An approach for solving an inverse spherical two-phase Stefan problem arising in modeling of electric contact phenomena

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In this paper, a model problem that can be used for mathematical modeling and investigation of arc phenomena in electrical contacts is considered. An analytical approach for the solution of a two-phase inverse spherical Stefan problem where along with unknown temperature functions heat flux function has to be determined is presented. The suggested solution method is obtained from a new form of integral error function and its properties that are represented in the form of series whose coefficients have to be determined. Using integral error function and collocation method, the solution of a test problem is obtained in exact form and approximately.

KEYWORDS

electrical contacts, integral error function, inverse two-phase Stefan problem

1 | INTRODUCTION

Partial differential equations play an important role for the development of models in heat conduction and investigated in various aspects (see, for example, literature[1-3] and the references therein). To realize the physical changes, some models need to be expressed as free or moving boundary problems. The theory of free boundaries has seen great progress in the last half century. For the general literature up to 2015, we refer to Friedman[4] and Chen et al.[5] Also, a long list of studies and literature therein are devoted to Stefan-type problems and their analytical and numerical solutions. [6-13]

Arcing processes are very rapid and include phase transformations. Thus, it is reasonable to use Stefan-type problems for mathematical modeling of this phenomena. Worth to say that exact solution of the problem allows to elucidate and enhance understanding of arcing processes and contribute to the development of the arc theory. Present study is devoted to theoretical investigation and mathematical modeling of arc phenomena in electrical contacts and appears as a continuation of recent studies where mathematical modeling of short arcing is considered. [14,15] While in 2 studies,[14,15] the analytical solutions of the one- and two-phase (direct) Stefan problems are found, in this paper, we consider an inverse Stefan problem for which along with unknown temperature functions, heat flux function has to be determined. In the considered model, heat flux depends on time variable; however, it is well known that besides time variable, heat flux
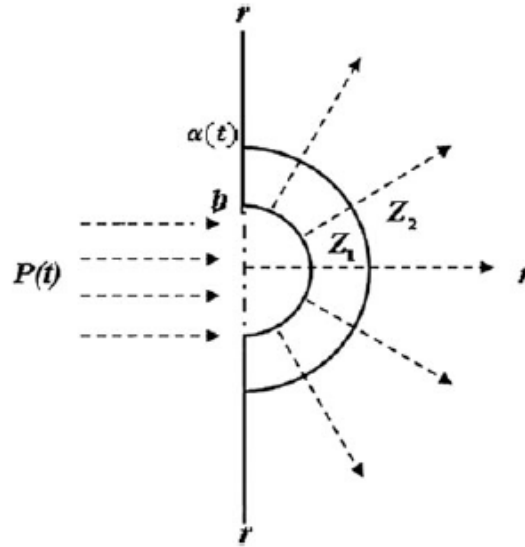
FIGURE 1 The contact spot model

depends on numerous factors like electron bombardment and diverse electric contact effects like tunnel, Joule, Thomson, and Peltier effects. Thus, this model does not claim to be universal for all electric contact phenomena occurring during opening or switching electric contacts.

In this study, we consider spherical model that agrees with Holm's so-called ideal sphere usually applied for electric contacts with small contact surface (radius of $b < 10^{-4}$m) and low electric current.[16]

1.1 | Problem statement

In a spherical model, the contact spot is given by the spherical surface of radius b. The heat flux $P(t)$ entering this surface melts the electric contact material (liquid zone $b < r < \alpha(t)$) and then passes further through the solid zone $\alpha(t) < r < \infty$ (for the illustration of the model, see Figure 1).

The heat equations for each zone are

$$\frac{\partial \theta_1}{\partial t} = a_1^2 \left(\frac{\partial^2 \theta_1}{\partial r^2} + \frac{2}{r} \frac{\partial \theta_1}{\partial r} \right), \quad b < r < \alpha(t),\ t > 0, \tag{1}$$

$$\frac{\partial \theta_2}{\partial t} = a_2^2 \left(\frac{\partial^2 \theta_2}{\partial r^2} + \frac{2}{r} \frac{\partial \theta_2}{\partial r} \right), \quad \alpha(t) < r < \infty,\ t > 0, \tag{2}$$

