MATHEMATICAL MODEL OF ELECTROMAGNETIC FIELD WITH SKIN-EFFECT IN CLOSED ELECTRICAL CONTACTS

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Abstract: The mathematical model describing electromagnetic field in closed electrical contacts is elaborated. It takes into account a non-uniformity of the current density on the contact spot which can be explained by three factors: 1) physical phenomena of the constriction, 2) influence of the contact cross-section radius, 3) skin effect in the case of the alternative current. The effect of each above factors in dependence of given parameters is discussed. The model is based on the Maxwell equations which are reduced then to the solution of dual integral equations and series. It is shown that the skin-effect should be taken into consideration at the current frequencies which are greater than $10^3 Hz$ and for the contact forces which are greater than $10^4 N$.

1. Introduction

Mathematical models describing electromagnetic field in electrical contacts are

based as a rule on the assumption about the uniformity of the current density on the contact spot. This assumption enables one to reduce the problem of finding of component of the el3ectrical field to the solution of the Neumann problem for the Laplace equation, which can be solved by standard methods [1]. Such model is approximate and can be used for the calculation of the contact resistance or for the estimation of an average temperature in the constriction zone only. But if the contact heat transfer is essentially non-stationary, for example, welding at high range of the current, then the local overheating at the edge of the contact spot can be explained by a non-uniformity of the current density along the radius of the contact spot only. Corresponding mathematical models describing electromagnetic and temperature fields in contacts are presented in the papers [2]-[4]. However the general model taking into account all factors responsible for the non-uniformity of electromagnetic field including the skin-effect should be elaborated.

Such non-uniformity can be stipulated by three factors: 1) physical phenomena of the constriction, 2) influence of the contact cross-section radius, 3) skin effect in the case of the alternative current. Let us estimate of an influence of each factor on the contact electromagnetic field.

At first we consider two semi-cylinders of the radius R occupying the regions $D_1(0 < r < R, \quad -\infty < z < 0)$ and $D_2(0 < r < R, \quad 0 < z < \infty)$ which have the common current conducting contact spot $D_0(0 < r < r_0, \quad z = 0)$. Electromagnetic field in electrical contacts can be described by the Maxwell equations

$$rot\overline{E} = -\mu\mu_0\frac{\partial\overline{H}}{\partial t} \tag{1}$$

$$rot\overline{H} = \frac{1}{p}\overline{E} \tag{2}$$

$$div\overline{H} = 0 \tag{3}$$

where $\overline{E}$ and $\overline{H}$ are electrical and magnetic strength, μ, μ_0, ρ are magnetic permeability, magnetic constant and electrical resistivity correspondingly. The vector of the electrical field $\overline{E}$ in the axisymmetric case has two components, the radial component $\overline{E}_r$ and the axial component E_z, while the vector of the magnetic field has the angle component H_φ only.

It can be derived from the Maxwell equations (1)-(3) that H_φ is satisfied the equation

$$\frac{\partial H_\varphi^{(i)}}{\partial t} = \frac{1}{\mu_0\mu}\left[\frac{\partial}{\partial z}\left(\rho_i\frac{\partial H_\varphi^{(i)}}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{\rho_i}{r}\frac{\partial}{\partial r}rH_\varphi^{(i)}\right)\right] \tag{4}$$

Here the index $i = 1$ and $i = 2$ correspond to the cathode D_1 and the anode D_2 respectively.

The boundary conditions can be written in the form

$$H_\varphi^{(i)}(r, z, 0) = 0 \tag{5}$$

$$\left.\frac{\partial H_\varphi^{(i)}}{\partial z}\right|_{\substack{z=0 \\ 0<r<r_0}} = 0 \tag{6}$$

$$\left.H_\varphi^{(i)}\right|_{\substack{z=0 \\ r_0<r<R}} = -\frac{I}{2\pi r} \tag{7}$$

$$\left.\frac{\partial H_\varphi^{(i)}}{\partial r}\right|_{r=R} = 0 \tag{8}$$

$$\left.\frac{\partial H_\varphi^{(i)}}{\partial z}\right|_{z=\infty} = 0 \tag{9}$$

where I is the electrical current. These conditions are obvious. In particular, the condition (7) states the law of the total current.

