The Likelihood of Braess' Paradox in Traffic Networks

Vadim Zverovich
University of the West of England, Bristol BS16 1QY, UK
vadim.zverovich@uwe.ac.uk

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Abstract

The well-known Braess' paradox illustrates situations when adding a new link to a traffic network might increase congestion in the network. In this article, we announce a number of new results devoted to the probability of Braess' paradox to occur in the classical network configuration, with particular emphasis on the Erlang distribution of parameters of the travel time function. This distribution is important in the context of traffic networks. However, other distributions will be analysed as well because Braess' paradox can be observed in various applied contexts such as telecommunication networks and power transmission networks. Our results revealed that typical probabilities for Braess' paradox to occur in the classical network configuration do not exceed 10%, and they are very low for some distributions of the parameters of travel time functions. If the classical network configuration consists of motorway sections and class A roads and the parameters of the travel time functions are modelled by the Erlang-2 distribution, then the probability of Braess' paradox to occur is 6%.

Keywords: Braess' paradox; probability; equilibrium flow; traffic network.

1 Introduction

The classical network configuration introduced by Braess [3] consists of three paths:

$$P_1 = a - b - d, \;\; P_2 = a - c - d, \;\; P_3 = a - b - c - d.$$

This network is denoted by N^+ and it has four nodes and five links, where a is the origin of all travel demand, and d is the destination of all demand (see Figure 1). The network N is N^+ with the link (b, c) removed. In 2006, Valiant and Roughgarden [16] showed that "the 'global' behaviour of an equilibrium flow in a large random network is similar to that in Braess' original four-node example". Thus, Braess' network configuration is of fundamental significance.

We assume that every link (i, j) in a network has a linear travel time function

$$\alpha_{ij} + \beta_{ij} f_{ij},$$

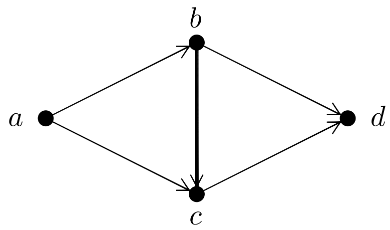

Figure 1: Braess' Network Configuration N/N^+, where $N = N^+ - (b, c)$.

where $\alpha_{ij} \geq 0$ is the free flow travel time for the link (i, j), $\beta_{ij} > 0$ is the delay parameter for (i, j) and $f_{ij} \geq 0$ is the flow on the link (i, j). Thus, Braess' network has the following linear travel time functions:

$$\begin{aligned} \alpha_1 + \beta_1 f_{ab} &\text{ for link } (a, b), \\ \alpha_2 + \beta_2 f_{bd} &\text{ for link } (b, d), \\ \alpha_3 + \beta_3 f_{bc} &\text{ for link } (b, c), \\ \alpha_4 + \beta_4 f_{ac} &\text{ for link } (a, c), \\ \alpha_5 + \beta_5 f_{cd} &\text{ for link } (c, d). \end{aligned}$$

A path P from the origin to the destination is said to have a *vanishing flow* if P carries no traffic from the origin to the destination. Note that some links in the path P may have a non-zero flow that contributes to the traffic of other paths. A path has a *non-vanishing flow* if it carries some traffic from the origin to the destination.

Definition 1 *A network with one origin and one destination is said to be at* equilibrium *if*

(a) The travel time on paths with non-vanishing flow is the same (it is denoted by T_{Eq}*) and*

(b) The travel time on paths with no flow is at least T_{Eq}*.*

This fundamental definition is, of course, a re-formulation of Wardrop's first principle [18] and it can be used to determine the equilibrium time and the equilibrium flow. At equilibrium, no user can decrease their route travel time by unilaterally switching routes [18]. In other words, if a network is not at equilibrium, then some users of the network (e.g. drivers) can switch their routes in order to improve their travel times.

