

# Braess' Paradox in a Generalised Traffic Network

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## Abstract

Braess' paradox illustrates situations when adding a new link to a transport network might lead to an equilibrium state in which travel times of users will increase. The classical network configuration introduced by Braess in 1968 to demonstrate the paradox is of fundamental significance because Valiant and Roughgarden showed in 2006 that 'the "global" behaviour of an equilibrium flow in a large random network is similar to that in Braess' original four-node example'. Braess' paradox has been studied mainly in the context of the classical problem introduced by Braess and his colleagues, assuming a certain type of symmetry in networks. Specifically, two pairs of links in those networks are assumed to have the same volume-delay functions. The occurrence of Braess' paradox for this specific case of network symmetry was investigated by Pas and Principio in 1997. Such a symmetry is not common in real-life networks because the parameters of volume-delay functions are associated with roads physical and functional characteristics, which typically differ from one link to another. This research provides an extension of previous studies on Braess' paradox by considering arbitrary volume-delay functions, i.e. symmetry properties are not assumed for any of the network's links and the occurrence of Braess' paradox is studied for a general configuration.

*Keywords:* Braess' paradox; equilibrium flow; traffic network.

## 1 Introduction

The traffic network can generally be described as a game, where a finite number of interdependent network users compete with one another by making simultaneous route choices. It is commonly assumed that network users non-cooperatively interact with one another in the traffic network in order to minimise their travel costs. This problem is commonly modelled as an  $N$ -person nonzero-sum game (see [6]), and its solution assumes the existence of equilibrium. The concept of equilibrium in the context of transport systems had appeared in the 1950s ([2, 19]) and is based on the general assumption that network users are making adjustments to their travel choices until a state of equilibrium is reached, i.e. when no individual can make a further improvement to their utility function as a result of any individual choice (thus making an alternative path choice will not lead to a reduction in individual's travel time). A specific situation investigated in this work is the effect of introducing a new link to a congested traffic network, and the likelihood of this additional capacity to improve the system's performance (measured by users' aggregated cost or travel time). The addition of a link to an existing transport system may lead to undesired situations.

The well-known Braess' paradox [3] illustrates situations when adding a new link to a transport network might not reduce congestion in the network but instead increase it. This is due to individual entities acting selfishly/separately when making their travel plan choices and hence forcing the system as a whole not to operate optimally. Deeper insight into this paradox from

the viewpoint of the structure and characteristics of networks may help transport planners to avoid the occurrence of Braess-like situations in real-life networks. Nagurney [11] proved that Braess' paradox disappears under higher demands, while in [9] it was shown how to avoid Braess' paradox by adding resources efficiently to a network. Braess' paradox can be observed in other applied contexts such as telecommunication networks and power transmission networks, and it has been studied for an evolutionary variational inequality model of the Internet [12]. Note that Sheffi [16] coined the term 'pseudo-paradoxes' to describe Braess' paradox and similar traffic flow phenomena.

Pas and Principio [13] investigated the existence of Braess' paradox in his classical network configuration, where travel times of links are specified in such a way that the resulting network is symmetrical. Their result describes the situations when Braess' paradox occurs in the above-mentioned network but is only limited to a specific type of networks in which the attributes associated with travel times of some links are symmetrical and there are further restrictions on free flow travel times and delay parameters (see Section 5). In the context of volume-delay functions and their parameters, it can be argued that symmetry properties of networks, commonly presented in analyses of the paradox are not very common in real-life situations. Although many previous studies have investigated Braess' paradox and various aspects of traffic equilibrium in asymmetric networks [7, 10, 14, 17, 20], the generalisation of Pas and Principio's results [13] devoted to the existence of Braess' paradox has not been previously studied.

The above-mentioned Pas-Principio's result is formulated in Corollary 1 (see Section 5) for the symmetric network configuration  $M/M^+$  of Figure 3. Note that free flow travel times for some links in  $M/M^+$  are assumed to be equal to zero. Another assumption is that the delay parameter for the link  $(b, c)$  is not arbitrary, it is equal to the delay parameter of the link  $(b, d)$ . In Corollary 2 we do not make those assumptions and thus generalise Pas-Principio's result. Indeed, in Corollary 2 there are six arbitrary parameters for travel time functions  $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ , while Corollary 1 only contains four such parameters  $(\alpha_2, \alpha_3, \beta_1, \beta_2)$ . These results are devoted to symmetric networks, where travel times of some links are same. A further generalisation of Pas-Principio's result is given in Theorems 1–4, where there are no such restrictions, i.e. the travel times of the links in the network are arbitrary linear functions without symmetric/asymmetric properties.

Valiant and Roughgarden [17] proved that Braess' paradox is likely to occur in a natural random network model. More precisely, for a given appropriate total flow, they showed that in almost all networks there is a set of links whose removal improves the travel time at equilibrium, i.e. Braess' paradox is widespread in this context. Thus, the fundamental significance of Valiant-Roughgarden's result is that the presence of Braess' paradox is not rare and exceptional, but rather widespread. Therefore, real-life networks must be analysed from the viewpoint of this phenomenon before adding/constructing a new link/road. Moreover, Valiant and Roughgarden [17] showed that 'the "global" behaviour of an equilibrium flow in a large random network is similar to that in Braess' original four-node example'. This important result means that the classical network configuration introduced by Braess in 1968 is fundamental and it should be analysed in depth.