The electrical field can be found from the magnetic field due to the Maxwell equations as

$$E_r^{(i)} = -\rho_i \frac{\partial H_\varphi^{(i)}}{\partial z}, \quad E_z^{(i)} = \frac{\rho_i}{r}\frac{\partial}{\partial r}rH_\varphi^{(i)} \tag{10}$$

2. Influence of the Current Constriction on the Electromagnetic Field

Let us suppose that $\rho_i = const, I = const$ and $R = \infty$ because $r_0 << R$. Then the equation (4) takes the same form for both $i = 1$ and $i = 2$:

$$\frac{\partial^2 H_\varphi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}rH_\varphi\right) = 0 \tag{11}$$

The condition (5) should be omitted. The solution of this equation with the conditions (6)-(9) can be represented in the form of the integral

$$H_\varphi(r, z) = \int_0^\infty erxp(-\lambda z)J_1(\lambda r)\nu(\lambda)\,d\lambda \tag{12}$$

which kernel corresponds to the eigenfunctions of the equation (11) and $\nu(\lambda)$ is unknown function. The equation (11) and the condition (9) are satisfied for any $\nu(\lambda)$. To satisfy the conditions (6) and (7) we get the dual integral equations for the function $\nu(\lambda)$:

$$
\begin{cases}
\int_0^\infty J_1(\lambda r)\nu(\lambda)\lambda\, d\lambda = 0, & 0 \leq r < r_0 \\[2mm]
\int_0^\infty J_1(\lambda r)\nu(\lambda)\lambda\, d\lambda = -\frac{I}{2\pi r}, & r_0 \leq r < \infty
\end{cases}
\tag{13}
$$

The solution of these equations can be found using Wiener-Hopf method [5]:

$$
\nu(\lambda) = -\frac{I}{2\pi r_0 \lambda}\sin(r_0\lambda)
\tag{14}
$$

After substitution of (14) into (12) and calculation of integral we get

$$
H_\varphi(r,z) = -\frac{I}{2\pi r}\left(1 - \frac{z}{\xi}\right)
\tag{15}
$$

where $\xi = \xi(r,z)$ is an equipotential defined from the equation

$$
\frac{r^2}{r_0^2 + \xi_2} + \frac{z^2}{\xi^2} = 1
\tag{16}
$$

i.e.

$$
\xi(r,z) = \frac{1}{\sqrt{2}}\sqrt{s + \sqrt{s^2 + 4r_0^2 z^2}}, \quad s = z^2 - r_0^2
\tag{17}
$$

Substituting the expression (15) into formulas (10) and using the relationship (16) we get the components of the electrical field:

$$
E_r^{(i)} = -\frac{I\rho_i}{2\pi(r_0^2 + \xi^2)}\frac{\partial\xi}{\partial r}, \quad E_z^{(i)} = -\frac{I\rho_i}{2\pi(r_0^2 + \xi^2)}\frac{\partial\xi}{\partial z}
\tag{18}
$$

where

$$
\frac{\partial\xi}{\partial r} = \frac{r\xi(r_0^2 + \xi^2)}{(r_0^2 + \xi^2)^2 - r^2 r_0^2}, \quad \frac{\partial\xi}{\partial z} = \frac{z}{\xi}\left[1 + \frac{r^2 r_0^2}{(r_0^2 + \xi^2)^2 - r^2 r_0^2}\right]
\tag{19}
$$

Using the relationship between the vector of electrical field $\overline{E}$, electric potential φ and current density $\overline{j}$

$$
\overline{E} = -grad\varphi = \rho\overline{j}
$$

it is not difficult now to write the expressions for the components of the current density in contacts :

$$j_r = -\frac{I}{2\pi(r_0^2 + \xi^2)}\frac{\partial \xi}{\partial r}, \quad j_z = -\frac{I}{2\pi(r_0^2 + \xi^2)}\frac{\partial \xi}{\partial z} \tag{20}$$

