A new result on observer-based sliding mode control design for a class of uncertain Itô stochastic delay systems

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Abstract

This paper develops a new observer-based sliding mode control (SMC) scheme for a general class of Itô stochastic delay systems (SDS). The key merit of the presented scheme lies in its simplicity and integrity in design process of the traditional sliding mode observer (SMO) strategy, i.e., the state observer and sliding surface design as well as the associated sliding mode controller synthesis. For guaranteeing to use the scheme, a new LMIs-based criterion is established to ensure the exponential stability of the underlying sliding mode dynamics (SMDs) in mean-square sense with H_∞ performance. A bench test example is provided to

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numerically demonstrate the efficacy of the scheme and illustrate the application procedure for potential readers/users with interest in their ad hoc applications and methodology expansion.
Keywords: Stochastic systems; state estimation; H_∞ performance; sliding mode control; time-delay

1. Introduction

It has been widely witnessed that control of stochastic systems has increasingly received much attention as one of the most practically meaningful systems in both academic research and application fields [1]. It is worthwhile pointing out that, in recent years, the study of stochastic delay systems (SDS) has inspired a new wave of research under a very crucial factor that time-delay may frequently occur during the whole operation process, such as chemical processes, networked control systems, etc. Accordingly, a great deal of work has been devoted to the stability and stabilization of It$\hat{o}$ SDS [2, 3]. Meanwhile, the associated control design problems have been explored for the systems in parallel with the development of system control theory, e.g., H_∞ control and filtering [4-6].

Due to its various attractive features such as quick response, good transient performance, particularly, the invariance against matched uncertainties, and wide applications to various complex systems [7-17], sliding mode control (SMC) [18, 19] has been well known as an effective robust control strategy for uncertain and incompletely modelled systems. It is noted that a growing interest has been devoted to the extension of SMC to accommodate the SDS. Also, it is a fact that uncertain nonlinearity may occur through the system control channels (i.e., the matched uncertainty in SMC theory [18]), due to the variation of the control components and the structural parameters as well as the existence of the inevitably external disturbance, and this will also affect the systems performance directly or indirectly, and even leads to instability. Some representative results regarding the SMC of SDS include SMC of uncertain SDS [14], SMC for uncertain SDS with H_∞ performance index [15] to deal with a limitation (i.e., there exist a matrix G with appropriate dimension satisfying $Gg(t, x(t), x(t-d)) = 0$ for all $t \geq 0$); further, robust SMC of uncertain SDS has been considered where such restrictive condition to the most existing results is removed in [16].

It should be noted that most existing results for the SDS are obtained upon the premise that the system states are accessible, despite the efficacy of SMC. In many cases, consider that the state variables may not be totally acquired or even knotty to be estimated via output measurement, the observer-based SMC, also called sliding mode observer (SMO) strategy [20], has been developed and excellently implemented in various cases [21-29]. In particular, by using the SMO approach, a class of Markovian jump systems against actuator faults with quantized measurements and unknown actuator faults was concerned in [21]. In view of a new observer-based SMC design, a class of nonlinear delay systems was investigated in [25], and recently in [26] robust H_∞ control for uncertain singular time-delay systems was studied via a novel SMO synthesis. It is noted that designs of the new sliding surface and/or new-form observers were developed in those works, which may not be extended to stochastic control systems for certain technical reasons directly. To the best of the authors' knowledge, vast majority of the existing routes for the SMO are that: state observer is designed to generate the original state with assistance of the control input and/or its compensator to restrain the uncertainties and nonlinearity of the system and make the closed-loop systems operate stably. In detail, the design principle leads to that the estimation error system does not contain the control input in general, and then the closed-loop systems can maintain the desirable characteristics on the predesigned sliding surface through the observer and its error system when the associated sliding mode controller is employed. As a result, the achievements using this SMO-idea have been widely applied for SDS [28, 29]. By following along the lines of [25, 26], a new SMO-based scheme is presented in this paper, which may be a worthy addition of the SMO approach for the SDS. The key novelty covers the following:

(a) A particular state observer is designed without any control terms compared with the existing results on SDS;

(b) A novel integral sliding surface design is established on the basis of the new observer and the outputs in such a way that specific sliding mode dynamics (SMDs) of the closed-loop systems is reconstructed;

(c) A sufficient criterion for expected performance of the underlying SMDs is proposed with an easy-to-test LMI framework;

(d) A novel associated reaching motion controller is then synthesized to adaptively ensure the sliding mode phase so as to accommodate the desirable effects of the control strategy.

As such, the proposed scheme is feasible for analysing the stability of the unmeasured system state through the original system itself and its error system. In the other words, an improved procedure is created from the fact that if the stability of the original system and its error system can be ensured, then the observer state can also tend to be stable as it is. All these features distinguish the present scheme from the existing literatures.

The rest of the paper is organized as follows. Section 2 describes the research problems and preliminaries. Section 3 presents the main results of the new scheme. Section 4 selects a bench test example to demonstrate the results with computational experiments. Section 5 draws brief summary of the study and potential research expansion.

Notations: Throughout the paper and unless specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a completed probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where Ω is a sample space, $\mathcal{F}$ is the σ-algebra of subset of the sample space, and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. $\mathcal{E}\{\cdot\}$ is the expectation operator with respect to the probability measure $\mathcal{P}$. If A is a vector or matrix, its transpose is denoted by A^{T}, and the symmetric elements of the matrix is denoted by " $*$ ". $X > Y$ means that the matrix $X - Y$ is positive definite. $\mathrm{sym}\{X\}$ is denoted as $\mathrm{sym}\{X\} = X + X^{\mathrm{T}}$. If M is a matrix, its operator norm is denoted by $\|M\| = \sup\{\|Mx\| : \|x\| = 1\}$, $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ represent its maximum and minimum eigenvalues, respectively. $\mathrm{Tr}\{\cdot\}$ denotes the trace of a matrix. $\mathrm{diag}\{\cdot\}$ represents a block-diagonal matrix. $\mathbb{L}_2[0, \infty)$ stands for the space of square integral vector functions over $[0, \infty)$. Let $d > 0$ and $C([-d,\ 0]; \mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued functions on $[-d,\ 0]$. Let $C^b_{\mathcal{F}_0}([-d,\ 0]; \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-d,\ 0]; \mathbb{R}^n)$-valued random variables, and $\mathbb{L}^2([a, b]; \mathbb{R}^n)$ be the family of all $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process $\{\mathcal{F}_t\}_{a\leq t\leq b}$ such that $\int_a^b \|f(t)\|^2 \mathrm{d}t < \infty$ a.s. Let $\mathcal{M}^2([a, b]; \mathbb{R}^n)$ be the family of processes $\{\mathcal{F}_t\}_{a\leq t\leq b}$ in $\mathbb{L}^2([a, b]; \mathbb{R}^n)$ such that $\mathcal{E}\{\int_a^b \|f(t)\|^2 \mathrm{d}t\} < \infty$.