with initial condition

$$\theta_1(b, 0) = T_m, \tag{3}$$

$$\theta_2(r, 0) = f(r), \tag{4}$$

$$f(b) = T_m,\ \alpha(0) = b,\ f(\infty) = 0, \tag{5}$$

subjected to boundary condition at $r = b$

$$-\lambda_1 \frac{\partial \theta_1(b, t)}{\partial r} = P(t) \tag{6}$$

and to free boundary

$$\theta_1(\alpha(t), t) = T_m, \tag{7}$$

$$\theta_2(\alpha(t), t) = T_m, \tag{8}$$

the Stefan's condition

$$-\lambda_1 \frac{\partial \theta_1(\alpha(t), t)}{\partial r} = -\lambda_2 \frac{\partial \theta_2(\alpha(t), t)}{\partial r} + L\gamma \frac{d\alpha}{dt}, \tag{9}$$

as well as the condition at infinity

$$\theta_2(\infty, t) = 0. \tag{10}$$

Here, θ_1 and θ_2 are unknown heat functions, $P(t)$ is an unknown heat flux coming from electric arc, T_m is a melting temperature of electric contact material, $f(r)$ is a given function, and a_1, a_2, λ_1, λ_2, L, and γ are given constants. In the equation, power balance is described by Stefan's condition (9). The function $\alpha(t)$ describing the interphase location is given in the inverse problem under consideration.

2 | PROBLEM SOLUTION

Suppose that the initial and free boundary conditions are analytic functions and they can be expanded in Taylor series as

$$f(r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(r-b)^n, \quad \alpha(t) = b + \sum_{n=1}^{\infty} a_n t^{n/2}. \tag{11}$$

We represent the solution of (1) to (10) in the following form

$$\theta_1(r, t) = \frac{1}{r}\sum_{n=0}^{\infty}\left(2a_1\sqrt{t}\right)^n\left[A_n i^n erfc\frac{r-b}{2a_1\sqrt{t}} + B_n i^n erfc\frac{b-r}{2a_1\sqrt{t}}\right],$$

$$\theta_2(r, t) = \frac{1}{r}\sum_{n=0}^{\infty}\left(2a_2\sqrt{t}\right)^n\left[C_n i^n erfc\frac{r-b}{2a_2\sqrt{t}} + D_n i^n erfc\frac{b-r}{2a_2\sqrt{t}}\right],$$

where coefficients A_n, B_n, C_n, and D_n have to be found. Here, $i^n erfcx$ is integral error function determined by the following recurrent formulas:

$$i^n erfcx = \int_x^{\infty} i^{n-1} erfcv\, dv, n = 1, 2, \cdots,$$

$$i^0 erfcx = erfcx = \frac{2}{\sqrt{\pi}}\int_x^{\infty} \exp(-v^2)dv.$$

Lemma 1. *The integral error function holds the following properties:*

1. $\lim\limits_{x \to \infty} \frac{i^n erfc(-x)}{x^n} = \frac{2}{n!}$,
2. $\lim\limits_{t \to 0}\left(2a_1\sqrt{t}\right)^n C_n i^n erfc\frac{r-b}{2a_1\sqrt{t}} = 0$,
3. $\lim\limits_{t \to 0}\left(2a_1\sqrt{t}\right)^n D_n i^n erfc\frac{b-r}{2a_1\sqrt{t}} = \frac{2}{n!}D_n(r-b)^n$.

The proof of the lemma can be given by the L'Hopital's rule and properties of $i^n erfcx$ function.

Theorem 2. *Let f be n times differentiable analytic function. Then*

$$\lim_{t \to 0} \theta_2(r, t) = \frac{1}{r}\sum_{n=0}^{\infty}\frac{2}{n!}D_n(r-b)^n.$$

Proof. Using Lemma 1, it is easy to see that

$$\lim_{t \to 0} \theta_2(r, t) = \lim_{t \to 0}\frac{1}{r}\sum_{n=0}^{\infty}\left(2a_2\sqrt{t}\right)^n D_n i^n erfc\frac{b-r}{2a_2\sqrt{t}}$$

$$= \frac{1}{r}\lim_{t \to 0}\sum_{n=0}^{\infty} D_n(r-b)^n \frac{i^n erfc\left(-\frac{r-b}{2a_2\sqrt{t}}\right)}{\left(\frac{r-b}{2a_2\sqrt{t}}\right)^n} = \frac{1}{r}\sum_{n=0}^{\infty}\frac{2}{n!}D_n(r-b)^n.$$