Using these formulas we can now to find the expressions for the power density of the Joule heat sources

$$q_i(r, z, r_0) = \rho_i(j_z^2 + j_r^2) = \frac{I^2\rho_i}{2\pi}[2(2\xi^2 - s)(r_0^2 + \xi^2)]^{-1} \tag{21}$$

or

$$q_i(r, z, r_0) = \frac{I^2\rho_i}{16\pi^2 r_0^2}q_0(r, z, r_0) \tag{22}$$

where

$$q_0(r, z, r_o) = \left[\frac{1}{r}\left(H - \frac{1}{H}\right)\right]^2, \quad H = \left[\frac{z^2 + (r + r_0)^2}{z^2 + (r - r_0)^2}\right]^{1/4} \tag{23}$$

In the neighborhood $r = 0$ it is more convenient another representation of the function $q_0(r, z, r_0)$

$$q_0(r, z, r_0) = \frac{16r_0^2}{\sqrt{z^2 + (r - r_0)^2}\sqrt{z^2 + (r + r_0)^2}} \times \tag{24}$$

$$\times \frac{1}{[\sqrt{z^2 + (r - r_0)^2} + \sqrt{z^2 + (r + r_0)^2}]^2}$$

It can be concluded from these expressions that maximum of the current density and the power density of the Joule heat sources are situated at the edge of the contact spot $z = 0, r = r_0$, where they are not bounded , however they have an integrable singularity. Such contact model is able to explain the phenomena of the ring-shape contact welding in the range of high current and high contact forces in contrast to the model with constant current density across a contact spot.

The components of current density on the contact plane are defined by the expressions:

$$j_r(r, 0, r_0) = 0, \quad j_z(r, 0, r_0) = \frac{I}{2\pi r_0}\frac{1}{\sqrt{r_0^2 - r^2}}, \quad r < r_0 \tag{25}$$

It can be derived from the relationships:

$$\lim_{z \to 0} \xi = \begin{cases} \sqrt{r^2 - r_0^2}, & r > r_0 \\ 0, & r < r_0 \end{cases}, \tag{26}$$

$$\lim_{z \to 0} \frac{\partial \xi}{\partial z} = \begin{cases} \frac{\xi}{\sqrt{r_0^2 - r^2}}, & r \geq r_0 \\ 0, & r > r_0 \end{cases}, \qquad \lim_{z \to 0} \frac{z}{\xi} = \begin{cases} \frac{1}{r_0}\sqrt{r_0^2 - r^2}, & r < r_0 \\ 0, & r \leq r_0 \end{cases}$$

It follows from these expressions that the portion of the current I passing through the circle of the radius r, $0 \leq r_1 \leq r$, $r < r_0$ can be defined by the formula

$$\frac{I}{r_0} \int_0^r \frac{r_1 dr_1}{\sqrt{r^2 - r_1^2}} = I\left[r - \sqrt{r_0^2 - r^2}\right]$$

In particular, $0.5I$ is passing through the circle of the radius $r = \sqrt{3}r_0/2 = 0.866r_0$

The picture of the electric field (and also the same picture of the stationary temperature field due to well-known Holms theorem about electrical-heat analogies [6]) in the constriction region corresponding cylindrical coordinate system is presented in Fig. 1
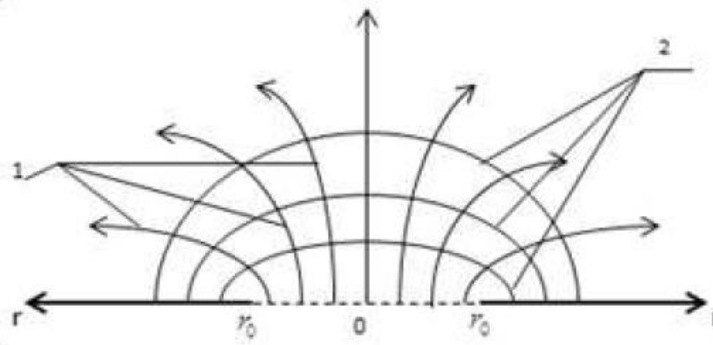

Fig.1 Electric field and stationary temperature field in the contact constriction region 1-passing heat flux (electrical current), 2-isotherms(equipotentials)

The contact circle of the radius is reproduced in the form of infinitely thin disk for the electrical current passing into semi-bounded contact. In the contrast to the radial spherical model equipotential and isothermal surfaces of contacting electrodes are the family of ellipsoids of revolution 1, while passing heat flux and electrical current 2 correspond to the confocal family of hyperboloids of revolution.