2. System description and preliminaries

Consider the following uncertain Itô-type stochastic delay systems (SDS) [15, 16] described by

$$\begin{cases} \mathrm{d}x(t) = \{(A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t-d) \\ \qquad\qquad + B[u(t) + f(t, x(t), x(t-d))] + Dv(t)\}\mathrm{d}t \\ \qquad\qquad + g(t, x(t), x(t-d))\mathrm{d}\omega(t), \\ y(t) = Cx(t), \\ x(\theta) = \phi(\theta),\ \theta \in [-d,\ 0] \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the system output, and $v(t) \in \mathrm{R}^q$ represents exogenous disturbance which belongs to $\mathbb{L}_2[0, \infty)$. $d > 0$ is the time-delay, and $\omega(t)$ is a standard scalar Brownian motion defined on a completed probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and satisfies $\mathcal{E}\{\mathrm{d}\omega(t)\} = 0$, $\mathcal{E}\{\mathrm{d}\omega^2(t)\} = \mathrm{d}t$. $\phi(t) \in C^b_{\mathcal{F}_0}([-d,\ 0]; \mathbb{R}^n)$ is the initial condition. A, A_d, B, C and D are known real matrices, B is of full column rank. $\Delta A(t)$ and $\Delta A_d(t)$ are norm bounded, i.e., $[\Delta A(t)\ \ \Delta A_d(t)] = MF(t)[N\ \ N_d]$, where M, N and N_d are constant matrices, and $F(t)$ is unknown matrix function satisfying $F^{\mathrm{T}}(t)F(t) \leq I$ for all $t \geq 0$.

For simplicity, denote the functions $f(t, x(t), x(t-d)) = f(t, x_t)$ and $g(t, x(t), x(t-d)) = g(t, x_t)$, respectively.

Assumption 1. $f(t, x_t)$ is unknown nonlinearity which represents the lumped perturbation of a physical plant through the control channel satisfying $\|f(t, x_t)\| \leq \alpha\|y(t)\|$, where $\alpha > 0$ is an unknown scalar [27].

Assumption 2. The diffusion gain function $g(t, x_t)$ may be unknown but there exist a matrix F to satisfy $\mathrm{Tr}\{g^{\mathrm{T}}(t, x_t)g(t, x_t)\} \leq \|Fy(t)\|^2$.

In the position, it is easy to verify that the SDS in (1) with $u(t) = 0$ and $v(t) = 0$, has a unique solution [1, 2]. The main objective of the study can be twofold: develop a new SMO scheme in such a way that **a**) reachability of the designed sliding surface for the closed-loop systems can be ensured within finite-time almost surely, and **b**) given a scalar $\gamma > 0$, the dynamics of the closed-loop systems during sliding mode phase are mean-square exponentially stable with $v(t) = 0$, and the inequality $\mathcal{E}\left\{\sup_{0\neq v(t)\in\mathbb{L}_2[0,\infty)} \|y(t)\|_2/\|v(t)\|_2\right\} < \gamma$, is held under zero initial condition.

3. Main results

This section presents the main results of the SMO enhanced adaptive control of SDS, which includes step by step details of the major analytical development. And the novelty of the developed scheme with comparison of relevant literatures is presented in the following Remarks 1-3, respectively.

3.1. Design of the observer

The following observer is introduced to estimate the state of the original system (1)

$$\begin{cases} \mathrm{d}\hat{x}(t) = [A\hat{x}(t) + A_d\hat{x}(t-d) + L(y(t) - C\hat{x}(t))]\mathrm{d}t, \\ \hat{y}(t) = C\hat{x}(t), \\ \hat{x}(\theta) = \hat{\phi}(\theta),\ \theta \in [-d,\ 0] \end{cases} \tag{2}$$

where $\hat{x}(t) \in \mathbb{R}^n$ represents the estimation of $x(t)$, $\hat{y}(t)$ is the output of the observer. $L \in \mathbb{R}^{n\times p}$ is the observer gain to be designed later.

Let $e(t) = x(t) - \hat{x}(t)$ be the error viriable. Thus, by subtracting (2) from (1), it gives the following estimation error system

$$\begin{aligned} \mathrm{d}e(t) \quad = \quad & \{(A - LC)e(t) + A_d e(t-d) + \Delta A(t)x(t) + \Delta A_d(t)x(t-d) \\ & + B[u(t) + f(t, x_t)] + Dv(t)\}\mathrm{d}t + g(t, x_t)\mathrm{d}\omega(t). \end{aligned} \tag{3}$$

Remark 1. The **first** advantage of the work is the simplicity in both observer and controller design of the SDS. Different from the current representative observer designs [23, 28, 29], the present observer does not involve any control terms (e.g., the control input or controller compensator), namely, only the observer gain L is to be determined in the paper. In other words, the controller is only used for the original system control without directly adjusting the observer.

3.2. Design of the integral-type sliding surface

Here, a new integral-type sliding surface function is defined as follows

$$\sigma(t) = G[e(t) - e(0)] + G[\hat{x}(t) - \hat{x}(0)] - \int_0^t G(A + BK)\hat{x}(\tau)\mathrm{d}\tau, \tag{4}$$

where the gain matrix $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is Hurwitz stable, and $G \in \mathbb{R}^{m\times n}$ satisfies that GB is nonsingular and $G = UC$, where the matrix U is to be solved. Then, in accordance with the equality condition, it follows that

$$\sigma(t) = U[y(t) - y(0)] - \int_0^t G(A + BK)\hat{x}(\tau)\mathrm{d}\tau.$$

Therefore, the design of the sliding surface can be feasible and only requires the current information. At this point, an optimization algorithm will be given to solve the matrix U in practical applications, please see Remark 5.

3.3. Establishment of sliding mode dynamics

To achieve the sliding motion, an equivalent controller [15, 16, 19] is to be adopted, i.e., $\mathcal{E}\sigma(t) = 0$ and $\frac{\mathrm{d}(\mathcal{E}\sigma(t))}{\mathrm{d}t} = 0$. Thus, $\mathcal{L}\sigma(t) = 0$ should be guaranteed from the condition $\mathcal{E}\{\mathrm{d}\omega(t)\} = 0$. In detail, $\sigma(t)$ is an Itô stochastic process satisfying the following request

$$\mathrm{d}\sigma(t) = G\mathrm{d}e(t) + G\mathrm{d}\hat{x}(t) - G(A + BK)\hat{x}(t)\mathrm{d}t = \mathcal{L}\sigma(t)\mathrm{d}t + Gg(t, x_t)\mathrm{d}\omega(t) \tag{5}$$

where

$$\mathcal{L}\sigma(t) = GAe(t) + GA_d x(t-d) + G\Delta A(t)x(t) + G\Delta A_d(t)x(t-d)$$
$$+GB[u(t) + f(t, x_t)] + GDv(t) - GBK\hat{x}(t).$$

Hence, the so-called equivalent controller can be determined as

$$u_{eq}(t) = K\hat{x}(t) - f(t, x_t) - (GB)^{-1}[GAe(t) + GA_d x(t-d)$$
$$+G\Delta A(t)x(t) + G\Delta A_d(t)x(t-d) + GDv(t)]. \tag{6}$$

Substituting (6) into (1), one can get the dynamic equation of the original system (1) during the sliding mode as

$$\mathrm{d}x(t) = \{[A + BK + B_G\Delta A(t)]x(t) + B_G[A_d + \Delta A_d(t)]x(t-d)$$
$$-(BK + B_A)e(t) + G_B v(t)\}\mathrm{d}t + g(t, x_t)\mathrm{d}\omega(t) \tag{7}$$

where $B_G = I - B(GB)^{-1}G$, $B_A = B(GB)^{-1}GA$, $G_B = B_G D$. Similarly, combining (6) and (3), the dynamic equation of the error system (3) during the sliding mode can be expressed as

$$\mathrm{d}e(t) = \{(A_B - LC - BK)e(t) + A_d e(t-d) + [BK + B_G\Delta A(t)]x(t)$$
$$+[B_G\Delta A_d(t) - B_d]x(t-d) + G_B v(t)\}\mathrm{d}t + g(t, x_t)\mathrm{d}\omega(t) \tag{8}$$

where $A_B = A - B_A$, $B_d = B(GB)^{-1}GA_d$.