$\square$

2.1 | Calculation of coefficients

By the theorem and equations (4) and (11), we can write

$$\lim_{t \to 0} \theta_2(r,t) = f(r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(r-b)^n.$$

Using Theorem 2 and letting $F(r) = rf(r)$, we have

$$\sum_{n=0}^{\infty} \frac{2}{n!} D_n(r-b)^n = \sum_{n=0}^{\infty} \frac{F^{(n)}(b)}{n!}(r-b)^n,$$

$$D_n = \frac{F^{(n)}(b)}{2}. \tag{12}$$

From (8), when we put $r = \alpha(t)$, then b will be canceled and there left only series $\alpha(t) = \sum_{n=1}^{\infty} \alpha_1 t^{n/2}$. In integral error function, if we take $\sqrt{t}$ out of the brackets,

$$i^n erfc \frac{\sqrt{t}(\alpha_1 + \alpha_2\sqrt{t} + \alpha_3 t + \dots)}{2a_1\sqrt{t}} = i^n erfc \frac{\alpha(t)}{2a_1},$$

where $\alpha(t)$ becomes

$$\alpha(t) = \alpha_1 + \alpha_2\sqrt{t} + \alpha_3 t + \dots = \sum_{n=1}^{\infty} \alpha_n t^{\frac{n-1}{2}}.$$

Let us take $\sqrt{t} = \tau$, and we obtain from Equation 8

$$\frac{1}{\alpha(\tau)} \sum_{n=0}^{\infty} \left(2a_2\sqrt{t}\right)^n \left[C_n i^n erfc\delta(\tau) + D_n i^n erfc(-\delta(\tau))\right] = T_m, \tag{13}$$

where $\delta(\tau) = \frac{\alpha(\tau)}{2a_2}$.

To calculate coefficient C_n, we apply Leibniz and Faa Di Bruno formulas and Bell polynomials. Using Leibniz formula, we have

$$\frac{\partial^k[2^{n/2}\tau^n i^n erfc\delta]}{\partial\tau^k}\bigg|_{\tau=0} = \begin{cases} 0, & \text{for } k < n \\ \frac{2^{n/2}k!}{(k-n)!}[i^n erfc\delta]^{(k-n)}, & \text{for } k \geq n \end{cases}.$$

Using Faa Di Bruno formula and Bell polynomials for a derivative of a composite function, we have

$$\frac{\partial^{(k-n)}[i^n erfc(\pm\delta)]}{\partial\tau^{k-n}}\bigg|_{\tau=0} = \sum_{m=1}^{k-n}(i^n erfc(\pm\delta))^{(m)}\bigg|_{\delta=0} B_{k-n,m}(\delta^{(k-n-m+1)}(\tau))\big|_{\tau=0},$$

where

$$B_{k-n,m} = \sum \frac{(k-n)!}{j_1!j_2! \dots j_{k-n-m+1}!} \cdot \delta_1^{j_1}\delta_2^{j_2}\delta_3^{j_3} \dots \delta_{k-n-m+1}^{j_{k-n-m+1}},$$

$$\delta_1 = \frac{\alpha_2}{2a_2}, \dots, \delta_{k-n-m+1} = \frac{\alpha_{k-n-m+2}}{2a_2}$$

and $j_1, j_2, \dots$ satisfy the following equations

$$j_1 + j_2 + \dots + j_{k-n-m+1} = m,$$

$$j_1 + 2j_2 + \dots + (k - n - m + 1)j_{k-n-m+1} = k - n$$

for $m \geq n$. Since

$$[i^n erfc(\pm\delta)]^{(m)}\big|_{\alpha=0} = (-1)^m i^{n-m} erfc 0 = (\mp 1)^m \frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m!)\sqrt{\pi}}$$