3. The Electric Potential and the Constriction Resistance

To get more information about considered electromagnetic system we find also the distribution of the electric potential $\varphi_i(r, z)$ and the constriction resistance R_c.

The boundary conditions for the potential far away from the contact zone can be written in the form

$$\varphi_1(r,z)\Big|_{\substack{z^2+r^2=\infty \\ r<0}} = -\frac{u_c}{2}, \quad \varphi_2(r,z)\Big|_{\substack{z^2+r^2=\infty \\ r>0}} = \frac{u_c}{2} \tag{27}$$

Then the potentials can be calculated by the expressions

$$\varphi_1(r,z) = -\frac{u_c}{2} - \int_{-\infty}^{z} E_z^{(1)}\, dz = -\frac{u_c}{2} + \frac{I\rho_1}{2\pi} \int_{-\infty}^{z} \frac{1}{r_0^2+\xi^2}\frac{\partial\xi}{\partial z}\, dz = \tag{28}$$

$$-\frac{u_c}{2} + \frac{I\rho_1}{2\pi}\left(\frac{\pi}{2} - \arctan\frac{\xi}{r_0}\right)$$

$$\varphi_2(r,z) = \frac{u_c}{2} + \int_{z}^{\infty} E_z^{(2)}\, dz = \frac{u_c}{2} - \frac{I\rho_2}{2\pi} \int_{z}^{\infty} \frac{1}{r_0^2+\xi^2}\frac{\partial\xi}{\partial z}\, dz = \tag{29}$$

$$\frac{u_c}{2} - \frac{I\rho_2}{2\pi}\left(\frac{\pi}{2} - \arctan\frac{\xi}{r_0}\right)$$

The constriction resistance $R_c^{(i)}$ of a semi-contact D_i can be defined by the formula [1]:

$$R_c^{(i)} = \frac{2\pi}{I^2} \int_0^{\infty} \left(E^{(i)} H_\varphi\right)\Big|_{z=0} r\, dr$$

Taking into account the expressions (18), (26), (7) we get

$$R_c^{(i)} = \frac{\rho_i}{2\pi} \int_{r_0}^{\infty} \frac{dr}{r\sqrt{r^2-r_0^2}} = \frac{\rho_i}{4r_0}$$

The total constriction resistance R_c and the contact voltage u_c can be found using the formulas

$$R_c^{(1)} + R_c^{(2)} = \frac{\rho_1+\rho_2}{4r_0}, \quad u_c = IR_c = \frac{\rho_1+\rho_2}{4r_0} \tag{30}$$

In the case of homogeneous contact materials ($\rho_1 = \rho_2$) we get from (28)-(29) the well-known formula for the potential distribution

$$\varphi_i(r,z) = (-1)^i\frac{u_c}{\pi}\arctan\frac{\xi}{r_0} \tag{31}$$

One can conclude from the expressions (20)-(21) that the current density is the same in each semi-contact D_i, while the power density of the Joule heat sources is different, and it will be greater in a semi-contact with greater value of ρ_i.