From the above discussions, both (7) and (8) can be recognised as the sliding mode dynamics (SMDs) of the closed-loop systems. Thus, stability of the system will be analysed through the SMDs via adaptive SMC.

Remark 2. Note that the system is in its sliding mode at the initial time. Different from those representative forms [14-16, 23, 28, 29], the error term $e(t)$ is introduced into the sliding surface function (4), i.e., $\sigma(t) = G[x(t) - x(0)] - \int_0^t G(A + BK)\hat{x}(\tau)\mathrm{d}\tau$, which results in additional items to suppress the impact of the uncertainty through the control channel (i.e., $f(t, x_t)$), with reference to Eq. (1), as can be seen from the derivative of the SMDs (7)-(8). Also, the design may exert its benefit to highlight the attractive feature of SMC that SMDs can be insensitive to all matched uncertainties. At this point, the sliding surface design may be regarded as the **second** merit in the paper.

3.4. Stability analysis of sliding motion with H_∞ performance

Denote

$z_1(t) \triangleq [A + BK + B_G\Delta A(t)]x(t) + B_G[A_d + \Delta A_d(t)]x(t-d) - (BK + B_A)e(t) + G_B v(t)$,

and

$z_2(t) \triangleq [A_B - LC - BK]e(t) + A_d e(t-d) + [BK + B_G\Delta A(t)]x(t) + [B_G\Delta A_d(t) - B_d]x(t-d) + G_B v(t)$, $g(t, x_t) \triangleq g(t)$.

Hence, the SMDs of the closed-loop systems are expressed by the concise form of

$$\begin{cases} \mathrm{d}x(t) = z_1(t)\mathrm{d}t + g(t)\mathrm{d}\omega(t), & \text{(9a)} \\ \mathrm{d}e(t) = z_2(t)\mathrm{d}t + g(t)\mathrm{d}\omega(t). & \text{(9b)} \end{cases}$$

Lemma 1. *If $x(t)$ and $e(t)$ are the solution of systems* (9)*, Υ_i $(i = 1, 2)$ are any compatible dimension matrices, then $\mathcal{E}\left\{x^{\mathrm{T}}(t-d)\Upsilon_1\left[\int_{t-d}^{t} g(s)\mathrm{d}\omega(s)\right]\right\} = 0$, and $\mathcal{E}\left\{e^{\mathrm{T}}(t-d)\Upsilon_2\left[\int_{t-d}^{t} g(s)\mathrm{d}\omega(s)\right]\right\} = 0, t \geq d$.*

Proof. The idea of the proof is the same as that of [3], with the details as it is.

In the following concern, a new delay-dependent sufficient criterion for the mean-square exponential stability of the SMDs with H_∞ disturbance attenuation level is derived by means of the linear matrix inequality (LMI) technique and the stochastic stability theory.

Theorem 1. *Given a scalar $\gamma > 0$, the SMDs in* (9) *on the sliding surface $\sigma(t) = 0$ is mean-square exponentially stable with $\mathscr{H}_\infty$ disturbance attenuation level γ provided that the following LMIs* (10) – (12) *can be satisfied with some symmetric definite matrices P, Q_1, Q_2, R_1 and R_2, matrix X, positive scalars μ, κ_i $(i = 1, 2)$ and ε_j $(j = 1, 2)$:*

$$P < \mu I, \tag{10}$$

$$dR_i < \kappa_i I, \tag{11}$$

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & R_1 & \Sigma_{14} & 0 & 0 & PG_B & \Pi_1 \\ * & \Sigma_{22} & 0 & -(PB_d)^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ * & * & -R_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{44} & PA_d & R_2 & PG_B & \Pi_2 \\ * & * & * & * & -Q_2 + R_2 & 0 & 0 & 0 \\ * & * & * & * & * & -R_2 & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & \Pi_3 \end{bmatrix} < 0, \tag{12}$$

where $\Sigma_{11} = sym\{PA+PBK\}+Q_1-R_1+(2\mu+\kappa_1+\kappa_2)C^{\mathrm{T}}F^{\mathrm{T}}FC+(\varepsilon_1+\varepsilon_2)N^{\mathrm{T}}N+C^{\mathrm{T}}C$, $\Sigma_{12} = PB_GA_d+(\varepsilon_1+\varepsilon_2)N^{\mathrm{T}}N_d$, $\Sigma_{14} = -PBK-PB_A+(PBK)^{\mathrm{T}}$, $\Sigma_{22} = -Q_1+R_1+(\varepsilon_1+\varepsilon_2)N_d^{\mathrm{T}}N_d$, $\Sigma_{44} = sym\{PA_B-XC-PBK\}+Q_2-R_2$, $\Pi_1 = [PB_GM\ \ 0]$, $\Pi_2 = [0\ \ PB_GM]$, $\Pi_3 = diag\{-\varepsilon_1 I, -\varepsilon_2 I\}$. *Moreover, the observer gain matrix is given by* $L = P^{-1}X$.

Proof. To begin with, in the light of Eq. (9), its integral form can be interpreted as

$$\begin{cases} x(t) = x(0) + \int_0^t z_1(s)\mathrm{d}s + \int_0^t g(s)\mathrm{d}\omega(s), & \text{(13a)} \\ e(t) = e(0) + \int_0^t z_2(s)\mathrm{d}s + \int_0^t g(s)\mathrm{d}\omega(s), \quad t \geq 0. & \text{(13b)} \end{cases}$$

In the position, from the above equations, it is followed that

$$\begin{cases} x(t) = x(t-d) + \int_{t-d}^t z_1(s)\mathrm{d}s + \int_{t-d}^t g(s)\mathrm{d}\omega(s), & \text{(14a)} \\ e(t) = e(t-d) + \int_{t-d}^t z_2(s)\mathrm{d}s + \int_{t-d}^t g(s)\mathrm{d}\omega(s), \quad t \geq d. & \text{(14b)} \end{cases}$$

Step 1. Let us consider the stability of the SMDs (9) with $v(t) = 0$. Choose the following Lyapunov-Krasovskii functional

$$\begin{aligned} V(x_t, e_t, t) \;=\; & x^{\mathrm{T}}(t)Px(t) + \int_{t-d}^t x^{\mathrm{T}}(s)Q_1x(s)\mathrm{d}s + \int_{-d}^0\int_{t+s}^t g^{\mathrm{T}}(\theta)R_1g(\theta)\mathrm{d}\theta\mathrm{d}s \\ & + e^{\mathrm{T}}(t)Pe(t) + \int_{t-d}^t e^{\mathrm{T}}(s)Q_2e(s)\mathrm{d}s + \int_{-d}^0\int_{t+s}^t g^{\mathrm{T}}(\theta)R_2g(\theta)\mathrm{d}\theta\mathrm{d}s \end{aligned}$$

for all $t \geq d$. By Itô formula [1], one has the stochastic differential as follows

$$\mathrm{d}V(x_t, e_t, t) = \mathcal{L}V(x_t, e_t, t)\mathrm{d}t + 2[x(t) + e(t)]^{\mathrm{T}}Pg(t)\mathrm{d}\omega(t)$$

where

$$\begin{aligned} \mathcal{L}V(x_t, e_t, t) = \; & 2x^{\mathrm{T}}(t)Pz_1(t) + x^{\mathrm{T}}(t)Q_1x(t) - x^{\mathrm{T}}(t-d)Q_1x(t-d) \\ & + dg^{\mathrm{T}}(t)R_1g(t) + 2\mathrm{Tr}\{g^{\mathrm{T}}(t)Pg(t)\} - \int_{t-d}^t g^{\mathrm{T}}(s)R_1g(s)\mathrm{d}s \\ & + 2e^{\mathrm{T}}(t)Pz_2(t) + e^{\mathrm{T}}(t)Q_2e(t) - e^{\mathrm{T}}(t-d)Q_2e(t-d) \\ & + dg^{\mathrm{T}}(t)R_2g(t) - \int_{t-d}^t g^{\mathrm{T}}(s)R_2g(s)\mathrm{d}s. \end{aligned}$$