In this paper, a natural generalisation of Braess' network is introduced in Section 2. Such a generalised network can be reduced to the classical network configuration  $N/N^+$ , where  $N^+$  is the network of Figure 1 and the network  $N$  is  $N^+$  with the link  $(b, c)$  removed. However, in contrast to the classical case with symmetrical travel times of links, the network configuration  $N/N^+$  has arbitrary linear travel time functions. Note that the conditions for Braess' paradox to occur in the generalised network and the corresponding reduced network are the same. The travel times at equilibria in the networks  $N$  and  $N^+$  are completely described in Section 3, while Section 4 provides necessary and sufficient conditions for the occurrence of Braess' paradox in these networks. In Section 5, Pas-Principio's result describing the existence of Braess' paradox in the symmetrical network configuration is obtained and generalised. The motivation for such

an extension was given by Pas and Principio in their conclusions in [13]. Also, the concept of a pseudo-paradox is introduced and the conditions for its occurrence in  $N/N^+$  are given. In particular, it is shown that in an asymmetrical network configuration the pseudo-paradox is happening for any total flow. A numerical example is analysed in Section 6, and conclusions are presented in Section 7. The proofs of all the lemmas and theorems are given in Appendices A and B, respectively.

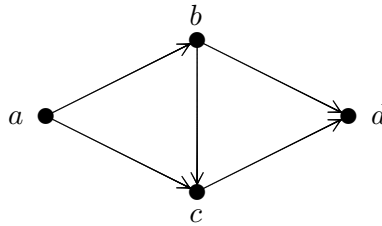
## 2 A Generalisation of Braess' Network Configuration and Notation

The classical network configuration introduced by Braess [3] consists of three paths:

$$P_1 = a - b - d, \quad P_2 = a - c - d, \quad P_3 = a - b - c - d.$$

This network is denoted by  $N^+$ , and it has four nodes and five links, where  $a$  is the origin of all travel demand, and  $d$  is the destination of all demand (see Figure 1). The network  $N$  is  $N^+$  with the link  $(b, c)$  removed.

In 2006 Valiant and Roughgarden [17] proved an important and interesting result that Braess' paradox is likely to happen in a natural random network model. Also, they showed that 'the "global" behaviour of an equilibrium flow in a large random network is similar to that in Braess' original four-node example'. Thus, Braess' network configuration is of fundamental significance.



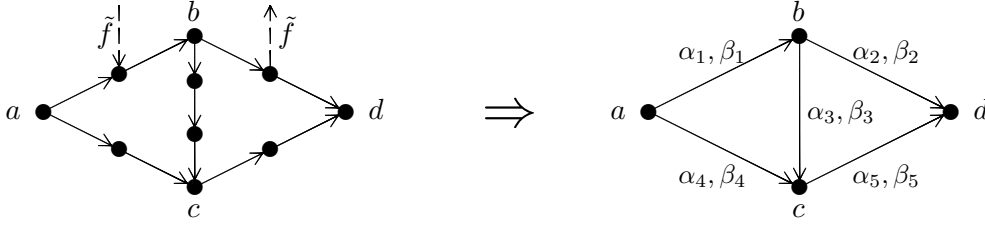
**Figure 1.** Braess' Network Configuration.

Let us consider a natural generalisation of Braess' network, where every link of the network in Figure 1 is replaced by a path of arbitrary length (i.e. a route with any number of links). Thus, the generalised network 'comprises' five paths of arbitrary length:  $(a, b)$ -path,  $(b, d)$ -path,  $(a, c)$ -path,  $(c, d)$ -path and  $(b, c)$ -path. This is illustrated by the first network in Figure 2, where the  $(b, c)$ -path is of length 3 and other four paths have length 2. Every link  $(i, j)$  in the resulting network has a linear travel time function

$$\alpha_{ij} + \beta_{ij} f_{ij},$$

where  $\alpha_{ij} \geq 0$  is the free flow travel time for the link  $(i, j)$ ,  $\beta_{ij} > 0$  is the delay parameter for  $(i, j)$ , and  $f_{ij} \geq 0$  is the flow on the link  $(i, j)$ . A fixed traffic coming from outside the network is allowed. For example, in Figure 2 the dashed arrows represent a fixed traffic  $\tilde{f}$  coming to the network and then going outside.

The assumption of a linear relationship between traffic volume on a link and the travel time on it (so-called 'volume-delay function') is common in the context of Braess' paradox. Although there is evidence to support such a linear approximation[18], many different types of volume-delay functions, e.g. BPR functions [5], have been applied (see [4] for a review article). However, the investigation of the non-linear case is not in the scope of this work.



**Figure 2.** A generalised network reduced to a four-node network  $N^+$ .

Suppose we want to decide whether Braess' paradox occurs when removing a link on the path going from  $b$  to  $c$ . If a particular link  $(i, j)$  has a fixed flow  $\tilde{f}$  coming from outside the network, i.e. not the internal flow  $f$  going from  $a$  to  $d$  through this link, then its travel time function can be written as follows:

$$\alpha_{ij} + \beta_{ij}f_{ij} = \alpha_{ij} + \beta_{ij}(f + \tilde{f}) = (\alpha_{ij} + \beta_{ij}\tilde{f}) + \beta_{ij}f = \tilde{\alpha}_{ij} + \beta_{ij}f.$$