by taking kth derivatives of both sides of (13) at $\tau = 0$, we have

$$\sum_{n=0}^{k} C_n (2a_2)^n \frac{k!}{(k-n)!} \sum_{m=1}^{k-n} (-1)^m i^{n-m} erfc(\delta_1) \sum \frac{(k-n)!}{j_1! j_2! \cdots j_{k-n-m+1}!} \delta_2^{j_1} \delta_3^{j_2} \cdots \delta_{k-n-m+2}^{j_{k-n-m+1}}$$

$$= T_m \alpha_k - \sum_{n=0}^{k} D_n (2a_2)^n \frac{k!}{(k-n)!} \sum_{m=1}^{k-n} (-1)^m i^{n-m} erfc(\delta_1) \sum \frac{(k-n)!}{j_1! j_2! \cdots j_{k-n-m+1}!} \delta_2^{j_1} \delta_3^{j_2} \cdots \delta_{k-n-m+2}^{j_{k-n-m+1}} \tag{14}$$

for $k = 0, 1, 2, \cdots$.

From expression (14), we express coefficients C_n. From (7), we have

$$\frac{1}{\alpha(\tau)} \sum_{n=0}^{\infty} \left(2a_1 \sqrt{t} \right)^n \left[A_n i^n erfc\xi(\tau) + B_n i^n erfc(-\xi(\tau)) \right] = T_m,$$

where

$$\xi(\tau) = \frac{\alpha(\tau)}{2a_1}. \tag{15}$$

In the same manner, we get recurrent formula from (15) where we express A_n in terms of B_n as

$$\sum_{n=0}^{k} A_n (2a_1)^n \frac{k!}{(k-n)!} \sum_{m=1}^{k-n} (-1)^m i^{n-m} erfc(\xi_1) \sum \frac{(k-n)!}{j_1! j_2! \cdots j_{k-n-m+1}!} \xi_2^{j_1} \xi_3^{j_2} \cdots \xi_{k-n-m+2}^{j_{k-n-m+1}}$$

$$= T_m \alpha_k - \sum_{n=0}^{k} D_n (2a_2)^n \frac{k!}{(k-n)!} \sum_{m=1}^{k-n} (-1)^m i^{n-m} erfc(\xi_1) \sum \frac{(k-n)!}{j_1! j_2! \cdots j_{k-n-m+1}!} \xi_2^{j_1} \xi_3^{j_2} \cdots \xi_{k-n-m+2}^{j_{k-n-m+1}}, \tag{16}$$

where

$$\xi_1 = \frac{\alpha_1}{2a_2}, \xi_2 = \frac{\alpha_2}{2a_2}, \cdots, \xi_{k-n-m+2} = \frac{\alpha_{k-n-m+2}}{2a_2}.$$

We can express coefficients A_n from this expression. In Stefan's condition (9), we take first derivative of

$$\frac{d\alpha}{dt} = \alpha_1 \frac{1}{2\sqrt{t}} + \alpha_2 + \frac{3}{2} \alpha_3 \sqrt{t} + 2\alpha_4 t + \ldots = \sum_{n=1}^{\infty} \frac{n}{2} \alpha_n t^{\frac{n-2}{2}},$$

and we take $\sqrt{t} = \tau$. So

$$\lambda_1 \frac{1}{\alpha^2(\tau)} \sum_{n=0}^{\infty} (2a_1\tau)^n \left[A_n i^n erfc\xi(\tau) + B_n i^n erfc(-\xi(\tau)) \right]$$

$$- \lambda_1 \frac{1}{\alpha(\tau)} \sum_{n=0}^{\infty} (2a_1\tau)^{n-1} \left[-A_n i^{n-1} erfc\xi(\tau) + B_n i^{n-1} erfc(-\xi(\tau)) \right]$$

$$= \lambda_2 \frac{1}{\alpha^2(\tau)} \sum_{n=0}^{\infty} (2a_2\tau)^n \left[C_n i^n erfc\delta(\tau) + D_n i^n erfc(-\delta(\tau)) \right] \tag{17}$$

$$- \lambda_2 \frac{1}{\alpha(\tau)} \sum_{n=0}^{\infty} (2a_1\tau)^{n-1} \left[-C_n i^{n-1} erfc\delta(\tau) + D_n i^{n-1} erfc(-\delta(\tau)) \right] + \alpha(\tau) L\gamma,$$

where $\alpha(\tau) = \sum_{n=1}^{\infty} \frac{n}{2} \alpha_n \tau^{n-2}$.