4. Influence of the Contact Cross-Section Radius

Let us consider now the situation when it is important to take into account the influence of the cross-section radius R of the contact on the electromagnetic field, for example by the modeling of the skin-effect. One can suppose approximately that the expressions (10), (15), (16) describe the field only inside the domain bounded by the contact cross-section $z = 0$, $\quad 0 < r < R$ and the supported ellipsoid $\frac{r^2}{R^2} + \frac{z^2}{R^2 - r_0^2} = 1$. Outside of this domain the magnetic field is radial and the electric field is axial (Fig.2)
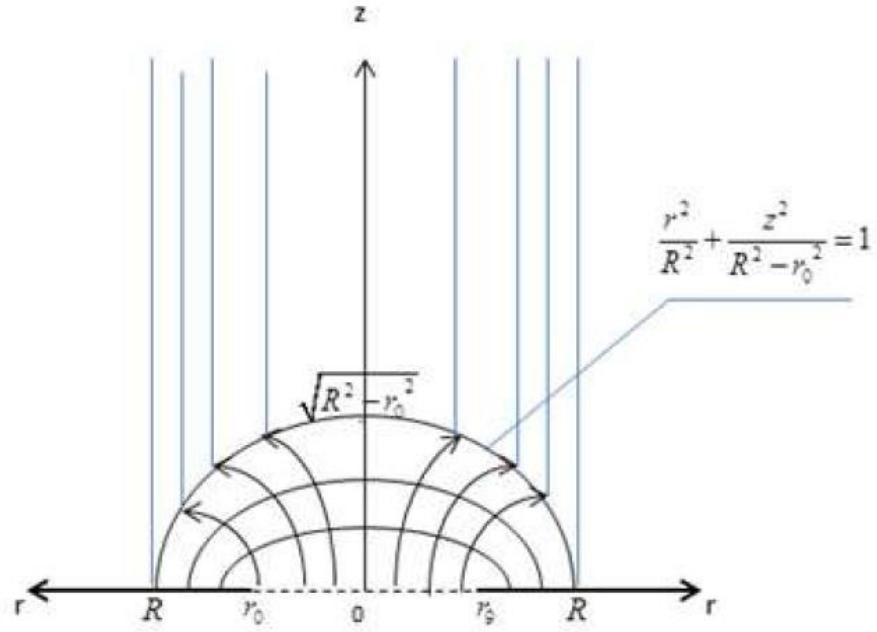

Fig2. The model for the correction of the electromagnetic field due to the bounded radius of the contact cross-section

To verify the correctness of this model we calculate the axial component of the electrodynamic force P_ν using the basic well-known formula

$$\overline{P}_{ed} = \mu\mu_0 \int_D \frac{1}{\rho} \overline{[EH]} dV$$

In accordance to this formula we get for the considered model

$$P_\nu = \frac{2\pi\mu\mu_0}{\rho} \int_0^{\sqrt{1 - \frac{z^2}{R^2 - r_0^2}}} E_r H_\varphi r \, dr =$$

$$= \frac{\mu\mu_0 I^2}{2\pi} \int_0^{\sqrt{R^2-r_0^2}} dz \int_0^{R\sqrt{1-\frac{z^2}{R^2-r_0^2}}} \left(1 - \frac{z}{\xi}\right) \frac{1}{r_0^2 + \xi^2} \frac{\partial\xi}{\partial r} dr =$$

$$= \frac{\mu\mu_0 I^2}{2\pi} \int_0^{\sqrt{R^2-r_0^2}} dz \int_0^{\sqrt{R^2-r_0^2}} \left(1 - \frac{z}{\xi} bigg\right) \frac{d\xi}{r_0^2 + \xi^2} =$$

$$= \frac{\mu\mu_0 I^2}{2\pi} \int_0^{\sqrt{R^2-r_0^2}} \left(\frac{1}{r_0} \arctan \frac{\sqrt{R^2 - r_0^2}}{r_0} - \right.$$

$$\left. - \frac{1}{r_0} \arctan \frac{z}{r_0} - \frac{z}{r_0^2} ln \frac{z}{\sqrt{z^2 + r_0^2}} \right) dz = \frac{\mu\mu_0 I^2}{4\pi} ln \frac{R}{r_0}$$

i.e.

$$P_\nu = \frac{\mu\mu_0 I^2}{4\pi} ln \frac{R}{r_0} \tag{32}$$

That coincides with the well-known formula obtained by Dwight [7].

To estimate the influence of the radius R on the electromagnetic field more accurately one should consider the equation (4) with the conditions (5)-(9) in the bounded domain. In this case the magnetic field should be written in the form of Bessel series

$$H_\nu(r, z) = -\frac{I}{2\pi} \sum_{n=1}^{\infty} C_n \exp\left(-\lambda_n \frac{z}{r_0}\right) J_1\left(\lambda_n \frac{r}{r_0}\right) \tag{33}$$

where λ_n are the roots of the equation

$$J_1'(\lambda_n \alpha) = 0, \quad \alpha = \frac{r_0}{R}$$

and the coefficients C_n should be found by the solution of the equations of dual series

$$\begin{cases} \sum_{n=1}^{\infty} C_n J_1\left(\lambda_n \frac{r}{r_0}\right) = \frac{1}{r}, & 0 < r < R \\\\ \sum_{n=1}^{\infty} C_n J_1\left(\lambda_n \frac{r}{r_0}\right) = 0, & 0 < r < r_0 \end{cases} \tag{34}$$