The updated function depends on the internal flow and it is linear, because the external flow  $\tilde{f}$  is fixed and hence  $\tilde{\alpha}_{ij}$  is a fixed number. Thus, the first step is to update all travel time functions taking into accounts external flows. Further, it is easy to see that the total travel time functions for the above paths are linear functions, since the travel time functions for links are linear and all the links belonging to one of the paths share the same internal flow. For instance, if such a path  $P$  with an internal flow  $f$  consists of two links  $(i, j)$  and  $(j, k)$ , then the total travel time function is as follows:

$$\alpha_{ij} + \beta_{ij}f + \alpha_{jk} + \beta_{jk}f = \alpha_{ij} + \alpha_{jk} + (\beta_{ij} + \beta_{jk})f = \alpha_P + \beta_P f.$$

Thus, if all the above paths are replaced by single links, then we obtain Braess' network with arbitrary linear travel time functions (see Figure 2):

$$\begin{aligned} \alpha_1 + \beta_1 f_{ab} & \text{ for link } (a, b); \\ \alpha_2 + \beta_2 f_{bd} & \text{ for link } (b, d); \\ \alpha_3 + \beta_3 f_{bc} & \text{ for link } (b, c); \\ \alpha_4 + \beta_4 f_{ac} & \text{ for link } (a, c); \\ \alpha_5 + \beta_5 f_{cd} & \text{ for link } (c, d). \end{aligned}$$

Note that in Braess' original example [3] and in many of the studies that followed it (e.g. [6]), the network is symmetric, i.e the time functions for the links  $(a, b)$  and  $(c, d)$  are the same as well as the time functions for the links  $(b, d)$  and  $(a, c)$ , and the free flow travel times for the links  $(a, b)$  and  $(c, d)$  are equal to zero. The occurrence of Braess' paradox in this symmetrical network configuration was described by Pas and Principio [13] (see Corollary 1).

In this paper, we consider a more general situation with arbitrary linear time functions. The existence of Braess' paradox in such a network can be decided by using Theorems 1–4, where the network  $N$  is  $N^+$  with the link  $(b, c)$  removed. Note that the conditions for Braess' paradox to occur in the above generalised network and the corresponding reduced network are the same.

Let  $Q > 0$  denote the total flow in  $N/N^+$ , i.e  $Q = f_{ab} + f_{ac} = f_{bd} + f_{cd}$ . Note that  $f_{ij}$  and  $Q$  are not necessarily integer numbers. Let us denote

$$\alpha_{ij} = \alpha_i + \alpha_j,$$

e.g.  $\alpha_{12}$  means  $\alpha_1 + \alpha_2$ , and  $\beta_{ij}$  is defined similarly. Also,

$$\alpha = \alpha_{45} - \alpha_{12}, \quad \bar{\alpha} = \alpha_4 - \alpha_{13}, \quad \hat{\alpha} = \alpha_2 - \alpha_{35},$$

and

$$\beta = \beta_{1245} = \beta_1 + \beta_2 + \beta_4 + \beta_5, \quad \beta_{ijk} = \beta_i + \beta_j + \beta_k.$$

The following equality will be used throughout the paper:

$$\alpha = \bar{\alpha} - \hat{\alpha}.$$

Further, we introduce the Braess numbers  $\mathcal{B}_i$  for  $i = 1, 2, 3, 4$ :

$$\mathcal{B}_1 = \beta_1\beta_5 - \beta_2\beta_4, \quad \mathcal{B}_2 = \beta_{135}\beta - \beta_{12}\beta_{45}, \quad \mathcal{B}_3 = \beta_{45}^2\beta_{134} - \beta_4^2\beta \quad \text{and} \quad \mathcal{B}_4 = \beta_{12}^2\beta_{235} - \beta_2^2\beta.$$

Also, two parameters  $\mu_1$  and  $\mu_2$  are defined as follows:

$$\mu_1 = \frac{\hat{\alpha}\beta_{14} - \alpha\beta_3}{\beta_3\beta_{45} + \beta_5\beta_{14}}, \quad \mu_2 = \frac{\bar{\alpha}\beta_{25} + \alpha\beta_3}{\beta_1\beta_{25} + \beta_3\beta_{12}}.$$

### 3 Equilibria in $N$ and $N^+$

It is well known that a user equilibrium always exists, and in a network without capacities, it is essentially unique (e.g. see [15]). A path  $P$  from the origin to the destination is said to have a *vanishing flow* if  $P$  carries no traffic from the origin to the destination. Note that some links in the path  $P$  may have a non-zero flow that contributes to the traffic of other paths. A path has a *non-vanishing flow* if it carries some traffic from the origin to the destination.

A network with one origin and one destination is said to be at equilibrium if

- (a) The travel time on paths with non-vanishing flow is the same, and it is denoted by  $T_{Eq}$ , and
- (b) The travel time on paths with no flow is at least  $T_{Eq}$ .

The equilibrium described above is associated with aggregated strategic behaviour of all road users, described as an  $N$ -person Nash equilibrium. The concept of Nash equilibrium is related to a Wardrop equilibrium. In fact, the Nash equilibrium in a network game with a finite number of players converges to a Wardrop equilibrium when the number of players increases [8]. In equilibrium, no user can decrease their route travel time by unilaterally switching routes [19]. In other words, if a network is not at equilibrium, then some users of the network (e.g. drivers) can switch their routes in order to improve their travel time. However, if a driver decides to switch to a better route, then the travel time for this route increases, and, after a certain period of time, it will become impossible to improve drivers' travel times by switching the routes. Thus, the equilibrium describes 'stable state' behaviour in a network, and no driver has any incentive to switch routes at equilibrium because it will not improve their current travel times.