If we multiply both sides of (17) by $\tau a(\tau)$ and use (7) and (8), we get the following expression

$$- \lambda_1 \sum_{n=0}^{\infty} (2a_1)^{n-1} \tau^n \left[-A_n i^{n-1} erfc(\xi(\tau)) + B_n i^{n-1} erfc(-\xi(\tau)) \right]$$

$$= T_m (\lambda_2 - \lambda_1) - \lambda_2 \sum_{n=0}^{\infty} (2a_2)^{n-1} \tau \left[-C_n i^{n-1} erfc(\delta(\tau)) + D_n i^{n-1} erfc(-\delta(\tau)) \right] + \beta(\tau) L\gamma, \tag{18}$$

where

$$\beta(\tau) = \sum_{n=1}^{\infty} (b + a_n \tau^n) \sum_{n=1}^{\infty} a_n \tau^n.$$

Taking k-times derivative of both sides of (18) at $\tau = 0$, we get recurrent formula for B_n coefficients.

By using all these expression on condition (6), we express coefficient of heat flux. From (6), we get

$$\frac{\lambda_1}{b^2} \sum_{n=0}^{\infty} \left(2a_1\sqrt{t}\right)^n [A_n i^n erfc0 + B_n i^n erfc0] + \frac{\lambda_1}{b} \sum_{n=0}^{\infty} \left(2a_1\sqrt{t}\right)^{n-1} [A_n i^{n-1} erfc0 - B_n i^{n-1} erfc0] = \sum_{n=0}^{\infty} P_n t^{n/2},$$

$$\begin{cases} P_0 = \frac{\lambda_1}{b} i^0 erfc0[A_1 - B_1 + \frac{1}{b}A_0 + \frac{1}{b}B_0], \\ P_1 = \frac{\lambda_1}{b}(2a_1)i^1 erfc0[A_2 - B_2 + \frac{1}{b}A_1 + \frac{1}{b}B_1], \\ P_2 = \frac{\lambda_1}{b}(2a_1)^2 i^2 erfc0[A_3 - B_3 + \frac{1}{b}A_2 + \frac{1}{b}B_2], \\ \vdots \\ P_n = \frac{\lambda_1}{b}(2a_1)^n i^n erfc0[A_{n+1} - B_{n+1} + \frac{1}{b}A_n + \frac{1}{b}B_n]. \end{cases} \tag{19}$$

Remark 3. For the convergence of temperature functions Θ_1, Θ_2, it is possible to follow the idea proposed in Holm.[15]

3 | APPROXIMATE SOLUTION OF A TEST PROBLEM

In this section, the collocation method that is practical for engineers is applied using 3 points $t_0 = 0$, $t_1 = 0.5$, and $t_2 = 1$.

To show the effectiveness of the method, we proposed the following problem. Solution is found both exactly and approximately.

Let us consider

$$\frac{\partial \theta_1}{\partial t} = a_1^2 \left(\frac{\partial^2 \theta_1}{\partial r^2} + \frac{2}{r} \frac{\partial \theta_1}{\partial r} \right), \quad b < r < \alpha(t),$$

$$\frac{\partial \theta_2}{\partial t} = a_2^2 \left(\frac{\partial^2 \theta_2}{\partial r^2} + \frac{2}{r} \frac{\partial \theta_2}{\partial r} \right), \quad \alpha(t) < r < \infty,$$

$$\theta_1(b, 0) = T_m,$$

$$\theta_2(r, 0) = f(r), \tag{20}$$

$$-\lambda_1 \frac{\partial \theta_1(b, t)}{\partial r} = P(t),$$

$$\theta_1(\alpha(t), t) = \theta_2(\alpha(t), t) = T_m, \tag{21}$$

$$-\lambda_1 \frac{\partial \theta_1(\alpha(t), t)}{\partial r} = -\lambda_2 \frac{\partial \theta_2(\alpha(t), t)}{\partial r} + L\gamma \frac{d\alpha}{dt}, \tag{22}$$

$$\theta_2(\infty, t) = 0.$$

We represent solution of the problem in the following form

$$\theta_l(r, t) = \frac{1}{r} \sum_{n=0}^{\infty} \left(2a_l\sqrt{t}\right)^n \left[A_n(C_n) i^n erfc\left(\frac{r-b}{2a_l\sqrt{t}}\right) + B_n(D_n) i^n erfc\left(\frac{b-r}{2a_l\sqrt{t}}\right)\right].$$

It is clear that from (20), we get D_n and from (21), we get C_n. Also, using (21), we can express A_n in terms of B_n. Thus, we can find B_n by (22).