Let $Q > 0$ denote the total flow in N/N^+, that is,

$$Q = f_{ab} + f_{ac} = f_{bd} + f_{cd}.$$

Note that f_{ij} and Q are not necessarily integer numbers. Let us denote

$$\alpha_{ij} = \alpha_i + \alpha_j, \quad \beta_{ij} = \beta_i + \beta_j,$$

for example, α_{12} means $\alpha_1 + \alpha_2$. Also,

$$\alpha = \alpha_{45} - \alpha_{12}, \qquad \bar{\alpha} = \alpha_4 - \alpha_{13}, \qquad \hat{\alpha} = \alpha_2 - \alpha_{35},$$

and

$$\beta = \beta_{1245} = \beta_1 + \beta_2 + \beta_4 + \beta_5, \qquad \beta_{ijk} = \beta_i + \beta_j + \beta_k.$$

The following identity will be used throughout the article:

$$\alpha = \bar{\alpha} - \hat{\alpha}.$$

Lemma 1 describes the equilibrium in the network N, which is N^+ with the link (b, c) removed. Note that the case (a) of this lemma corresponds to the situation when the path P_1 has a vanishing flow and P_2 has a non-vanishing flow in N. In case (b) the path P_1 has a non-vanishing flow and P_2 has a vanishing flow, and in case (c) no path has a vanishing flow. Also, the cases (a) and (b) in this lemma are mutually exclusive because one of the numbers $-\alpha/\beta_{45}$ and α/β_{12} is negative, or they both are equal to zero.

Lemma 1 [20] *In the network N, the travel time at equilibrium is as follows:*

(a) $T_{Eq} = \alpha_{45} + Q\beta_{45}$ *if* $0 < Q \leq -\alpha/\beta_{45}$;

(b) $T_{Eq} = \alpha_{12} + Q\beta_{12}$ *if* $0 < Q \leq \alpha/\beta_{12}$;

(c) $T_{Eq} = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta$ *if* $Q > \max\{\alpha/\beta_{12}; -\alpha/\beta_{45}\}$.

The equilibrium in N^+ is described by seven cases in Lemma 2. It may be pointed out that these cases correspond to the following situations in N^+:

(a) The only path with non-vanishing flow is P_3;

(b) The only path with non-vanishing flow is P_2;

(c) The only path with non-vanishing flow is P_1;

(d) The only path with vanishing flow is P_1;

(e) The only path with vanishing flow is P_2;

(f) The only path with vanishing flow is P_3;

(g) No path has a vanishing flow.

It is not difficult to see that some of the cases in Lemma 2 are mutually exclusive, hence the equilibrium in a particular network N^+ is described by some of the presented seven cases. For example, if $\alpha_i = \beta_i = 1$ for $1 \leq i \leq 5$, then the equilibrium is given by just one case (f).

Let us define the Braess numbers $\mathcal{B}_i$ for $i = 1, 2, 3, 4$:

$$\mathcal{B}_1 = \beta_1\beta_5 - \beta_2\beta_4, \quad \mathcal{B}_2 = \beta_{135}\beta - \beta_{12}\beta_{45},$$

$$\mathcal{B}_3 = \beta_{45}^2\beta_{134} - \beta_4^2\beta, \quad \mathcal{B}_4 = \beta_{12}^2\beta_{235} - \beta_2^2\beta.$$

We will also need two parameters μ_1 and μ_2:

$$\mu_1 = \frac{\hat{\alpha}\beta_{14} - \alpha\beta_3}{\beta_3\beta_{45} + \beta_5\beta_{14}}, \qquad \mu_2 = \frac{\bar{\alpha}\beta_{25} + \alpha\beta_3}{\beta_1\beta_{25} + \beta_3\beta_{12}}.$$

Lemma 2 [20] *In the network N^+, the travel time at equilibrium is as follows:*

(a) $T_{Eq}^+ = \alpha_{135} + Q\beta_{135}$ *if* $0 < Q \le \min\{\hat{\alpha}/\beta_{35}; \bar{\alpha}/\beta_{13}\}$;

(b) $T_{Eq}^+ = \alpha_{45} + Q\beta_{45}$ *if* $0 < Q \le \min\{-\alpha/\beta_{45}; -\bar{\alpha}/\beta_4\}$;

(c) $T_{Eq}^+ = \alpha_{12} + Q\beta_{12}$ *if* $0 < Q \le \min\{\alpha/\beta_{12}; -\hat{\alpha}/\beta_2\}$;