The following lemma describes the equilibrium in the network  $N$ , which is  $N^+$  with the link  $(b, c)$  removed. Note that in Lemma 1 the case (a) corresponds to the situation when the path  $P_1$  has a vanishing flow and  $P_2$  has a non-vanishing flow in  $N$ . In case (b) the path  $P_1$  has a non-vanishing flow and  $P_2$  has a vanishing flow, and in case (c) no path has a vanishing flow. Also, the cases (a) and (b) in this lemma are mutually exclusive because one of the numbers  $-\alpha/\beta_{45}$  and  $\alpha/\beta_{12}$  is negative, or they both are equal to zero. The proof of Lemma 1 is given in Appendix A.

**Lemma 1** *In the network  $N$ , the travel time at equilibrium is as follows:*

- (a)  $T_{Eq} = \alpha_{45} + Q\beta_{45}$  if  $0 < Q \leq -\alpha/\beta_{45}$ ;
- (b)  $T_{Eq} = \alpha_{12} + Q\beta_{12}$  if  $0 < Q \leq \alpha/\beta_{12}$ ;

$$(c) \ T_{Eq} = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta \quad \text{if } Q > \max\{\alpha/\beta_{12}; -\alpha/\beta_{45}\}.$$

The equilibrium in  $N^+$  is described by seven cases in Lemma 2. It may be pointed out that these cases correspond to the following situations in  $N^+$ : (a) the only path with non-vanishing flow is  $P_3$ ; (b) the only path with non-vanishing flow is  $P_2$ ; (c) the only path with non-vanishing flow is  $P_1$ ; (d) the only path with vanishing flow is  $P_1$ ; (e) the only path with vanishing flow is  $P_2$ ; (f) the only path with vanishing flow is  $P_3$ ; (g) no path has a vanishing flow.

Also, it is not difficult to see that some of the cases in Lemma 2 are mutually exclusive, so the equilibrium in a particular network  $N^+$  is described by some of the presented seven cases. For example, if  $\alpha_i = \beta_i = 1$  for  $1 \leq i \leq 5$ , then the equilibrium is given by just one case (f).

**Lemma 2** *In the network  $N^+$ , the travel time at equilibrium is as follows:*

- (a)  $T_{Eq}^+ = \alpha_{135} + Q\beta_{135} \quad \text{if } 0 < Q \leq \min\{\hat{\alpha}/\beta_{35}; \bar{\alpha}/\beta_{13}\};$
- (b)  $T_{Eq}^+ = \alpha_{45} + Q\beta_{45} \quad \text{if } 0 < Q \leq \min\{-\alpha/\beta_{45}; -\bar{\alpha}/\beta_4\};$
- (c)  $T_{Eq}^+ = \alpha_{12} + Q\beta_{12} \quad \text{if } 0 < Q \leq \min\{\alpha/\beta_{12}; -\hat{\alpha}/\beta_2\};$
- (d)  $T_{Eq}^+ = \alpha_{45} + Q\beta_{45} - (\bar{\alpha} + Q\beta_4)\beta_4/\beta_{134} \quad \text{if } \max\{\bar{\alpha}/\beta_{13}; -\bar{\alpha}/\beta_4\} < Q \leq \mu_1;$
- (e)  $T_{Eq}^+ = \alpha_{12} + Q\beta_{12} - (\hat{\alpha} + Q\beta_2)\beta_2/\beta_{235} \quad \text{if } \max\{\hat{\alpha}/\beta_{35}; -\hat{\alpha}/\beta_2\} < Q \leq \mu_2;$
- (f)  $T_{Eq}^+ = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta \quad \text{if } Q > \max\{\alpha/\beta_{12}; -\alpha/\beta_{45}\} \quad \text{and}$

$$\mathcal{B}_1 \geq \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{Q};$$

$$(g) \ T_{Eq}^+ = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta + g\mathcal{B}_1/\beta, \quad \text{where}$$

$$g = \frac{\bar{\alpha}\beta - \alpha\beta_{14} - Q\mathcal{B}_1}{\beta_3\beta + \beta_{14}\beta_{25}},$$

$$\text{if } Q > \max\{\mu_1; \mu_2\} \quad \text{and}$$

$$\mathcal{B}_1 < \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{Q}.$$

The proof of Lemma 2 is given in Appendix A.

## 4 Braess' Paradox in $N$ and $N^+$

Braess' paradox is said to occur in the network configuration  $N/N^+$  for a given total flow  $Q$  if

$$T_{Eq}^+ > T_{Eq},$$

where  $T_{Eq}$  and  $T_{Eq}^+$  are travel times at equilibria in  $N$  and  $N^+$ , respectively. Thus, Braess' paradox describes a situation when adding a new link to a network makes a general performance worse.

The following mega-theorem describes all possible situations when Braess' paradox may occur in  $N/N^+$  in terms of their paths. In fact, it says that Braess' paradox may occur in  $N/N^+$  only if, at equilibria, both  $P_1$  and  $P_2$  have a non-vanishing flow in  $N$ , and  $P_3$  has a non-vanishing flow in  $N^+$ .