Let $\alpha(t) = b + a\sqrt{t}$ and $\theta_l = \frac{1}{r}\Phi(r, t)$ where $i = 1, 2$. Taking derivative, we get

$$\frac{\partial \theta_l}{\partial r} = \frac{1}{r}\Phi'(r, t) - \frac{1}{r^2}\Phi(r, t).$$

Let $f(r) = r$. Using (21), we have

$$C_0 i^0 erfc\frac{1}{2a_2} + D_0 i^0 erfc\left(-\frac{1}{2a_1}\right) + 2a_2\sqrt{t}\left[C_1 i^1 erfc\frac{1}{2a_2} + D_1 i^1 erfc\left(-\frac{1}{2a_2}\right)\right]$$

$$+\left(2a_2\sqrt{t}\right)^2\left[C_2 i^2 erfc\frac{1}{2a_2} + D_2 i^2 erfc\left(-\frac{1}{2a_2}\right)\right] + \ldots = T_m(b + \alpha\sqrt{t}),$$

$$C_0 = \frac{bT_m - D_0 i^0 erfc\left(-\frac{1}{2a_2}\right)}{i^0 erfc\frac{1}{2a_2}},$$

$$C_1 = \frac{\alpha T_m - D_1 i^1 erfc\left(-\frac{1}{2a_2}\right)}{i^1 erfc\frac{1}{2a_2}},$$

$$C_n = -\frac{D_n i^n erfc\left(-\frac{1}{2a_2}\right)}{i^n erfc\frac{1}{2a_2}}, \quad \text{where} \quad i = 2, 3, \cdots.$$

Let us transform (22) to obtain

$$-\lambda_1\left[\frac{1}{r}\Phi_1'(r, t) - \frac{1}{r^2}\Phi_1(r, t)\right]_{r=a(\tau)} = -\lambda_2\left[\frac{1}{r}\Phi_2'(r, t) - \frac{1}{r^2}\Phi_2(r, t)\right]_{r=a(\tau)} + \frac{L\gamma}{2\tau},$$

$$-\lambda_1\left[a(\tau)\sum_{n=0}^{\infty}(2a_1\tau)^{n-1}\left[-A_n i^{n-1}erfc\left(\frac{1}{2a_1}\right) + B_n i^{n-1}erfc\left(-\frac{1}{2a_1}\right)\right]\right.$$

$$\left. -\sum_{n=0}^{\infty}(2a_1\tau)^n\left[A_n i^n erfc\left(\frac{1}{2a_1}\right) + B_n i^n erfc\left(-\frac{1}{2a_1}\right)\right]\right]$$

$$= -\lambda_2\left[a(\tau)\sum_{n=0}^{\infty}(2a_2\tau)^{n-1}\left[-C_n i^{n-1}erfc\left(\frac{1}{2a_2}\right) + D_n i^{n-1}erfc\left(-\frac{1}{2a_2}\right)\right]\right.$$

$$\left. -\sum_{n=0}^{\infty}(2a_2\tau)^n\left[C_n i^n erfc\left(\frac{1}{2a_2}\right) + D_n i^n erfc\left(-\frac{1}{2a_2}\right)\right]\right] + \frac{L\gamma}{2a_2\tau}.$$

Multiply both sides on τ, we get

$$-\lambda_1\left[a(\tau)\tau\Phi_1' - \tau\Phi_1\right] = -\lambda_2\left[a(\tau)\tau\Phi_2' - \tau\Phi_2\right] + \frac{L\gamma}{2a_2}.$$