In view of Proposition 1(c) in Appendix, Lemma 1 and the similar technical channel in [3], it follows that

$$\begin{aligned}\mathcal{E}\left\{\int_{t-d}^{t} g^{\mathrm{T}}(s)R_1 g(s)\mathrm{d}s\right\} &= \mathcal{E}\left\{\left(x(t)-\int_{t-d}^{t} z_1(s)\mathrm{d}s\right)^{\mathrm{T}} R_1 \left(x(t)-\int_{t-d}^{t} z_1(s)\mathrm{d}s\right)\right\} \\ &\quad -\mathcal{E}\left\{x^{\mathrm{T}}(t-d)R_1 x(t-d)\right\}, \end{aligned} \tag{15}$$

and

$$\begin{aligned}\mathcal{E}\left\{\int_{t-d}^{t} g^{\mathrm{T}}(s)R_2 g(s)\mathrm{d}s\right\} &= \mathcal{E}\left\{\left(e(t)-\int_{t-d}^{t} z_2(s)\mathrm{d}s\right)^{\mathrm{T}} R_2 \left(e(t)-\int_{t-d}^{t} z_2(s)\mathrm{d}s\right)\right\} \\ &\quad -\mathcal{E}\left\{e^{\mathrm{T}}(t-d)R_2 e(t-d)\right\}. \end{aligned} \tag{16}$$

Then, one can get from (15) and (16)

$$\begin{aligned} &\mathcal{E}\{\mathrm{d}V(x_t,e_t,t)\} = \mathcal{E}\{\mathcal{L}V(x_t,e_t,t)\} \\ =&\,\mathcal{E}\left\{2x^{\mathrm{T}}(t)P[(A+BK)x(t)-B_G A_d x(t-d)-(BK+B_A)e(t)]\right\} \\ &+\mathcal{E}\left\{x^{\mathrm{T}}(t)Q_1 x(t)-x^{\mathrm{T}}(t-d)Q_1 x(t-d)\right\} \\ &-\mathcal{E}\left\{\left(x(t)-\int_{t-d}^{t} z_1(s)\mathrm{d}s\right)^{\mathrm{T}} R_1\left(x(t)-\int_{t-d}^{t} z_1(s)\mathrm{d}s\right)-x^{\mathrm{T}}(t-d)R_1 x(t-d)\right\} \\ &+\mathcal{E}\left\{2e^{\mathrm{T}}(t)P[(A_B-LC-BK)e(t)+A_d e(t-d)+BKx(t)-B_d x(t-d)]\right\} \\ &+\mathcal{E}\left\{e^{\mathrm{T}}(t)Q_2 e(t)-e^{\mathrm{T}}(t-d)Q_2 e(t-d)\right\} \\ &-\mathcal{E}\left\{\left(e(t)-\int_{t-d}^{t} z_2(s)\mathrm{d}s\right)^{\mathrm{T}} R_2\left(e(t)-\int_{t-d}^{t} z_2(s)\mathrm{d}s\right)-e^{\mathrm{T}}(t-d)R_2 e(t-d)\right\} \\ &+\mathcal{E}\left\{2x^{\mathrm{T}}(t)P[B_G\Delta A(t)x(t)+B_G\Delta A_d(t)x(t-d)]\right\} \\ &+\mathcal{E}\left\{2e^{\mathrm{T}}(t)P[B_G\Delta A(t)x(t)+B_G\Delta A_d(t)x(t-d)]\right\} \\ &+\mathcal{E}\left\{dg^{\mathrm{T}}(t)R_1 g(t)+dg^{\mathrm{T}}(t)R_2 g(t)+2\mathrm{Tr}\{g^{\mathrm{T}}(t)Pg(t)\}\right\}. \end{aligned} \tag{17}$$

Furthermore, the following inequalities are easily obtained

$$\begin{aligned} &2x^{\mathrm{T}}(t)PB_G[\Delta A(t)x(t)+\Delta A_d(t)x(t-d)] \\ =&\ 2x^{\mathrm{T}}(t)PB_G MF(t)[Nx(t)+N_d x(t-d)] \\ \le&\ \varepsilon_1^{-1}x^{\mathrm{T}}(t)PB_G MM^{\mathrm{T}}B_G^{\mathrm{T}}Px(t)+\varepsilon_1[Nx(t)+N_d x(t-d)]^{\mathrm{T}} \\ &\cdot[Nx(t)+N_d x(t-d)], \end{aligned} \tag{18}$$

and

$$
\begin{aligned}
&2e^{\mathrm{T}}(t)PB_G[\Delta A(t)x(t)+\Delta A_d(t)x(t-d)]\\
\le\ &\varepsilon_1^{-2}e^{\mathrm{T}}(t)PB_GMM^{\mathrm{T}}B_G^{\mathrm{T}}Pe(t)+\varepsilon_2[Nx(t)+N_dx(t-d)]^{\mathrm{T}}\\
&\cdot[Nx(t)+N_dx(t-d)].
\end{aligned}
\tag{19}
$$

Moreover, by (10) (11) and Lemma 2 in the Appendix, the following inequalities are held

$$
\mathrm{Tr}\{g^{\mathrm{T}}(t)Pg(t)\}\le\lambda_{\max}(P)\|Fy(t)\|^2\le\mu x^{\mathrm{T}}(t)C^{\mathrm{T}}F^{\mathrm{T}}FCx(t), \tag{20}
$$

$$
dg^{\mathrm{T}}(t)R_ig(t)\le\kappa_i\|Fy(t)\|^2=\kappa_ix^{\mathrm{T}}(t)C^{\mathrm{T}}F^{\mathrm{T}}FCx(t),\ \ (i=1,\ 2). \tag{21}
$$