**Mega-Theorem.** *Braess' paradox may occur in  $N/N^+$  in the following cases only:*

- (a) *At equilibria, both  $N$  and  $N^+$  have no paths with vanishing flow.*
- (b) *At equilibria,  $N$  has no path with vanishing flow, and  $P_3$  is the only path with non-vanishing flow in  $N^+$ .*
- (c) *At equilibria,  $N$  has no path with vanishing flow, and  $P_1$  is the only path with vanishing flow in  $N^+$ .*
- (d) *At equilibria,  $N$  has no path with vanishing flow, and  $P_2$  is the only path with vanishing flow in  $N^+$ .*

In the following theorems, we formulate the necessary and sufficient conditions for the existence of the paradox for all the cases of Mega-Theorem. These theorems are proved in Appendix B.

**Theorem 1** *Suppose that at equilibria both  $N$  and  $N^+$  have no paths with vanishing flow. Then Braess' paradox occurs in  $N/N^+$  if and only if the Braess number  $\mathcal{B}_1$  is positive and*

$$\max \left\{ \frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \mu_1; \mu_2 \right\} < Q < \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{\mathcal{B}_1}.$$

**Theorem 2** *Suppose that at equilibria  $N$  has no path with vanishing flow and  $P_3$  is the only path with non-vanishing flow in  $N^+$ . Then Braess' paradox occurs in  $N/N^+$  if and only if the Braess number  $\mathcal{B}_2$  is positive and*

$$\max \left\{ \frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\hat{\alpha}\beta_{45} + \bar{\alpha}\beta_{12}}{\mathcal{B}_2} \right\} < Q \leq \min \left\{ \frac{\hat{\alpha}}{\beta_{35}}; \frac{\bar{\alpha}}{\beta_{13}} \right\}.$$

**Theorem 3** *Suppose that at equilibria  $N$  has no path with vanishing flow and  $P_1$  is the only path with vanishing flow in  $N^+$ . Then Braess' paradox occurs in  $N/N^+$  if and only if the Braess number  $\mathcal{B}_3$  is positive and*

$$\max \left\{ \frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\bar{\alpha}}{\beta_{13}}; \frac{-\bar{\alpha}}{\beta_4}; \frac{\bar{\alpha}\beta_4\beta - \alpha\beta_{134}\beta_{45}}{\mathcal{B}_3} \right\} < Q \leq \mu_1.$$

**Theorem 4** *Suppose that at equilibria  $N$  has no path with vanishing flow and  $P_2$  is the only path with vanishing flow in  $N^+$ . Then Braess' paradox occurs in  $N/N^+$  if and only if the Braess number  $\mathcal{B}_4$  is positive and*

$$\max \left\{ \frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\hat{\alpha}}{\beta_{35}}; \frac{-\hat{\alpha}}{\beta_2}; \frac{\hat{\alpha}\beta_2\beta + \alpha\beta_{235}\beta_{12}}{\mathcal{B}_4} \right\} < Q \leq \mu_2.$$

It might be pointed out that if  $\mathcal{B}_1 \geq 0$ , then  $\mathcal{B}_2$ ,  $\mathcal{B}_3$  and  $\mathcal{B}_4$  are positive numbers, because

$$\mathcal{B}_2 = \beta_{12}\beta_{13} + \beta_{35}\beta_{45} + \mathcal{B}_1,$$

$$\mathcal{B}_3 = \beta_5^2\beta_{134} + \beta_4(\beta_3\beta_{455} + \beta_5\beta_{14} + \mathcal{B}_1),$$

$$\mathcal{B}_4 = \beta_1^2\beta_{235} + \beta_2(\beta_1\beta_{335} + \beta_2\beta_{13} + \mathcal{B}_1).$$

Moreover, Theorems 3 and 4 are mutually exclusive in the sense that they cannot provide intervals for  $Q$  simultaneously. This is true because the inequalities  $\bar{\alpha}/\beta_{13} < \mu_1$  and  $\hat{\alpha}/\beta_{35} < \mu_2$  are inconsistent. Note also that if, for example, Theorems 1–3 provide non-empty intervals for  $Q$ , then the interval with highest values of  $Q$  is given by Theorem 1, the interval with smallest values of  $Q$  is provided by Theorem 2, and Theorem 3 yields the interval with mid-range values of  $Q$ .

Note that the original assumption  $\beta_i > 0$  for all  $i$  can be relaxed by allowing  $\beta_i = 0$  for some  $i$ . This can be done by introducing  $+\infty$  and  $-\infty$  when a non-zero number is divided by zero. For example, let us consider Arnott-Small's example [1]:

$$\alpha_1 = \alpha_5 = 0, \quad \alpha_2 = \alpha_4 = 15, \quad \alpha_3 = 7.5, \quad \beta_1 = \beta_5 = 0.01, \quad \beta_2 = \beta_3 = \beta_4 = 0.$$

Using the formulas in Section 2, we obtain

$$\alpha = 0, \quad \bar{\alpha} = \hat{\alpha} = 7.5, \quad \beta = 0.02, \quad \mu_1 = \mu_2 = 750.$$

Now let us apply Theorems 1–4 to Arnott-Small's example, as shown in the following table:

Theorem	Braess Number	Lower Bound for $Q$	Upper Bound for $Q$	Interval for $Q$
1	$\mathcal{B}_1 = 10^{-4}$	$\max\{0; 0; 750; 750\} = 750$	1500	$]750; 1500[$
2	$\mathcal{B}_2 = 3 \times 10^{-4}$	$\max\{0; 0; 500\} = 500$	$\min\{750; 750\} = 750$	$]500; 750]$
3	$\mathcal{B}_3 = 10^{-6}$	$\max\{0; 0; 750; -7.5/0; 0\} = 750$	750	$\emptyset$
4	$\mathcal{B}_4 = 10^{-6}$	$\max\{0; 0; 750; -7.5/0; 0\} = 750$	750	$\emptyset$

Table 1: Practical application of Theorems 1–4 to Arnott-Small's example.