(d) $T_{Eq}^+ = \alpha_{45} + Q\beta_{45} - (\bar{\alpha} + Q\beta_4)\beta_4/\beta_{134}$ *if*

$$\max\{\bar{\alpha}/\beta_{13}; -\bar{\alpha}/\beta_4\} < Q \le \mu_1;$$

(e) $T_{Eq}^+ = \alpha_{12} + Q\beta_{12} - (\hat{\alpha} + Q\beta_2)\beta_2/\beta_{235}$ *if*

$$\max\{\hat{\alpha}/\beta_{35}; -\hat{\alpha}/\beta_2\} < Q \le \mu_2;$$

(f) $T_{Eq}^+ = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta$ *if* $Q > \max\{\alpha/\beta_{12}; -\alpha/\beta_{45}\}$ *and*

$$\mathcal{B}_1 \ge \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{Q};$$

(g) $T_{Eq}^+ = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta + g\mathcal{B}_1/\beta$, *where*

$$g = \frac{\bar{\alpha}\beta - \alpha\beta_{14} - Q\mathcal{B}_1}{\beta_3\beta + \beta_{14}\beta_{25}},$$

if $Q > \max\{\mu_1; \mu_2\}$ *and*

$$\mathcal{B}_1 < \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{Q}.$$

The next definition is devoted to Braess' paradox [3] in the classical network configuration N/N^+; however, the same definition is valid if N/N^+ represents any network configuration. Basically, the paradox describes a situation when adding a new link to a network makes a general performance worse.

Definition 2 Braess' paradox *is said to occur in the network configuration N/N^+ for a given total flow Q if*

$$T_{Eq}^+ > T_{Eq},$$

where T_{Eq} and T_{Eq}^+ are travel times at equilibria in N and N^+, respectively.

Thus, Braess' paradox illustrates situations when adding a new link to a transport network might not reduce congestion in the network but instead increase it.

Proposition 1 describes all possible situations when Braess' paradox may occur in N/N^+ in terms of their paths. In fact, it says that Braess' paradox may occur in N/N^+ only if, at equilibria, both P_1 and P_2 have a non-vanishing flow in N, and P_3 has a non-vanishing flow in N^+.

Proposition 1 [20] *Braess' paradox may occur in N/N^+ in the following cases only:*

(a) At equilibria, both N and N^+ have no paths with vanishing flow.

(b) At equilibria, N has no path with vanishing flow, and P_3 is the only path with non-vanishing flow in N^+.

(c) At equilibria, N has no path with vanishing flow, and P_1 is the only path with vanishing flow in N^+.

(d) At equilibria, N has no path with vanishing flow, and P_2 is the only path with vanishing flow in N^+.

In the following theorems, the necessary and sufficient conditions for the existence of the paradox are formulated. These theorems correspond to the four cases of Proposition 1.

Theorem 1 [20] *Suppose that at equilibria both N and N^+ have no paths with vanishing flow. Then Braess' paradox occurs in N/N^+ if and only if the Braess number $\mathcal{B}_1$ is positive and*

$$\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \mu_1; \mu_2\right\} < Q < \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{\mathcal{B}_1}.$$

Theorem 2 [20] *Suppose that at equilibria N has no path with vanishing flow and P_3 is the only path with non-vanishing flow in N^+. Then Braess' paradox occurs in N/N^+ if and only if the Braess number $\mathcal{B}_2$ is positive and*

$$\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\hat{\alpha}\beta_{45} + \bar{\alpha}\beta_{12}}{\mathcal{B}_2}\right\} < Q \leq \min\left\{\frac{\hat{\alpha}}{\beta_{35}}; \frac{\bar{\alpha}}{\beta_{13}}\right\}.$$

Theorem 3 [20] *Suppose that at equilibria N has no path with vanishing flow and P_1 is the only path with vanishing flow in N^+. Then Braess' paradox occurs in N/N^+ if and only if the Braess number $\mathcal{B}_3$ is positive and*

$$\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\bar{\alpha}}{\beta_{13}}; \frac{-\bar{\alpha}}{\beta_4}; \frac{\bar{\alpha}\beta_4\beta - \alpha\beta_{134}\beta_{45}}{\mathcal{B}_3}\right\} < Q \leq \mu_1.$$