For $\tau = 0$,

$$-\lambda_1\left[bA_0 i^{-1}erfc\left(\frac{1}{2a_1}\right) + bB_0 i^{-1}erfc\left(-\frac{1}{2a_1}\right)\right] = -\lambda_2\left[bC_0 i^{-1}erfc\left(\frac{1}{2a_2}\right) + bD_0 i^{-1}erfc\left(-\frac{1}{2a_2}\right)\right] + \frac{L\gamma}{2a_2},$$

and from (21), we get

$$A_0 = \frac{T_m - B_0 i^0 erfc\left(-\frac{1}{2a_1}\right)}{i^0 erfc\left(\frac{1}{2a_1}\right)}.$$

Let

$$-A_n(C_n)i^{n-1}erfc\frac{1}{2a_t} + B_n(D_n)i^{n-1}erfc\left(-\frac{1}{2a_t}\right) > = \Psi_t,$$

$$A_n(C_n)i^n erfc\frac{1}{2a_t} + B_n(D_n)i^n erfc\left(-\frac{1}{2a_t}\right) = \Psi_t$$

and

$$-\lambda_1 \left[(b + \alpha\tau) \sum_{n=0}^{\infty} (2a_1)^{n-1}\tau^n\Psi_1' - \sum_{n=0}^{\infty} (2a_1)^n\tau^{n+1}\Psi_1 \right]$$

$$= -\lambda_2 \left[(b + \alpha\tau) \sum_{n=0}^{\infty} (2a_2)^{n-1}\tau^n\Psi_2' - \sum_{n=0}^{\infty} (2a_2)^n\tau^{n+1}\Psi_2 \right] + \frac{L\gamma}{2a_2},$$

$$-\lambda_1 \left[b\sum_{n=0}^{\infty} (2a_1)^{n-1}\tau^n\Psi_1' + \sum_{n=0}^{\infty} \tau^{n+1}\left[(2a_1)^{n-1}\alpha\Psi_1' - (2a_1)^n\Psi_1 \right] \right]$$

$$= -\lambda_2 \left[b\sum_{n=0}^{\infty} (2a_2)^{n-1}\tau^n\Psi_2' + \sum_{n=0}^{\infty} \tau^{n+1}\left[(2a_2)^{n-1}\alpha\Psi_2' - (2a_1)^n\Psi_2 \right] \right] + \frac{L\gamma}{2a_2}.$$

For the first derivative at $\tau = 0$, we get

$$-\lambda_1 \left[b\Psi_1'\big|_{n=1} + (2a_1)^{-1}\alpha\Psi_1'\big|_{n=0} - \Psi_1\big|_{n=0} \right] = -\lambda_2 \left[b\Psi_2'\big|_{n=1} + (2a_2)^{-1}\alpha\Psi_2'\big|_{n=0} - \Psi_2\big|_{n=0} \right]. \tag{23}$$

From (21) and (23), we get

$$A_1 = \frac{T_m\alpha - B_1 i^1 erfc\left(-\frac{1}{2a_1}\right)}{i^1 erfc\left(\frac{1}{2a_1}\right)},$$

$$-\lambda_1 \left[b\left(-A_1 i^0 erfc\frac{1}{2a_1} + B_1 i^0 erfc\left(-\frac{1}{2a_1}\right) \right) + (2a_1)^{-1}\alpha\left(-A_0 i^{-1} erfc\left(\frac{1}{2a_1}\right) + B_0 i^{-1} erfc\left(\frac{1}{2a_1}\right) \right) \right.$$
$$\left. - \left(A_0 i^0 erfc\left(\frac{1}{2a_1}\right) + B_0 i^0 erfc\left(-\frac{1}{2a_1}\right) \right) \right]$$

$$= -\lambda_2 \left[b\left(-C_1 i^0 erfc\frac{1}{2a_2} + D_1 i^0 erfc\left(-\frac{1}{2a_2}\right) \right) + (2a_2)^{-1}\alpha\left(-C_0 i^{-1} erfc\left(\frac{1}{2a_2}\right) + D_0 i^{-1} erfc\left(\frac{1}{2a_2}\right) \right) \right.$$
$$\left. - \left(C_0 i^0 erfc\left(\frac{1}{2a_2}\right) + D_0 i^0 erfc\left(-\frac{1}{2a_2}\right) \right) \right]$$
$$+ \frac{L\gamma}{2a_2}.$$

We find coefficients A_n, B_n from (12) and (21) when $n \leq 3$.