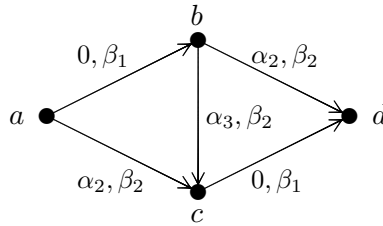
Notice that the notation  $]500; 750]$  in the last column of the table means an interval of real numbers from 500 (excluded) to 750 (included). Thus, by Theorems 1 and 2, Braess' paradox occurs if  $500 < Q < 1500$ , while Theorems 3 and 4 provide no intervals for  $Q$ . In calculating the lower bounds for  $Q$  in these theorems we have division by zero, but this problem is overcome by putting  $-7.5/0 = -\infty$ .

## 5 Symmetrical/Asymmetrical Networks and the Pseudo-Paradox

Let us consider the classical case of a symmetrical network presented by Braess [3] and discussed in Pas and Principio [13] and other papers. Using the notation introduced in the previous sections, it can be seen as a particular case of the network configuration  $N/N^+$  where time functions are symmetrical for links that do not share nodes with each other ( $(a, b)$  and  $(c, d)$ ,  $(a, c)$  and  $(b, d)$ ), the free flow travel times for the links  $(a, b)$  and  $(c, d)$  are equal to zero, and the delay parameter for  $(b, c)$  is equal to the delay parameter of the links  $(b, d)$  and  $(a, c)$ , i.e.

$$\alpha_1 = \alpha_5 = 0, \quad \alpha_2 = \alpha_4, \quad \beta_1 = \beta_5, \quad \beta_2 = \beta_3 = \beta_4.$$

Also, it is assumed that  $\alpha_2 > \alpha_3$  and  $\beta_1 > \beta_2$ . This network configuration is denoted by  $M/M^+$  (see Figure 3).



**Figure 3.** The symmetric network  $M^+$  ( $\alpha_2 > \alpha_3$  and  $\beta_1 > \beta_2$ ).

Pas and Principio [13] determined the occurrence of Braess' paradox in the symmetrical network configuration  $M/M^+$ . This result follows directly from Theorems 1–4:

**Corollary 1 ([13])** *Braess' paradox occurs in  $M/M^+$  if and only if*

$$\frac{2(\alpha_2 - \alpha_3)}{3\beta_1 + \beta_2} < Q < \frac{2(\alpha_2 - \alpha_3)}{\beta_1 - \beta_2}.$$



**Proof:** For the network configuration  $M/M^+$ , we have

$$\alpha = 0, \quad \bar{\alpha} = \hat{\alpha} = \alpha_2 - \alpha_3, \quad \mu_1 = \mu_2 = (\alpha_2 - \alpha_3)/\beta_{12}, \quad \mathcal{B}_1 = \beta_1^2 - \beta_2^2.$$

The Braess number  $\mathcal{B}_1$  is positive because  $\beta_1 > \beta_2$  in  $M/M^+$ . Under the conditions of Theorem 1, Braess' paradox occurs if

$$\frac{\alpha_2 - \alpha_3}{\beta_{12}} < Q < \frac{2\beta_{12}(\alpha_2 - \alpha_3)}{\beta_1^2 - \beta_2^2} = \frac{2(\alpha_2 - \alpha_3)}{\beta_1 - \beta_2}.$$

Now,

$$\mathcal{B}_2 = \beta_1^2 + 2\beta_{12}^2 - \beta_2^2 = \beta_{12}(3\beta_1 + \beta_2),$$

which is a positive number. Therefore, by Theorem 2,

$$\frac{2(\alpha_2 - \alpha_3)}{3\beta_1 + \beta_2} < Q \leq \frac{\alpha_2 - \alpha_3}{\beta_{12}}.$$

Note that the lower bound is less than the upper bound because  $\beta_1 > \beta_2$ . Thus, the above inequalities can be written together as

$$\frac{2(\alpha_2 - \alpha_3)}{3\beta_1 + \beta_2} < Q < \frac{2(\alpha_2 - \alpha_3)}{\beta_1 - \beta_2}.$$

The upper and lower bounds of Theorems 3 and 4 provide no intervals for  $Q$ . ■

Now let  $\tilde{M}/\tilde{M}^+$  denote the above network configuration  $M/M^+$  without the assumption that  $\alpha_2 > \alpha_3$  and  $\beta_1 > \beta_2$ . A proof similar to that of Corollary 1 shows that Braess' paradox occurs in  $\tilde{M}/\tilde{M}^+$  in the following cases only:

(a) If  $\beta_1 > \beta_2$  and

$$\frac{\alpha_2 - \alpha_3}{\beta_{12}} < Q < \frac{2(\alpha_2 - \alpha_3)}{\beta_1 - \beta_2};$$

(b) If

$$\frac{2(\alpha_2 - \alpha_3)}{3\beta_1 + \beta_2} < Q \leq \frac{\alpha_2 - \alpha_3}{\beta_{12}}.$$

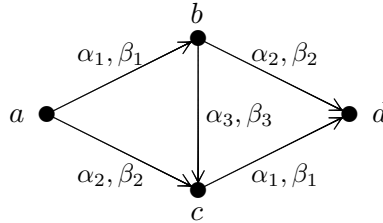
The both cases imply that  $\alpha_2 > \alpha_3$ . Another implicit relationship is obtained from (b) if we require that

$$\frac{2(\alpha_2 - \alpha_3)}{3\beta_1 + \beta_2} < \frac{\alpha_2 - \alpha_3}{\beta_{12}},$$

which is equivalent to  $\beta_1 > \beta_2$ . Thus, even though the network configuration  $\tilde{M}/\tilde{M}^+$  extends  $M/M^+$ , the conditions for the occurrence of Braess' paradox are the same.

