(Da.)$ The thresholds for γ values between 0 and 1 are contained between the $\gamma = 0$ and $\gamma = 1$ lines.

The numerical results for Poiseuille flow in Figure 2 correspond to comparable studies on Brinkman flow [39].

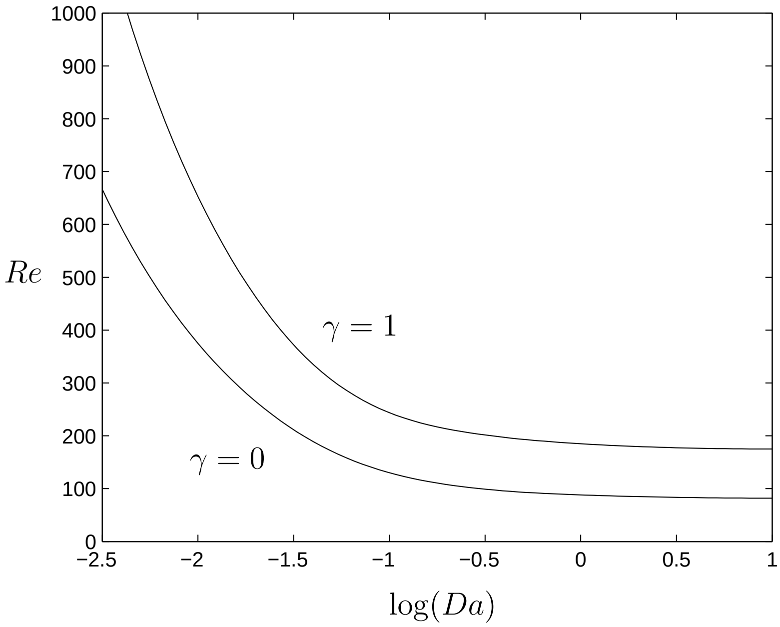

Figure 3: Visual representation of the couette flow nonlinear stability thresholds with critical Reynolds number Re plotted against $\log(Da.)$ The thresholds for γ values between 0 and 1 are contained between the $\gamma = 0$ and $\gamma = 1$ lines.

Although there is some quantitative differences with Poiseuille flow, the couette flow nonlinear stability thresholds follow a similar formation.

References

[1] H. C. Brinkman, *A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles*, Applied Scientific Research A1 (1947), 27–34.

[2] H. C. Brinkman, *On the permeability of the media consisting of closely packed porous particles*, Applied Scientific Research A1 (1947), 81–86.

[3] H. Darcy, *La Fontaines Publiques de La Ville de Dijon*, Victor Dalmont (1846).

[4] P. Forchheimer, *Wasserbewegung durch Boden*, Zeits. V. deutsch. Ing **45** (1901), 1782–1788.

[5] D. Munaf, D. Lee, A. S. Wineman, K. R. Rajagopal, *A boundary value problem in groundwater motion analysis-comparisons based on Darcy's*

law and the continuum theory of mixtures, Mathematical Modeling and Methods in Applied Science **3** (1993), 231–248.

[6] S. C. Subramaniam, K. R. Rajagopal, *A note on the ow through porous solids at high pressures*, Computers and Mathematics with Applications **53** (2007), 260–275.

[7] K. Kannan, K. R. Rajagopal, *Flow through porous media due to high pressure gradients*, Applied Mathematics and Computations **199** (2008), 748–759.

[8] K. R. Rajagopal, *Hierarchy of models for the flow of fluids through porous media*, Mathematical Model and Methods in the Applied Sciences **17** (2007), 215–252.

[9] C. Truesdell, *Sulle basi della termomeccanica*, Rendiconti dei Lincei **22** (1957), 33–38.

[10] C. Truesdell, *Sulla basi della termomeccanica*, Rendiconti dei Lincei **22** (1957), 158–166.

[11] R. M. Bowen, *Mechanics of Mixtures*, in Continuum Physics, ed. A. C. Eringen, Vol III, Academic Press (1976).

[12] R. J. Atkin, R. E. Craine, *Continuum theory of mixtures: basic theory and historical developments*, Quarterly Journal of Mechanics and Applied Mathematics **29** (1976), 209–234.

[13] I. Samohyl, *Thermodynamics of Irreversible processes in Fluid Mixtures*, Teubner (1987).