Such type of the equations is considered in the paper [4]. Using its results one can find that the coefficients C_n can be calculated by the formula

$$C_n = \frac{2\sqrt{2}}{\sqrt{\pi}} \alpha^2 = \frac{J_{3/2}(\lambda_n)}{\sqrt{\lambda_n} J_2^2(\lambda_n/\alpha)} \frac{1}{1 - (4\pi^2)(1,368\alpha + 0.216\alpha^2)} \tag{35}$$

with the error not greater than α^3.

If $R \to \infty$, then the series (33) can be summarized and transforms into the expression (15).

5. Skin Effect

Let us consider now the case of the alternative periodic current $I(t) = I_0 \exp(i\omega t)$ passing through homogeneous contact with constant resistivity ($\rho_1 = \rho_2 = \rho = const$). We suppose also that $R = \infty$. Then the equation (4) takes the form

$$\frac{\partial H_\varphi}{\partial t} = k^2 \left[\frac{\partial^2 H_\varphi}{\partial t^2} + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} r H_\varphi \right) \right] \tag{36}$$

where $k^2 = \rho/(\mu\mu_0)$. If the time of passing of the current is sufficiently long, then this equation should be considered without initial condition (5) but with the boundary conditions (6)-(8) where $R = \infty$.

We represent a solution of this problem in the form

$$H_\varphi(r, z, t) = \frac{I_0}{2\pi} \exp(i\omega t) \int_0^\infty \exp(-\sqrt{\lambda^2 - k^2}) J_1(\lambda r) \mu(\lambda) \, d\lambda \tag{37}$$

where $k^2 = -i\omega/k^2$ and $\mu(lambda)$ is an unknown function determined from the conditions (6)-(7), which give the dual integral equations

$$\begin{cases} \int_0^\infty \sqrt{\lambda^2 - k^2} J_1(\lambda r) \mu(\lambda) d\lambda = 0, & 0 < r < r_0 \\[2mm] \int_0^\infty J_1(\lambda r) \mu(\lambda) d\lambda = \frac{1}{r}, & r_0 < r < \infty \end{cases} \tag{38}$$

The project of a solution of these equations can be represented in the form

$$\mu(\lambda) = J_0(\lambda r_0) + \int_0^{r_0} \varphi(t) \left(\frac{\sin \lambda t}{\lambda t} - \cos \lambda t \right) dt \tag{39}$$

It is not difficult to conclude that the second equation in (38) is satisfied automatically for any function $\varphi(t)$ due to the properties of a discontinuous integral of the Weber-Shaftheitlin type. Substituting (39) into the first equation of (38) and using the formula

$$\lambda r^2 J_1(\lambda r) = \frac{d}{dr} \left[r^2 J_2(\lambda r) \right]$$

we get

$$r^2 \psi(r) = \frac{d}{dr} \int_0^\infty r^2 J_2(\lambda r) \frac{\sqrt{\lambda^2 - k^2}}{\lambda} \, d\lambda \int_0^{r_0} \varphi(t) \left(\frac{\sin \lambda t}{\lambda t} - \cos \lambda t \right) dt$$

where

$$\psi(r) = - \int_0^\infty \frac{\sqrt{\lambda^2 - k^2}}{\lambda} J_1(\lambda r) J_0(\lambda r_0) \lambda \, d\lambda$$

Integrating with respect r from to 0 to r, changing the order of integration and calculating the inner integral we get

$$\int_0^r \frac{t^2 \varphi(t) dt}{\sqrt{r^2 - t^2}} = \int_0^r r^2 \psi(r) dr + \int_0^{r_0} K_0(r,t) \varphi(t) dt \tag{40}$$

where

$$K_0(r,t) = r^2 \int_0^\infty \left(1 - \frac{\sqrt{\lambda^2 - k^2}}{\lambda}\right) J_0(\lambda r) \left(\frac{\sin \lambda t}{\lambda t} - \cos \lambda t\right) d\lambda \tag{41}$$