Let us further extend the network configuration  $\tilde{M}/\tilde{M}^+$  by allowing any non-negative free flow travel time  $\alpha_1$  for the links  $(a, b)$  and  $(c, d)$  and any positive delay parameter  $\beta_3$  for the link  $(b, c)$ . In other words, the symmetrical network configuration  $S/S^+$  of Figure 4 is obtained from  $N/N^+$  when time functions are symmetrical for links that do not share nodes with each other:

$$\alpha_1 = \alpha_5, \quad \alpha_2 = \alpha_4, \quad \beta_1 = \beta_5, \quad \beta_2 = \beta_4 \quad (\text{see Figure 4}).$$



**Figure 4.** The symmetric network  $S^+$ .

**Corollary 2** *Braess' paradox occurs in the symmetrical network configuration  $S/S^+$  if and only if  $\beta_1 > \beta_2$  and*

$$\frac{2(\alpha_2 - \alpha_{13})}{3\beta_1 + 2\beta_3 - \beta_2} < Q < \frac{2(\alpha_2 - \alpha_{13})}{\beta_1 - \beta_2}. \quad (1)$$

**Proof:** For the network configuration  $S/S^+$ , we have

$$\alpha = 0, \quad \bar{\alpha} = \hat{\alpha} = \alpha_2 - \alpha_{13}, \quad \mu_1 = \mu_2 = (\alpha_2 - \alpha_{13})/\beta_{13}, \quad \mathcal{B}_1 = \beta_1^2 - \beta_2^2 = \beta_{12}(\beta_1 - \beta_2).$$

By Theorem 1, Braess' paradox occurs if  $\mathcal{B}_1$  is positive, i.e.  $\beta_1 > \beta_2$ , and

$$\frac{\alpha_2 - \alpha_{13}}{\beta_{13}} < Q < \frac{2(\alpha_2 - \alpha_{13})}{\beta_1 - \beta_2}.$$

Now,

$$\mathcal{B}_2 = \beta_{131}2\beta_{12} - \beta_{12}^2 = \beta_{12}(3\beta_1 + 2\beta_3 - \beta_2).$$

Therefore, by Theorem 2, Braess' paradox occurs if  $3\beta_1 + 2\beta_3 > \beta_2$  and

$$\frac{2(\alpha_2 - \alpha_{13})}{3\beta_1 + 2\beta_3 - \beta_2} < Q \leq \frac{\alpha_2 - \alpha_{13}}{\beta_{13}}.$$

This implies that  $\alpha_2 > \alpha_{13}$ . Also, it is easy to see that the lower bound is less than the upper bound only if  $\beta_1 > \beta_2$ , which is stronger than  $3\beta_1 + 2\beta_3 > \beta_2$ . Thus, the above inequalities can be written together as

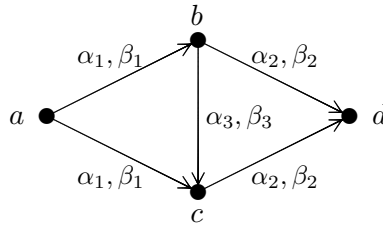
$$\frac{2(\alpha_2 - \alpha_{13})}{3\beta_1 + 2\beta_3 - \beta_2} < Q < \frac{2(\alpha_2 - \alpha_{13})}{\beta_1 - \beta_2}.$$

The upper and lower bounds of Theorems 3 and 4 provide no intervals for  $Q$ . ■

In Corollary 2 there is an implicit assumption that  $\alpha_2 > \alpha_{13}$  because  $Q$  is a positive number (if  $\alpha_2 \leq \alpha_{13}$ , then (1) provides no interval for  $Q$ ). We will see in Corollary 6 what is happening with the times at equilibria in  $S/S^+$  if  $Q$  exceeds the upper bound in (1), where  $\alpha_2 > \alpha_{13}$  and  $\beta_1 > \beta_2$ .