3.1 | Test problem

Let $f(r) = r, T_m = 0, \lambda_1 = \lambda_2 = 1, a_1 = a_2 = 1, L = \alpha = \gamma = 1$. Mathcad 15 is used for computations, and we get approximate values for $A_0 = 2.96 \times 10^{-6}, A_1 = 0.341, A_2 = -9.72, B_0 = -9.335 \times 10^{-7}, B_1 = -0.057$, and $B_2 = 1$ whereas exact values are $A_0 = 2.96 \times 10^{-6}, A_1 = 0.341, A_2 = -9.72, B_0 = -9.335377 \times 10^{-7}, B_1 = -0.05678483$, and $B_2 = 1.00000000001$.

In Figure 2, the graphs of both reconstructed exact (**exact_Flux_P(0,t)**) and approximate (**approx_Flux_P(0,t)**) flux functions are shown.
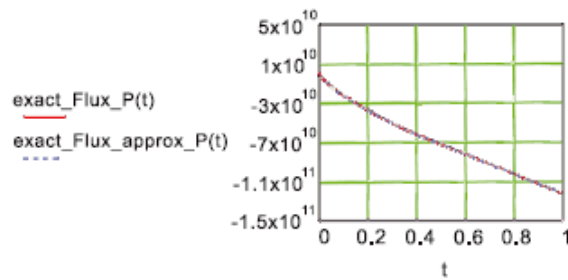

FIGURE 2   Exact and approximate values of flux function [Colour figure can be viewed at wileyonlinelibrary.com]
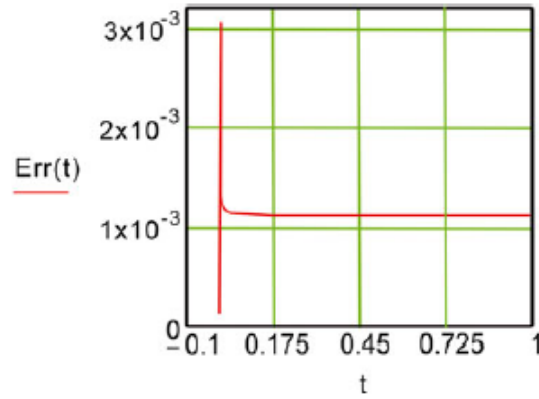
FIGURE 3 Relative error [Colour figure can be viewed at wileyonlinelibrary.com]
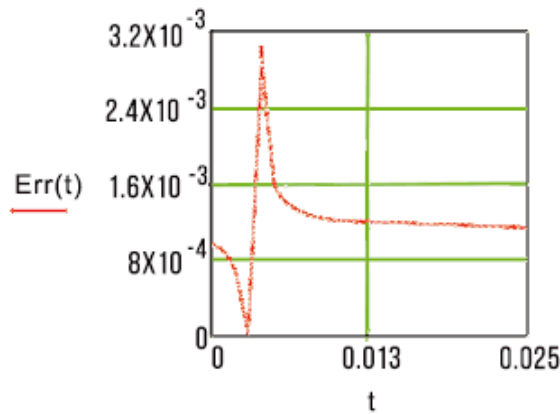

FIGURE 4 Relative error for small t values [Colour figure can be viewed at wileyonlinelibrary.com]

In Figures 3 and 4, we illustrate the graph of relative error function calculated by following formula that is less than 0.0032% at point $x = 0, 0 \leq t \leq 0.025$

$$Error_{rel} = \frac{|\text{exact flux-approximate flux}| \cdot 100}{\text{exact flux}}.$$

4 | CONCLUSION

A mathematical model describing heat propagation in electric contacts is constructed on the base of two-phase spherical inverse Stefan problem. The heat source $P(t)$ which is occurred by arcing, bridging, etc. can be determined from Equation 19. Temperature functions Θ_1, Θ_2 that are given in the form of series are determined whose coefficients A_n, B_n, C_n, and D_n are also determined from Equations 14, 16, and 17. In the test problem, we used maximum principle for error estimate; the deviation does not exceed 3.2×10^{-5} for 3 points. For better precision, more points has to be taken and better computer characteristics are required.

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