Theorem 4 [20] *Suppose that at equilibria N has no path with vanishing flow and P_2 is the only path with vanishing flow in N^+. Then Braess' paradox occurs in N/N^+ if and only if the Braess number $\mathcal{B}_4$ is positive and*

$$\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\hat{\alpha}}{\beta_{35}}; \frac{-\hat{\alpha}}{\beta_2}; \frac{\hat{\alpha}\beta_2\beta + \alpha\beta_{235}\beta_{12}}{\mathcal{B}_4}\right\} < Q \leq \mu_2.$$

It might be pointed out that if $\mathcal{B}_1 \geq 0$, then $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ are positive numbers because

$$\mathcal{B}_2 = \beta_{12}\beta_{13} + \beta_{35}\beta_{45} + \mathcal{B}_1, \tag{1}$$

$$\mathcal{B}_3 = \beta_5^2\beta_{134} + \beta_4(\beta_3\beta_{455} + \beta_5\beta_{14} + \mathcal{B}_1), \tag{2}$$

$$\mathcal{B}_4 = \beta_1^2\beta_{235} + \beta_2(\beta_1\beta_{335} + \beta_2\beta_{13} + \mathcal{B}_1). \tag{3}$$

Moreover, Theorems 3 and 4 are mutually exclusive in the sense that they cannot provide intervals for Q simultaneously. This is true because the inequalities $\bar{\alpha}/\beta_{13} < \mu_1$ and $\hat{\alpha}/\beta_{35} < \mu_2$ are inconsistent. Note also that if, for example, Theorems 1–3 provide non-empty intervals for Q, then the interval with highest values of Q is given by Theorem 1, the interval with smallest values of Q is provided by Theorem 2 and Theorem 3 yields the interval with mid-range values of Q.

The original assumption $\beta_i > 0$ for all i can be relaxed by allowing $\beta_i = 0$ for some i. This can be done by introducing $+\infty$ and $-\infty$ when a non-zero number is divided by zero.

2 Likelihood of Braess' Paradox

We develop a new technique to show that the likelihood of Braess' paradox to occur in the classical network configuration is rather small. This is demonstrated for different distributions of parameters of travel time functions for links in a network, with particular emphasis on the Erlang distribution because of its importance for traffic networks. For example, we prove mathematically that the probability of Braess' paradox to happen does not exceed 0.129 when the parameters follow the Erlang-3 distribution. Similar estimates are true for the exponential distribution, the χ^2-distribution, the uniform and other distributions. Our simulation results for different distributions revealed that typical probabilities for Braess' paradox to occur in the classical network configuration do not exceed 10%, and they are very low for some distributions of the parameters of travel time functions. If the classical network configuration consists of motorway sections and class A roads and the parameters of the travel time functions are modelled by the Erlang-2 distribution, then the probability of Braess' paradox to occur is 6%.

The focus of this section is on the probability of Braess' paradox to occur in the classical network configuration when a single link is added/removed, which is consistent with the original definition of the paradox. Under other assumptions, Valiant and Roughgarden [16] proved that Braess' paradox is likely to occur in a natural random network model. More precisely, they showed that in almost all networks there is a set of links whose removal improves the travel time at equilibrium for a given appropriate total flow.

Let us re-formulate and simplify Theorems 1–4 by replacing the condition $\mathcal{B}_i > 0$ for $i = 2, 3, 4$ by the condition $\mathcal{B}_1 > 0$. Actually, the latter is a stronger condition as discussed after Theorem 4. This modification is given in Theorem 5. Notice that some of the intervals in this theorem may be empty. If all four intervals are empty (or $\mathcal{B}_1 \leq 0$), then there is no Braess' paradox.

Theorem 5 *Braess' paradox occurs in the network configuration N/N^+ if and only if the Braess number $\mathcal{B}_1$ is positive and the total flow Q belongs to the following intervals:*

(A) $\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \mu_1; \mu_2\right\} < Q < \frac{\hat{\alpha}\beta_{14}+\bar{\alpha}\beta_{25}}{\mathcal{B}_1};$

(B) $\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\hat{\alpha}\beta_{45}+\bar{\alpha}\beta_{12}}{\mathcal{B}_2}\right\} < Q \leq \min\left\{\frac{\hat{\alpha}}{\beta_{35}}; \frac{\bar{\alpha}}{\beta_{13}}\right\};$