The left side of the equation (40) contains the integral operator of the Abel type capable the conversion, thus we get

$$\varphi(r) - \int_0^{r_0} K(r,t) \varphi(t) dt = F(r) \tag{42}$$

where

$$K(r,t) = \frac{2}{\pi r^2} \frac{d}{dr} \int_0^r \sqrt{r^2 - u^2} \frac{\partial K_0(u,t)}{\partial u} du \tag{43}$$

and $F(r)$ after simplification takes the form

$$F(r) = -\sqrt{\frac{2}{\pi}} \frac{1}{r^2} \frac{d}{dr} \int_0^\infty \frac{\sqrt{\lambda^2 - k^2}}{\lambda} J_{5/2}(\lambda r) J_0(\lambda r_0) \frac{d\lambda}{\lambda} \tag{44}$$

The equation (42) belongs to the type of second-order Fredholm integral equations which can be solved by the iterations:

$$\varphi(r) = \sum_{n=0}^\infty \varphi_n(r), \varphi_0(r) = F(r), \varphi_n(r) = F(r) + \int_0^{r_0} K(r,t) \varphi_{n-1}(t) dt. \tag{45}$$

Thus the solution of the problem (36), (6)-(8) is given by the expressions (37), (39), (45), (44), (43), (41).

In the case of a week skin-effect

$$|kr_0| << 1 \tag{46}$$

which is usually related to an ordinary real situation in electrical contacts it is more convenient to replace the expression (39) by the following expression

$$\mu(\lambda) = \frac{\sin \lambda r_0}{\lambda r_0} + \int_0^{r=0} \varphi(t) \left(\frac{\sin \lambda t}{\lambda t} - \cos \lambda t\right).$$

Taking into account that in this case $\mu(\lambda) \approx k^2/(2\lambda^2)$ and integrating the expression (37) we get the following formula for the magnetic field

$$H_\varphi(r,z,t) = -\frac{I_0}{2\pi}e^{i\omega t}\left[\frac{1}{r}\left(1 - \frac{z}{\xi}\right) + \frac{k^2 z}{6r\xi}(2\xi^2 - 2z\xi + z^2 + r^2)\right].$$

If $\omega = 0$, then this expression transforms to the expression (15). Substituting it into the formula (10) we can find the components of the electrical field, thus the component of the current density. In particular, the current density on the contact spot due to (26) is defined by the expression

$$j_z(r,0,t) = \frac{1}{r}\frac{\partial}{\partial r}rH_\varphi(r,0,t) = \tag{47}$$

$$-\frac{I_0 \exp(i\omega t)}{2\pi r_0\sqrt{r_0^2 - r^2}}\left[1 + \frac{k^2 r_0^2}{3}\left(1 - \frac{3r^2}{2r_0^2}\right)\right].$$

Here the second term in the square brackets gives the additional component for the increasing of the current density at the edge of the contact spot due to the skin-effect. Let us estimate now the range of the correctness of the condition (46). Taking into account that electrical resistivity has approximately the order from $10^{-8}\Omega \cdot m$ to $10^{-7}\Omega \cdot m$ and $\mu = 1$ for non-magnetic materials we can conclude that the thickness of the skin-layer is

$$\frac{1}{k} = \sqrt{\frac{\rho}{\omega\mu\mu_0}} = \sqrt{\frac{\rho}{2\pi\nu\mu\mu_0}} = \frac{c}{\sqrt{\nu}}$$

where $0.03 \leq c \leq 0.1$.

If the current frequency $\nu = 50Hz$ and $10^{-8}m \leq r_0 \leq 10^{-4}m$, then $7\cdot10^{-7} \leq |kr_0| \leq 3\cdot10^{-2}$, thus the condition (40) is satisfied. The same conclusion remains to be correct for ferromagnetics when their temperature is lower than the Curie point. Thus the skin effect in this situation is negligible. However if the current frequency has the order of $10^3 Hz$ or the contact force has the order of $10^4 N$, i.e. the radius of the contact spot r_0 has the order greater than $10^{-2}m$, then the skin-effect should be taken into consideration.

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Acknowledgements

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