Substituting (18)-(21) into (17) results in

$$
\mathcal{E}\{\mathcal{L}V(x_t,e_t,t)\}\le\mathcal{E}\left\{\zeta^{\mathrm{T}}(t)\Theta\zeta(t)\right\}
$$

where $\zeta^{\mathrm{T}}(t)=[x^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t-d)\ \ \left(\int_{t-d}^{t}z_1(s)\mathrm{d}s\right)^{\mathrm{T}}\ \ e^{\mathrm{T}}(t)\ \ e^{\mathrm{T}}(t-d)\ \ \left(\int_{t-d}^{t}z_2(s)\mathrm{d}s\right)^{\mathrm{T}}]$,

$$
\Theta=\begin{bmatrix}
\Theta_{11} & \Theta_{12} & R_1 & \Theta_{14} & 0 & 0\\
* & \Theta_{22} & 0 & -(PB_d)^{\mathrm{T}} & 0 & 0\\
* & * & -R_1 & 0 & 0 & 0\\
* & * & * & \Theta_{44} & PA_d & R_2\\
* & * & * & * & -Q_2+R_2 & 0\\
* & * & * & * & * & -R_2
\end{bmatrix}
$$

with $\Theta_{11}=\mathrm{sym}\{PA+PBK\}+Q_1-R_1+(2\mu+\kappa_1+\kappa_2)C^{\mathrm{T}}F^{\mathrm{T}}FC+\varepsilon_1^{-1}PB_GMM^{\mathrm{T}}B_G^{\mathrm{T}}P+(\varepsilon_1+\varepsilon_2)N^{\mathrm{T}}N$, $\Theta_{12}=PB_GA_d+(\varepsilon_1+\varepsilon_2)N^{\mathrm{T}}N_d$, $\Theta_{14}=-P(BK+B_A)+(PBK)^{\mathrm{T}}$, $\Theta_{22}=-Q_1+R_1+(\varepsilon_1+\varepsilon_2)N_d^{\mathrm{T}}N_d$, $\Theta_{44}=\mathrm{sym}\{P(A_B-LC-BK)\}+Q_2-R_2+\varepsilon_2^{-1}PB_GMM^{\mathrm{T}}B_G^{\mathrm{T}}P$. Denote $X=PL$, and it is observed that (12) implies $\Theta<0$ by the Schur complement lemma. Hence, it is tenable that there exists a positive scalar η such that $\mathcal{E}\{\mathcal{L}V(x_t,e_t,t)\}\le-\eta(\|x(t)\|^2+\|e(t)\|^2)$. Now, an auxiliary function is introduced: $J(t)=e^{\lambda t}V(x_t,e_t,t)$ with its infinitesimal operator $\mathcal{L}$ given by $\mathcal{L}J(t)=\lambda e^{\lambda t}V(x_t,e_t,t)+e^{\lambda t}\mathcal{L}V(x_t,e_t,t)$. With the method similar to [3] and Ch. 4 of [1], one can verify that the SMDs is mean-square exponentially stable by Definition 1.
Step 2. Under zero initial condition, the inequality $\mathcal{E}\left\{\sup_{0\neq v(t)\in\mathbb{L}_2[0,\infty)}\|y(t)\|_2/\|v(t)\|_2\right\}<\gamma$, will be verified further. Based on the similar procedure above, it follows

$$
\mathcal{E}\left\{\mathcal{L}V(x_t,e_t,t)+y^{\mathrm{T}}(t)y(t)-\gamma^2v^{\mathrm{T}}(t)v(t)\right\}\le\mathcal{E}\left\{\xi^{\mathrm{T}}(t)\Xi\xi(t)\right\}, \tag{22}
$$

where $\xi^{\mathrm{T}}(t) = [x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-d)\ \left(\int_{t-d}^{t} z_1(s)\mathrm{d}s\right)^{\mathrm{T}}\ e^{\mathrm{T}}(t)\ e^{\mathrm{T}}(t-d)\ \left(\int_{t-d}^{t} z_2(s)\mathrm{d}s\right)^{\mathrm{T}}\ v^{\mathrm{T}}(t)]$,

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & R_1 & \Xi_{14} & 0 & 0 & PG_B \\ * & \Xi_{22} & 0 & -(PB_d)^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & -R_1 & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & PA_d & R_2 & PG_B \\ * & * & * & * & -Q_2+R_2 & 0 & 0 \\ * & * & * & * & * & -R_2 & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} with$$

$\Xi_{11} = \mathrm{sym}\{PA + PBK\} + Q_1 - R_1 + \varepsilon_1^{-1} PB_G MM^{\mathrm{T}} B_G^{\mathrm{T}} P + (2\mu + \kappa_1 + \kappa_2) C^{\mathrm{T}} F^{\mathrm{T}} FC + (\varepsilon_1 + \varepsilon_2) N^{\mathrm{T}} N + C^{\mathrm{T}} C$, $\Xi_{12} = PB_G A_d + (\varepsilon_1 + \varepsilon_2) N^{\mathrm{T}} N_d$, $\Xi_{14} = -PBK - PB_A + (PBK)^{\mathrm{T}}$, $\Xi_{22} = -Q_1 + R_1 + (\varepsilon_1 + \varepsilon_2) N_d^{\mathrm{T}} N_d$, $\Xi_{44} = \mathrm{sym}\{PA_B - XC - PBK\} + Q_2 - R_2 + \varepsilon_2^{-1} PB_G MM^{\mathrm{T}} B_G^{\mathrm{T}} P$. At this point, using the Schur complement lemma again, one can obtain that (12) is equivalent to $\Xi < 0$. In other words, according to the LMIs (10)-(12), it leads to

$$\mathcal{E}\{\mathcal{L}V(x_t, e_t, t)\} \le \mathcal{E}\left\{-y^{\mathrm{T}}(t)y(t) + \gamma^2 v^{\mathrm{T}}(t)v(t)\right\}. \tag{23}$$

To this end, integrating both sides of (23) with respect to t from 0 to ∞ turns out

$$\begin{aligned} 0 < \mathcal{E}\{V(\infty)\} &= \mathcal{E}\left\{\int_0^{\infty} \mathcal{L}V(x_t, e_t, t)\right\} \\ &\le \mathcal{E}\left\{\int_0^{\infty} -y^{\mathrm{T}}(t)y(t)\mathrm{d}t\right\} + \int_0^{\infty} \gamma^2 v^{\mathrm{T}}(t)v(t)\mathrm{d}t, \end{aligned} \tag{24}$$

thereby completing the proof.

Remark 3. On the basis of Lemma 1, a new delay-dependent criterion for the mean-square exponential stability of the SMDs is presented. The **third** highlight of the result is the new technical procedure of stability analysis of the closed-loop systems: **a)** with the application of the observer (2), it is clearly found that the control input $u(t)$ is embedded onto the system (3), which could be seen as virtual controller for the error system; **b)** stability of the original system (1) can be achieved from that of the closed-loop systems constituted by the system (1) and error system (3) during the sliding mode via the SMO scheme, while the system performance of the existing works is completed via the observer and error system. In other words, the perspective of systematic analysis distinguishes from those existing SMO schemes of the SDS [23, 28, 29].

Remark 4. It is worth mentioning that, the SMC approaches applied to SDS have been reported in [14-16], where the sliding surface and controller design both rely on the availability of the system state. Nevertheless, there may be the general case of the unmeasured states in practical plants. Moreover, SMO methods have been properly employed to investigate various stochastic systems, e.g., [28, 29]. Yet, the key problem that the nonlinearity and/or perturbation may appear through the control channel has not been fully probed due to some difficulties. In this paper, the matched uncertainty (i.e., $f(t, x_t)$) of the system is involved.

Remark 5. In the position, a general optimization algorithm is proposed for determining the equality constraint $G = UC$ and the LMIs in (10)-(12). In detail, similar to the algorithm in [27], the following optimal minimization problem is shown to solve the undetermined parameters in Theorem 1 and the matrix U:

$$\min \beta, \quad \text{subject to} \quad \begin{bmatrix} -\beta I & (G - UC)^{\mathrm{T}} \\ * & -I \end{bmatrix} < 0 \ \text{ and } \ (10) - (12).$$

The verbatim argument is omitted here for brevity.