(C) $\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\bar{\alpha}}{\beta_{13}}; \frac{-\bar{\alpha}}{\beta_4}; \frac{\bar{\alpha}\beta_4\beta-\alpha\beta_{134}\beta_{45}}{\mathcal{B}_3}\right\} < Q \leq \mu_1;$

(D) $\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\hat{\alpha}}{\beta_{35}}; \frac{-\hat{\alpha}}{\beta_2}; \frac{\hat{\alpha}\beta_2\beta+\alpha\beta_{235}\beta_{12}}{\mathcal{B}_4}\right\} < Q \leq \mu_2.$

Let the delay parameters be arranged in a 2×2 matrix B, and let $\hat{\alpha}$ and $\bar{\alpha}$ be presented as a 2-dimensional vector $\underline{\alpha}$:

$$B = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_4 & \beta_5 \end{pmatrix}, \qquad \underline{\alpha} = \begin{pmatrix} \hat{\alpha} \\ \bar{\alpha} \end{pmatrix}.$$

The next important result, which follows from Theorem 5, gives necessary and sufficient conditions for Braess' paradox to occur in the classical network configuration N/N^+. Although this result does not provide the interval of values for the total flow where the paradox is happening, it is very helpful for finding the probability of Braess' paradox to occur. It is interesting to note that the delay parameter β_3 for link (b, c), which is added/removed in the network configuration, plays no role in the occurance of Braess' paradox.

Theorem 6 *The statements (a), (b) and (c) are equivalent:*

(a) Braess' paradox occurs in the classical network configuration N/N^+.

(b) The determinant of B is positive, and the linear transformation B applied to $\underline{\alpha}$ yields a vector with positive components, that is:

$$|B| > 0 \quad \textit{and} \quad B\underline{\alpha} > \underline{0}.$$

(c) The following inequalities are satisfied:

$$\beta_1\beta_5 > \beta_2\beta_4, \tag{4}$$

$$\beta_1\hat{\alpha} + \beta_2\bar{\alpha} > 0, \tag{5}$$

$$\beta_4\hat{\alpha} + \beta_5\bar{\alpha} > 0. \tag{6}$$

In what follows, we shall assume that the parameters of the travel time functions in a network are random continuous variables. More precisely, free flow travel times ($\alpha_i \geq 0$) for links follow specified probability distributions, and delay parameters ($\beta_i > 0$) have some general distribution. Let us define the random variables Ψ and Φ:

$$\Psi = \min\{\alpha_2 - \alpha_5; \alpha_4 - \alpha_1\} \quad \text{and} \quad \Phi = \max\{\alpha_2 - \alpha_5; \alpha_4 - \alpha_1\}.$$

Lemma 3 *The probability of Braess' paradox to occur in the classical network configuration N/N^+ satisfies the following bounds:*

$$0.5\,\mathbf{P}[\Psi - \alpha_3 > 0] \leq \mathbf{P}[\textit{Braess' paradox occurs}] \leq 0.5\,\mathbf{P}[\Phi - \alpha_3 > 0].$$

We start with a relatively simple case when free flow travel times for links are uniformly distributed. Without loss of generality, we assume that those random variables have support on the interval [0,1].

Theorem 7 *Let the free flow travel times (α_i) for links in the classical network configuration N/N^+ follow the uniform distribution on* [0,1], *and let $\mathbf{P}_{\mathrm{UN}}$ denote the probability of Braess' paradox to occur in such a network configuration. Then*

$$0.025 \leq \mathbf{P}_{\mathrm{UN}} < 0.142.$$

The following theorem is devoted to the situation when free flow travel times for links are exponentially distributed.

Theorem 8 *Let the free flow travel times (α_i) for links in the classical network configuration N/N^+ follow the exponential distribution, and let $\mathbf{P}_{\mathrm{EX}}$ denote the probability of Braess' paradox to occur in such a network configuration. Then*

$$0.041 < \mathbf{P}_{\mathrm{EX}} < 0.209.$$

For traffic networks consisting of motorway sections, class A roads or a mixture of both, statistical tests showed that the distribution of parameters of travel time functions follow the Erlang-k distribution for small values of k as well as some other distributions, which will be discussed in the section devoted to simulation.