[14] K. R. Rajagopal, L. Tao, *Mechanics of Mixtures*, World Scientific Press, Singapore (1995).

[15] G. Johnson, M. Massoudi, K. R. Rajagopal, *A review of interaction mechanisms in fluid-solid flows*, DOE Report, DOE/PETC/TR-90/9, Pittsburgh (1990).

[16] O. Reynolds, *On the dynamical theory of incompressible viscous fluids and the determination of the criterion*, Phil. Trans. Roy. Soc. London A **186** (1895), 123–164.

[17] W. F. McOrr, Proceedings of the Royal Irish Academy **27** (1907), 69.

[18] J. L. Synge, *Hydrodynamical stability*, Semi-centennial publications of the Amer. Math. Soc. **2** (1938), 227–269.

[19] J. Kampe de Feriet, *Sur la decroissance de l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine borné ayant pour frontière des parois solides fixes*, Ann. Soc. Sci. Bruxelles **63** (1949), 35–46.

[20] R. Berker, *Inégalité vérifiée par l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine spatial borné*, Bull. Tech. Univ. Istanbul **2** (1949), 41–50.

[21] T. Y. Thomas, *On the stability of viscous fluids*, Univ. Calif. Publ. Math. New Series **2** (1944), 13–43.

[22] E. Hopf, *On non-linear partial differential equations*, Lecture series of the symposium on partial differential equations, University of California (1955), 7–11.

[23] J. Serrin, *On the stability of viscous fluid motions*, Archive for Rational Mechanics and Analysis **3** (1959), 1–13.

[24] Y. Qin, P. N. Kaloni, *Convective instabilities in anisotropic porous media*, Stud. Appl. Math. **91** (1999), 189–204.

[25] Y. Qin, P. N. Kaloni, *Spatial decay estimates for plane flows in the Brinkman-Forchheimer model*, Q. Appl. Math. **56** (1998), 71–87.

[26] Y. Qin, J. Guo, P. N. Kaloni, *Double diffusive penetrative convection in a porous media*, Intl. Engng. Sci. **33** (1995), 303–312.

[27] J. Guo, P. N. Kaloni, *Double diffusive convection in porous medium, non-linear stability and the Brinkman effect*, Stud. Appl. Math. **94** (1995) 351–358.

[28] F. Franchi, B. Straughan, *Structural stability for the Brinkman equation in porous media*, Mathematical Methods in the Applied Sciences **19** (1996), 1335–1347.

[29] L. E. Payne, B. Straughan, *Stability in initial time geometry for the Brinkman and Darcy equations of flow in porous media*, J. Math. Pures Appl. **25** (1996), 225–271.

[30] A. Oberbeck, *Ueber die Wärmeleitung der Flüssigkeiten bei Berücksichtigung der Strömungen infolge von Temperaturdifferenzen*, Ann. Phys. Chem. **1** (1879), 271–292.

[31] A. Oberbeck, *Uber die Bewegungsercheinungen der Atmosphere*, Sitz. Ber. K. Preuss Academy (1888), 383–395.

[32] J. Boussinesq, *Theorie Analytique de la Chaleur*, Gauthier-Villars (1903).

[33] K. R. Rajagopal, M. Ruzicka, A. R. Srinivasa, *On the Oberbeck-Boussinesq equations*, Mathematical Models and Methods in the Applied Sciences **6** (1996), 1157–1167.

[34] B. Straughan, *Stability and wave motion in porous media*, Appl. Math. Sci. Ser. **165**, Springer-Verlag (2008).

[35] L. E. Payne, H. F. Weinberger, *An optimal Poincaré inequality for convex domains*, Archive for Rational Mechanics and Analysis **5** (1960), 286–292.

[36] O. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York (1969).

[37] S. Rionero, *Metodi variazionali per la stabilitá asintotica in media in magnetoidrodinamica*, Ann. Mate. Pura Appl. **78** (1968), 339–364.

[38] J. J. Dongarra, B. Straughan, D. W. Walker, *Chebyshev tau-QZ algorithm methods for calculating spectra of hydrodynamic stability problems*, Appl. Numer. Math. **22** (1996), 399–434.

[39] A. A. Hill, B. Straughan, *Stability of Poiseuille flow in a porous medium*, In: Rannacher R, Sequeira A, eds., Advances in Mathematical Fluid Mechanics. Springer Berlin Heidelberg, pp. 287–293.