3.5. Adaptive SMC law synthesis

In this section, the focus is devoted to an adaptive SMC law synthesis so that the systems can start its sliding motion with specific performance as expected and maintain it on the predesigned sliding surface almost surely. In our controller design, the potential technical difficulty lies in unmeasured information of the state and the relevant error information as well as the unknown perturbation entering the control channel of the system. In details, the following discussions are made step by step so as to provide a clearer route for the controller design:

(1) The following assumption is first proposed to facilitate the design.

Assumption 3. [8] There exists an unknown positive scalar q satisfying the inequality $\|x(t-d)\| \leq q\|x(t)\|$.

(2) Since the system states $x(t)$ are not completely available, it follows that its error term $e(t)$ may not be precisely estimated as well. With the relationships among the system states $x(t)$, the error $e(t)$, the outputs $y(t)$ and $\hat{y}(t)$, we may assume that there exist unknown scalars $\upsilon_i > 0$ $(i = 1, 2, 3)$ satisfying $\|x(t)\| \leq \upsilon_1\|y(t)\|$ and $\|e(t)\| \leq \upsilon_2\|y(t)\| + \upsilon_3\|\hat{y}(t)\|$. Then, the esitimation as follows can be presented with Assumption 1 and 3, reasonably, i.e., unknown scalars $c_i > 0$ $(i = 1, 2)$ can

be found to satisfy that:

$$
\begin{aligned}
\mathscr{L} &= \|GA\|\|e(t)\| + \|GA_d\|\|x(t-d)\| + \|G\Delta A(t)\|\|x(t)\| \\
&\quad + \|G\Delta A_d(t)\|\|x(t-d)\| + \|GB\|\|f(t,x_t)\| \\
&\leq c_1\|y(t)\| + c_2\|\hat{y}(t)\|, \ t \geq 0.
\end{aligned}
$$

(3) Consider that the estimate bounds c_1 and c_2 are not available in practical design. Let $\hat{c}_i(t)$ be the estimates of c_i with the errors being $\tilde{c}_i(t) = \hat{c}_i(t) - c_i$ $(i = 1, 2)$.

Based upon the above statement, the following gives a pertinent result of the controller design.

Theorem 2. *Suppose that the integral sliding surface function is designed by* (4). *The finite-time reachability of the sliding mode can be guaranteed almost surely, if the adaptive SMC law in* (25) *is employed*

$$
\begin{aligned}
u(t) &= K\hat{x}(t) - (GB)^{-1}[\hat{c}_1(t)\|y(t)\| + \hat{c}_2(t)\|\hat{y}(t)\| + \|GD\|\mu(t) + \rho \\
&\quad + \lambda_{max}(G^{\mathrm{T}}G)\|Fy(t)\|^2/\|\sigma(t)\|]sgn(\sigma(t)),
\end{aligned} \tag{25}
$$

where the updating laws are designed by $\dot{\hat{c}}_1(t) = \lambda_1\|y(t)\|$, $\dot{\hat{c}}_2(t) = \lambda_2\|\hat{y}(t)\|$ *and* $\lambda_i > 0$ *(*$i = 1, 2$*) are constants as the adaptive gains chosen by the designer,* ρ *is a positive constant, and* $\mu(t)$ *is the uniform bound of* $v(t)$*.*

Proof. Choose a Lyapunov function candidate as follows

$$
\tilde{V}(t, \sigma(t)) = (\sigma^{\mathrm{T}}(t)\sigma(t))^{\frac{1}{2}} + 0.5[\lambda_1^{-1}\tilde{c}_1^2(t) + \lambda_2^{-1}\tilde{c}_2^2(t)].
$$

By the Itô formula and operational rules of the derivatives, it follows that

$$
\mathrm{d}\tilde{V}(t, \sigma(t)) = \mathcal{L}\tilde{V}(t, \sigma(t))\mathrm{d}t + \frac{\sigma^{\mathrm{T}}(t)}{\|\sigma(t)\|}Gg(t)\mathrm{d}\omega(t)
$$

where

$$\begin{aligned}\mathcal{L}\tilde{V}(t,\sigma(t)) &= \frac{\sigma^{\mathrm{T}}(t)}{\|\sigma(t)\|}\{GAe(t) + GA_d x(t-d) + G\Delta A x(t) + G\Delta A_d(t) \\ &\quad \cdot x(t-d) + GB[u(t) + f(t,x_t)] + GDv(t) - GBK\hat{x}(t)\} \\ &\quad + \frac{1}{2}g^{\mathrm{T}}(t)G^{\mathrm{T}}\left\{\frac{I_m}{\|\sigma(t)\|} - \frac{\sigma(t)\sigma^{\mathrm{T}}(t)}{\|\sigma(t)\|^3}\right\}Gg(t) \\ &\quad + \lambda_1^{-1}\tilde{c}_1(t)\dot{\tilde{c}}_1(t) + \lambda_2^{-1}\tilde{c}_2(t)\dot{\tilde{c}}_2(t). \end{aligned} \tag{26}$$

Substituting (25) into (26) and employing some inequality techniques yields

$$\begin{aligned}\mathcal{L}\tilde{V}(t,\sigma(t)) &\leq \frac{1}{\|\sigma(t)\|}\|\sigma(t)\|\{\|GA\|\|e(t)\| + \|GA_d\|\|x(t-d)\| \\ &\quad + \|G\Delta A(t)\|\|x(t)\| + \|G\Delta A_d(t)\|\|x(t-d)\| \\ &\quad + \|GB\|\|f(t,x_t)\| + \|GDv(t)\|\} - \frac{1}{\|\sigma(t)\|}[\hat{c}_1(t)\|y(t)\| \\ &\quad + \hat{c}_2(t)\|\hat{y}(t)\| + \|GD\|\mu(t) + \rho + \lambda_{\max}(G^{\mathrm{T}}G) \\ &\quad \cdot\|Fy(t)\|^2/\|\sigma(t)\|]\sigma^{\mathrm{T}}(t)\mathrm{sgn}(\sigma(t)) + \frac{1}{\|\sigma(t)\|} \\ &\quad \cdot\|g^{\mathrm{T}}(t)G^{\mathrm{T}}Gg(t)\| + \lambda_1^{-1}\tilde{c}_1(t)\dot{\tilde{c}}_1(t) + \lambda_2^{-1}\tilde{c}_2(t)\dot{\tilde{c}}_2(t) \\ &\leq c_1\|y(t)\| + c_2\|\hat{y}(t)\| - \{\hat{c}_1(t)\|y(t)\| + \hat{c}_2(t)\|\hat{y}(t)\| + \rho \\ &\quad + \lambda_{\max}(G^{\mathrm{T}}G)\|Fy(t)\|^2/\|\sigma(t)\|\} + \frac{1}{\|\sigma(t)\|}\|g^{\mathrm{T}}(t)G^{\mathrm{T}}Gg(t)\| \\ &\quad + \lambda_1^{-1}\tilde{c}_1(t)\dot{\tilde{c}}_1(t) + \lambda_2^{-1}\tilde{c}_2(t)\dot{\tilde{c}}_2(t). \end{aligned}$$