Our next theorem is devoted to the situation when free flow travel times for links follow the Erlang-2 distribution. Similar to the previous results, the formulated lower and upper bounds are true for any distribution of the delay parameters (β_i).

Theorem 9 *Let the free flow travel times (α_i) for links in the classical network configuration N/N^+ follow the Erlang-2 distribution $\Gamma(2,\theta)$, and let $\mathbf{P}_{\mathrm{E2}}$ denote the probability of Braess' paradox to occur in such a network configuration. Then*

$$0.025 < \mathbf{P}_{\mathrm{E2}} < 0.163.$$

In the following theorem, we analyse the Erlang-3 distribution for free flow travel times.

Theorem 10 *Let the free flow travel times (α_i) for links in the classical network configuration N/N^+ follow the Erlang-3 distribution $\Gamma(3,\theta)$, and let $\mathbf{P}_{\mathrm{E3}}$ denote the probability of Braess' paradox to occur in such a network configuration. Then*

$$0.016 < \mathbf{P}_{\mathrm{E3}} < 0.129.$$

Our next theorem considers the Erlang-4 distribution for free flow travel times.

Theorem 11 *Let the free flow travel times (α_i) for links in the classical network configuration N/N^+ follow the Erlang-4 distribution $\Gamma(4,\theta)$, and let $\mathbf{P}_{\mathrm{E4}}$ denote the probability of Braess' paradox to occur in such a network configuration. Then*

$$0.01 < \mathbf{P}_{\mathrm{E4}} < 0.103.$$

It may be pointed out that the theorems devoted to the exponential and Erlang distributions are relevant to the χ^2-distribution, which is a particular case of those distributions if we put $\theta = 2$.

Let us consider a generalisation of the previous results for the Erlang-k distribution, which converges to the normal distribution for large values of k. As illustrated in the next section, this distribution might be important in the context of road networks because a number of consecutive motorway sections can be modelled by a single link. The parameters of such a link could be described by the Erlang-k distribution, where k might have large values.

We will only consider the upper bound because the lower bound becomes too small (< 0.01) for $k \geq 5$. Also, for the sake of simplicity, we sacrifice a non-dominant term ($-0.5 \int_0^{+\infty} f_\Omega(z)dz$) in the upper bound, which actually constitutes the aforementioned lower bound.

In contrast to previous theorems, the generalisation will not directly provide an explicit numerical upper bound for Braess' paradox to occur. Instead, we will deduce some formula that can be used for calculation of the numerical upper bound. Note that in the third sum of this formula, the upper limit $(k-2-f-l)$ may be equal to -1, in which case the entire sum is equal to 0.

Theorem 12 *Let the free flow travel times (α_i) for links in the classical network configuration N/N^+ follow the Erlang-k distribution $\Gamma(k,\theta)$, and let $\mathbf{P}_{\mathrm{E}k}$ denote the probability of Braess' paradox to occur in such a network configuration. Then*

$$\mathbf{P}_{\mathrm{E}k} < \sum_{l=0}^{k-1} \sum_{f=0}^{k-1-l} \frac{\binom{k-1+f}{f}}{2^{2k+f}} \left\{ 1 + \sum_{j=0}^{k-2-f-l} 2^{s_j+1-k} \binom{2k-3-s_j}{k-1} \frac{s_j}{k-1-s_j} \right\},$$

where $s_j = f + l + j$.

It is an interesting conjecture that the bound of Theorem 12 tends to 0 as $k \to +\infty$. We will only present numerical values of this bound for some values of the shape parameter k:

$$\mathbf{P}_{\mathrm{E5}} < 0.090, \quad \mathbf{P}_{\mathrm{E6}} < 0.072, \quad \mathbf{P}_{\mathrm{E7}} < 0.058, \quad \mathbf{P}_{\mathrm{E8}} < 0.047,$$

$$\mathbf{P}_{\mathrm{E9}} < 0.038, \quad \mathbf{P}_{\mathrm{E10}} < 0.031, \quad \mathbf{P}_{\mathrm{E11}} < 0.026, \quad \mathbf{P}_{\mathrm{E12}} < 0.021,$$

$$\mathbf{P}_{\mathrm{E13}} < 0.017, \quad \mathbf{P}_{\mathrm{E14}} < 0.014, \quad \mathbf{P}_{\mathrm{E15}} < 0.012, \quad \mathbf{P}_{\mathrm{E16}} < 0.010,$$

$$\mathbf{P}_{\mathrm{E28}} < 0.001.$$

2.1 Simulation Results

In this section, we present results of a computer simulation for the probability of Braess' paradox to occur in the classical network configuration N/N^+. The simulation is based on Theorem 6 and a random generation of the parameters of the travel time functions from specified probability distributions. More precisely, for a given probability distribution, the inverse of its cumulative probability density function is used to generate instances of the free flow travel times (α_i), and a similar procedure is used to generate instances of delay parameters (β_i). Up to 1 million instances were generated for calculation of each probability presented in the following tables; typically, more instances were needed for small probabilities to reduce simulation errors. The probabilities are given to one significant figure if they are less than 0.1 and to two significant figures otherwise.

Table 1: Probabilities for Braess' paradox to occur for different distributions of free flow travel times and delay parameters.

Delay Parameter,	**Free Flow Travel Times, α_i:**					
β_i**:**	**Unif.**	**Exp.**	**Erlang-2**	**Erlang-4**	**Erlang-16**	**Erlang-28**
Uniform	0.05	0.09	0.06	0.03	0.0006	0.00002
Exponential	0.05	0.09	0.06	0.03	0.0007	0.00003
Erlang-2	0.05	0.09	0.06	0.03	0.0007	0.00002
Erlang-4	0.05	0.10	0.06	0.03	0.0006	0.00002
Erlang-16	0.05	0.10	0.06	0.03	0.0006	0.00002
Erlang-28	0.05	0.10	0.07	0.03	0.0005	0.00001
Weibull(8,1)	0.06	0.11	0.07	0.03	0.0006	0.00002
Lognormal(0,1)	0.05	0.09	0.06	0.03	0.0006	0.00003
Beta(6,2)	0.05	0.10	0.07	0.03	0.0006	0.00001

Table 1 provides probabilities for Braess' paradox to occur for different distributions of free flow travel times and delay parameters. An important observation is that for a given distribution of free flow travel times, the choice of the distribution of delay parameters practically does not affect the likelihood of Braess' paradox to occur. Some influence of the delay parameters can be observed when free flow travel times follow the Erlang-28 distribution, however the probabilities themselves are very small.

Taking into account the observation made above, it seems reasonable to analyse situations when probability distributions for free flow travel times and delay parameters are the same. Our statistical tests for free flow travel times of links in road networks showed that they can be modelled by the Erlang-k distribution for small values of k as well as the Weibull and lognormal distributions. This is of no surprise because those distribution can be very similar for some sets of their parameters. Also, the Erlang-k distribution converges to the normal distribution for large values of k. On the other hand, with different parameters, the lognormal and Erlang-k distributions can be extremely right-skewed, which make them very different from the symmetric normal distribution, whereas the Weibull distribution can be skewed to any side depending on its parameters.

The next two tables show probabilities of Braess' paradox to occur when the parameters of travel time functions follow the Weibull distribution or the lognormal distribution. As can be seen from Table 2, the probabilities practically do not depend on the scale parameter λ if the shape parameter is fixed. Similarly, the probabilities in Table 3 do not depend on the M-parameter for a given S-parameter.

Table 2: Probabilities for the Weibull distribution Weibull(k,λ).

Shape parameter, k:	**Scale parameter, λ:** 1	10	50
2	0.03	0.03	0.03
3	0.01	0.01	0.01
4	0.003	0.003	0.003
5	0.0005	0.0006	0.0005
6	0.00009	0.0001	0.00009
7	0.00002	0.00002	0.00003
10	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$

It is interesting to note that the highest probability of 14% in all simulations is achieved for the lognormal distribution when the S-parameter is at least 5, that is, when the distribution is extremely right-skewed. Such a distribution, however, is not common for modelling the parameters of travel time functions of roads and it might be of rather theoretical interest.

Table 3: Probabilities for the lognormal distribution Lognormal(M,S).