Then, in view of Lemma 2 and the updating laws, it is obtained that

$$\begin{aligned}\mathcal{L}\tilde{V}(t,\sigma(t)) &\leq c_1\|y(t)\| + c_2\|\hat{y}(t)\| - \hat{c}_1(t)\|y(t)\| - \hat{c}_2(t)\|\hat{y}(t)\| \\ &\quad -\rho - \lambda_{\max}(G^{\mathrm{T}}G)\|Fy(t)\|^2/\|\sigma(t)\| + \lambda_{\max}(G^{\mathrm{T}}G) \\ &\quad \cdot\frac{1}{\|\sigma(t)\|}\|g^{\mathrm{T}}(t)g(t)\| + \lambda_1^{-1}\tilde{c}_1(t)\dot{\tilde{c}}_1(t) + \lambda_2^{-1}\tilde{c}_2(t)\dot{\tilde{c}}_2(t) \\ &\leq -\rho. \end{aligned} \tag{27}$$

Thus, by integrating (27) from 0 to t and taking expectation for both sides, one can test that

$$\mathscr{E}\|\sigma(t)\| \leq \mathscr{E}\tilde{V}(t,\sigma(t)) \leq \mathscr{E}\tilde{V}(0,\sigma(0)) - \rho t,$$

which implies $\mathscr{E}\|\sigma(t)\| = 0$ for all $t \geq t_f = \frac{\mathscr{E}\tilde{V}(0,\ \sigma(0))}{\rho}$, i.e., $\|\sigma(t)\| = 0$ a.s. [15, 16]. The proof is completed.

Remark 6. The novel adaptive sliding mode controller in (25) is developed for the system, which not only resolves the problems discussed by the existing literatures [14-16, 28], but also provides a new adaptive memory-less controller design (i.e., controller does not rely on the information of the delayed state) upon the premise of unmeasured states, whereas the designs in [14-16, 29] are all memory controllers. Moreover, novel adaptive laws are introduced for the controller design based on the information among $x(t)$, $e(t)$, $y(t)$ and $\hat{y}(t)$, by which the unknown bounds c_1 and c_2 could be well tracked, respectively. It is worth mentioning that the controller design can be satisfied from theoretical aspect, and note that there may be singularity when $\sigma(t) = 0$, thus a sufficient small positive scalar ε could be introduced for avoiding the case (i.e., $\|\sigma(t)\| + \varepsilon$) in such a way that the method can be exhibited in practical examples. To this end, the adaptive controller can maintain desirable system performance via a test example in the sequel.

Remark 7. As is seen, a novel SMO-based robust $\mathscr{H}_\infty$ control scheme is shown for uncertain SDS in (1) via Theorem 1 and 2. The detailed design procedure is summarized for practical applications below:

Step 1. Get the gain matrix K such that $A + BK$ is Hurwitz stable.

Step 2. Solve the gain matrix L of the LMIs in (10)-(12), then the observer is given by (2).

Step 3. Obtain the parametric matrix U such that $G = UC$. Then the sliding surface is designed by (4) with the gain matrix K.

Step 4. Select positive constants λ_1, λ_2 and ρ such that the SMC law is synthesized in (25).

4. Illustrative example

In this section, a specific bench test example is provided to further demonstrate the performance of the developed scheme in terms of computational experiments.

Example 1

Consider the mathematical model of a water-quality dynamic systems [15, 16] subject to environmental noises and external disturbance with the form in (1), where $x(t) = [x_1(t) \;\; x_2(t)]^{\mathrm{T}} \in \mathbb{R}^2$, $x_1(t)$ and $x_2(t)$ stand for the concentrations of two main types of pollutant sources, namely algae and ammonia products, respectively; $u(t)$ is the implemented control action. The aim is to apply the SMO design to the stochastic model described by the following data

$$A = \begin{bmatrix} -2 & 1 \\ -2 & -2 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^{\mathrm{T}}, D = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}, F = 0.8$$

$$M = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.5 \end{bmatrix}, N = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}, N_d = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}, F(t) = \begin{bmatrix} 0.5sin(2t) & 0 \\ 0 & sint \end{bmatrix}.$$

To facilitate the design, we choose matrices $K = \begin{bmatrix} -2.4634 & -5.5172 \end{bmatrix}$, and $G = \begin{bmatrix} 1 & 2 \end{bmatrix}$ such that $U = 1$. The diffusion function $g(t)$, external disturbance $v(t)$ and uncertain function $f(t, x_t)$ are chosen by

$$g(t) = \begin{bmatrix} 0.25 & 0.5 \\ 0.1 & -0.2 \end{bmatrix} x(t), \;\; v(t) = -sin(2t)e^{-0.5t},$$

$$f(t, x_t) = -\sqrt{2}sint \cdot x_1(t) + (1 - \sqrt{3}cos^2(t))x_2(t).$$

For $\gamma = 1$, $d = 0.5$, the LMIs (10)-(12) yield the following feasible solutions:

$$P = \begin{bmatrix} 8.4009 & 0.4687 \\ 0.4687 & 8.2699 \end{bmatrix}, Q_1 = \begin{bmatrix} 13.4765 & 2.2890 \\ 2.2890 & 13.9975 \end{bmatrix}, Q_2 = \begin{bmatrix} 11.0357 & 2.3376 \\ 2.3376 & 15.1893 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 4.7061 & 0.0149 \\ 0.0149 & 3.5801 \end{bmatrix}, R_2 = \begin{bmatrix} 4.3336 & 0.1114 \\ 0.1114 & 4.4980 \end{bmatrix}, X = \begin{bmatrix} 25.1464 \\ 32.8143 \end{bmatrix},$$
$$\mu = 11.1142, \varepsilon_1 = 8.4671, \varepsilon_2 = 8.6035, \kappa_1 = 6.4494, \kappa_2 = 6.5501.$$

It is also worth pointing out that the associated minimum H_∞ performance index is computed as $\gamma_{min} = 0.0544$, which is an important paremeter for H_∞ control design. To this end, the integral sliding surface function can be designed as

$$\sigma(t) = y(t) - y(0) + \int_0^t \begin{bmatrix} 12.1585 & 16.7930 \end{bmatrix} \hat{x}(s)\mathrm{d}s.$$

At this point, Consider that variable structural control system has its nonlinear characteristics, chattering phenomenon may always exist due to the sign function in the SMC law. Then, the methods [30 − 32], such as using hyperbolic tangent function, terminal sliding mode control strategy and boundary layer method may be employed to attenuate or present the chattering. In particular, we use the boundary layer method to reduce chattering in the simulation, i.e., the sign function $\mathrm{sgn}(\sigma(t))$ in controller is replaced by a continuous approximation as $\sigma(t)/(\|\sigma(t)\| + \theta)$ where $\theta > 0$ is a small constant. Thus, the associated controller is given as

$$u(t) = \begin{bmatrix} -2.4634 & -5.5172 \end{bmatrix} \hat{x}(t) - 0.4[\hat{c}_1(t)\|y(t)\| + \hat{c}_2(t)\|\hat{y}(t)\|$$
$$+0.6\|v(t)\| + 2.75 + 5\|Fy(t)\|^2/(\|\sigma(t)\| + 10^{-4})]\sigma(t)/(\|\sigma(t)\| + 10^{-3})$$

with the updating laws given by $\dot{\hat{c}}_1(t) = 2.0\|y(t)\|$, $\dot{\hat{c}}_2(t) = 2.0\|\hat{y}(t)\|$. Herein, a further comparison of stability results is performed between the work of [28] and the present paper, see Table 1. To this end, given the initial conditions $x(\theta) = \begin{bmatrix} 1 & -1.5 \end{bmatrix}^{\mathrm{T}}$, and $\hat{x}(\theta) = \begin{bmatrix} -1 & -1 \end{bmatrix}^{\mathrm{T}}$, $\theta \in [-0.5, \ 0]$, simulation results are shown in Figures 1-5. Among them, Figures 1 and 2 denote the responses of the system states and the observer. The evolutions of designed sliding surface function and control input are given by Figures 3 and 4 with the case that y-axis takes from −10 to 10 in Figure 4, and the adaptive values are provided by Figure 5, respectively. It is noted that the uncertainty (i.e., $f(t, x_t)$) appears through the control channel in this model, whereas the counterpart of [28] may not work