M-parameter:	**S-parameter:** 0.5	1	3	5	50
0	0.03	0.09	0.13	0.14	0.14
1	0.03	0.09	0.13	0.14	0.14
5	0.03	0.09	0.13	0.14	0.14

The beta distribution may be important in the context of Braess' paradox because of the variety of shapes provided by this distribution, taking into account that Braess' paradox can be observed in various applied contexts. Table 4 confirms another general observation that if a distribution is very left-skewed (e.g. $A = 5$, $B = 0.5$ for the beta distribution), then the corresponding probability of Braess' paradox to occur is very small; and the highest values of the probability are achieved for very right-skewed distributions.

Table 4: Probabilities for the beta distribution Beta(A,B).

A-parameter:	**B-parameter:** 0.5	1	3	5	50
0.5	0.07	0.09	0.10	0.11	0.11
1	0.03	0.05	0.07	0.08	0.09
3	0.0006	0.003	0.01	0.02	0.04
5	0.00001	0.0001	0.001	0.004	0.02

The normal distribution with a 'large' negative tail might not be appropriate to model the random behaviour of free flow travel times. However, because of its importance, some normal distributions can be used in the context of Braess' paradox as a first approximation, and hence we will only consider one family of the normal distribution with a rather 'small' negative tail, that is, when $\mu/\sigma \geq 3$. Note that in our simulations all generated instances with at least one negative parameter were rejected.

Table 5: Probabilities for the normal distribution $\mathcal{N}(\mu, \sigma^2)$.

	Standard Deviation, σ:				
Mean, μ:	**20**	**24**	**30**	**40**	**48**
120	$< 10^{-5}$	0.0001	0.001	0.007	-
240	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$	0.00009
480	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$

Table 6: (Cntd.) Probabilities for the normal distribution $\mathcal{N}(\mu, \sigma^2)$.

	Standard Deviation, σ:				
Mean, μ:	**60**	**80**	**96**	**120**	**160**
120	-	-	-	-	-
240	0.001	0.007	-	-	-
480	$< 10^{-5}$	$< 10^{-5}$	0.00008	0.001	0.007

As can be seen in Tables 5 and 6, the probability of Braess' paradox to occur for the normal distribution of the parameters is approximately the same is the ratio μ/σ is fixed. A further observation for the normal distribution (and perhaps any distribution) with a fixed positive mean is that the smaller the standard deviation, the smaller the probability of Braess' paradox to occur. This is easy to understand for a very small standard deviation because in this case, with very high probability, the parameters α_i would be very close to the positive mean, and hence $\hat{\alpha}$ and $\bar{\alpha}$ would be negative and there would be no Braess' paradox by Theorem 6, that is, the probability of Braess' paradox to happen would be very small.

A marvellous open problem would be to investigate the probability of Braess' paradox to occur in large traffic networks when a single link is added. Our hypothesis is that such probabilities are rather small. Some insight could be gained from the generalised traffic network discussed in [20]. Suppose that each of the (a, b)-path, (b, d)-path, (a, c)-path, (c, d)-path and (b, c)-path consists of eight motorway sections, and the parameters of travel time functions for each section are modelled by the Erlang-2 distribution. Using the addition property of this distribution, such a generalised network can be reduced to the classical network configuration, where the parameters of travel time functions for each link are modelled by the Erlang-16 distribution. From Table 1, the probability of Braess' paradox to occur in the generalised network when a link on the (b, c)-path is removed is 0.0006. Another interesting addition to the aforementioned open problem would be to consider non-linear BPR functions.

Based on the results of this section, we can conclude that typical probabilities for Braess' paradox to occur in the classical network configuration do not exceed 10%, and they are very low for some distributions of the parameters of travel time functions. If the classical network configuration consists of motorway sections and class A roads and the parameters of the travel time functions are modelled by the

Erlang-2 distribution, then the probability of Braess' paradox to occur is 6%. Also, let us summarize three observations made in this section:

1. For a given distribution of free flow travel times, the choice of the distribution of delay parameters practically does not affect the likelihood of Braess' paradox to occur.

2. If the distribution of the parameters of travel time functions is very left-skewed, then the corresponding probability of Braess' paradox to occur is very small. The highest values of the probability are achieved for very right-skewed distributions.

3. If the distribution of the parameters of travel time functions has a fixed positive mean, then the smaller the standard deviation, the smaller the probability of Braess' paradox to occur.

3 References

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