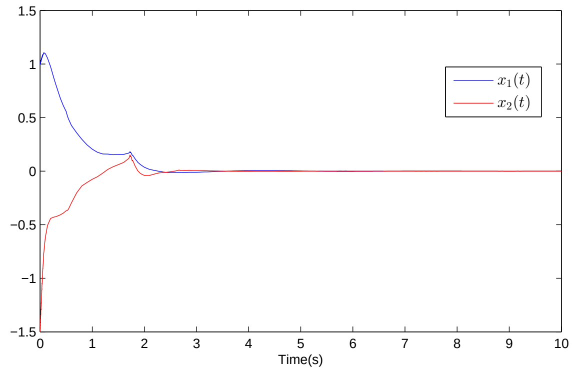

Figure 1: Curve of the system states

under some harsh conditions. The result implies that the optimization and control problem can be achieved when the system states are unmeasured or incompletely available via the proposed scheme, which is in accordance with the analysis in the paper.

Table 1: Comparison of the stability results by different methods.

Methods	Theorem 1	Theorem 4 in [28]
Gain L	$\begin{bmatrix} 2.7807 & 3.8103 \end{bmatrix}^{\mathrm{T}}$	Infeasible

5. Conclusions

The problems of SMO design for uncertain SDS with unmeasured states, nonlinearity and external disturbance have been studied in this paper. The key fea-

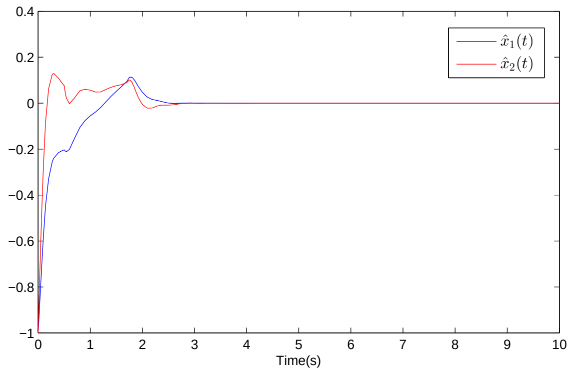

Figure 2: Curve of the state observer

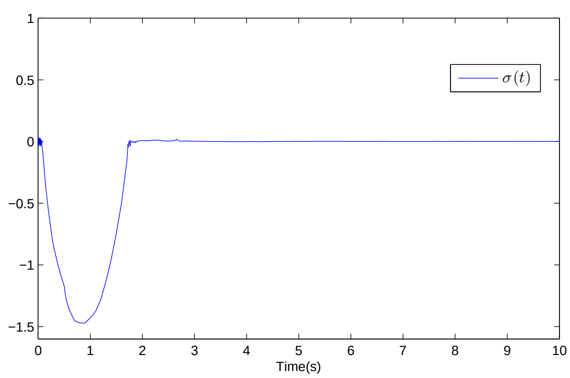

Figure 3: Curve of the sliding surface function

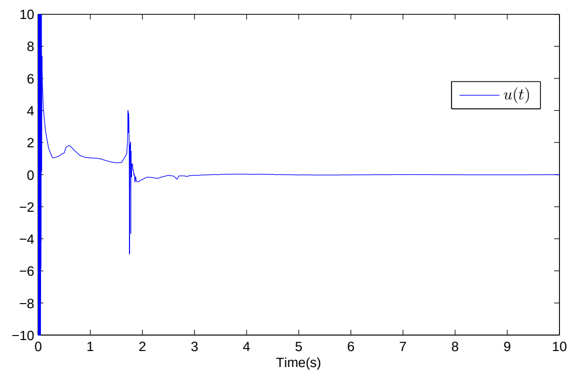

Figure 4: Curve of the control input

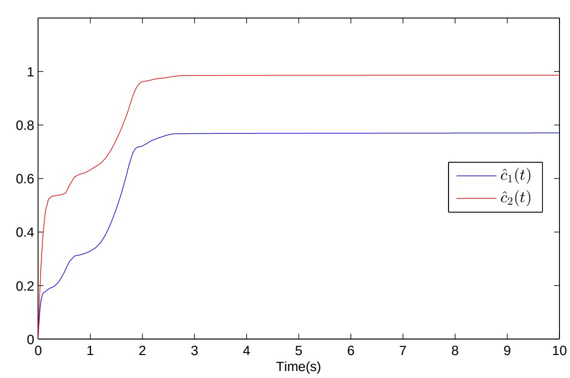

Figure 5: Curve of the adaptive estimations

tures of the scheme lie in the design of a particular state observer, integral-type sliding surface and the associated adaptive SMC law for the SDS. By Lemma 1, the easy-to-check LMIs condition has been established to ensure the mean-square exponential stability of the SMDs enforced on the sliding surface. If a non-affine model, e.g., nonlinear rational model or called total nonlinear model [33] with time-delay is concerned, how to deal with uncertainties may be a hot topic to be further studied, while combining some novel approaches, e.g., U-model-based control system design [34].

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (No. 61273188, 61374079, 61473097, 61603204, 41306002), the Natural Science Foundation of Shandong province under grant ZR2016FP03 and the Qingdao Application Basic Research Project under grant 16-5-1-22-jch.

Appendix.

Definition 1. [1, 15] The system (1) with $u(t) = 0$ and $v(t) = 0$ is said to be exponentially stable in mean-square, if there exist a scalar $\lambda > 0$ such that

$$\lim_{t\to\infty} \sup \frac{1}{t} \log \mathcal{E}\{\|x(t,\phi)\|^2\} \leq -\lambda$$

for all admissible uncertainties.

Proposition 1. [3] Let $\{h(t)\}_{t_0 \le t \le T} \in \mathcal{M}^2([t_0, T])$, then

(a) $\{\int_{t_0}^{t} h(s)\mathrm{d}\omega(s) : t_0 \le t \le T\}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t_0 \le t \le T}$;

(b) $\int_{t_0}^{t} h(s)\mathrm{d}\omega(s)$ is $\mathcal{F}_t$-measurable, $t_0 \le t \le T$;

(c) $\mathcal{E}\{\| \int_{t_0}^{T} g(t)\mathrm{d}\omega(t)\|^2\} = \mathcal{E}\{\int_{t_0}^{T} \|g(t)\|^2 \mathrm{d}t\}$, where $\{g(t)\}_{t_0 \le t \le T} \in \mathcal{M}^2([t_0, T]; \mathbb{R}^n)$.

Lemma 2. [15] *For a pair of constant matrices* $G \in \mathbb{R}^{p\times p}$ *and* $M \in \mathbb{R}^{p\times q}$*, if* $G \ge 0$*, then* $\mathrm{Tr}(M^{\mathrm{T}}GM) \le \lambda_{max}(G)\mathrm{Tr}(M^{\mathrm{T}}M)$.

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