The asymmetrical network configuration  $A/A^+$  of Figure 5 is obtained from  $N/N^+$  when the time functions for the links  $(a, b)$  and  $(a, c)$  are the same as well as the time functions for the links  $(b, d)$  and  $(c, d)$ , i.e.

$$\alpha_1 = \alpha_4, \quad \alpha_2 = \alpha_5, \quad \beta_1 = \beta_4, \quad \beta_2 = \beta_5 \quad (\text{see Figure 5}).$$



**Figure 5.** The asymmetric network  $A^+$ .

**Corollary 3** *Braess' paradox cannot occur in the asymmetrical network configuration  $A/A^+$ .*

**Proof:** For the network configuration  $A/A^+$ , we have

$$\alpha = 0, \quad \bar{\alpha} = \hat{\alpha} = -\alpha_3, \quad \mathcal{B}_1 = \beta_1\beta_2 - \beta_2\beta_1 = 0, \quad \mu_1 \leq 0, \quad \mu_2 \leq 0.$$

It is easy to see that Theorems 1–4 provide no intervals, so Braess' paradox is impossible.  $\blacksquare$

Using Lemmas 1 and 2 we see that the equilibria in  $A/A^+$  are described by the cases (c) and (f), respectively, i.e. in  $A$  no path has a vanishing flow, and in  $A^+$  the only path with vanishing flow is  $P_3$ . Also, the flow  $Q$  is distributed evenly between  $P_1$  and  $P_2$  and

$$T_{Eq} = T_{Eq}^+ = \alpha_{12} + 0.5\beta_{12}Q.$$

Thus, the travel times at equilibria in  $A$  and  $A^+$  are equal for any  $Q$ . This observation is important because adding a new link to  $A$  does not improve the general performance, even though Braess' paradox is not occurring.

We say that the pseudo-paradox occurs in the network configuration  $N/N^+$  if

$$T_{Eq}^+ = T_{Eq}$$

for an interval of values for  $Q$  (as opposite to a single point). In other words, we exclude single values of  $Q$  when going from the situation “Braess' paradox does not occur” to “Braess' paradox occurs”.

Thus, the pseudo-paradox describes a situation when adding a new link to a network does not change the general performance for a range of values of the total flow. As seen above, the pseudo-paradox occurs in the asymmetrical network configuration  $A/A^+$  for any  $Q > 0$ . In the authors' view, the pseudo-paradox is a more common phenomenon than Braess' paradox.

**Corollary 4** *For the network configuration  $N/N^+$ , the pseudo-paradox occurs if*

- (a)  $0 < Q \leq \min\{-\alpha/\beta_{45}; -\bar{\alpha}/\beta_4\};$
- (b)  $0 < Q \leq \min\{\alpha/\beta_{12}; -\hat{\alpha}/\beta_2\};$
- (c)  $Q > \max\{\alpha/\beta_{12}; -\alpha/\beta_{45}\} \quad \text{and} \quad Q\mathcal{B}_1 \geq \hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25};$
- (d)  $\mathcal{B}_1 = 0, \quad \hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25} > 0 \quad \text{and} \quad Q > \max\{\alpha/\beta_{12}; -\alpha/\beta_{45}; \mu_1; \mu_2\}.$

**Proof:** The first three cases follow immediately from the cases (a,b), (b,c), (c,f) of the proof of theorems as discussed in the first paragraph of Appendix B. The last case is similar to the case (c,g), where  $T_{Eq}^+ = T_{Eq}$  must be satisfied. This is equivalent to  $g\mathcal{B}_1/\beta = 0$ . Since  $g > 0$ , we obtain  $\mathcal{B}_1 = 0$ , and therefore  $\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25} > 0$ . In addition, there are lower bounds for  $Q$  in Lemma 1 (c) and Lemma 2 (g).  $\blacksquare$

The application of Corollary 4 (c) to the asymmetrical network configuration  $A/A^+$  confirms the above observation:

**Corollary 5** *The pseudo-paradox occurs in the asymmetrical network configuration  $A/A^+$  for any  $Q > 0$ .*

By Corollary 2, Braess' paradox occurs in the symmetrical network configuration if  $\beta_1 > \beta_2$ ,  $\alpha_2 > \alpha_{13}$ , and the total flow  $Q$  is between the lower and upper bounds in (1). Corollary 4 (c) allows us to see what is happening with the times at equilibria if  $Q$  exceeds the upper bound:

**Corollary 6** *Suppose that Braess' paradox occurs in the symmetrical network configuration  $S/S^+$ , i.e.  $\beta_1 > \beta_2$  and  $\alpha_2 > \alpha_{13}$ . Then  $S/S^+$  is experiencing the pseudo-paradox for any*

$$Q \geq \frac{2(\alpha_2 - \alpha_{13})}{\beta_1 - \beta_2}.$$

**Proof:** We know that  $\alpha = 0$ ,  $\bar{\alpha} = \hat{\alpha} = \alpha_2 - \alpha_{13}$  and  $\mathcal{B}_1 = \beta_{12}(\beta_1 - \beta_2) > 0$ . Therefore, the second inequality in Corollary 4 (c) is equivalent to

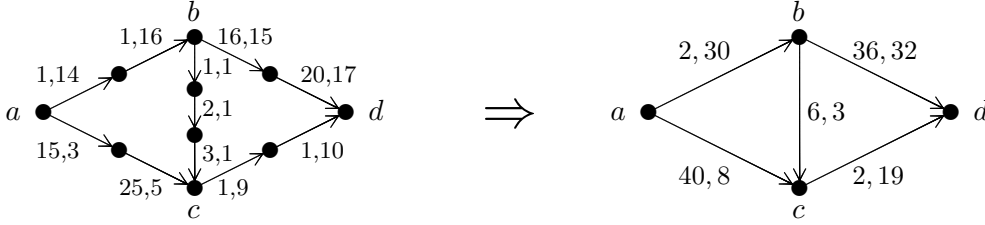
$$Q\beta_{12}(\beta_1 - \beta_2) \geq 2(\alpha_2 - \alpha_{13})\beta_{12},$$

as required. ■

Thus, under the conditions of Corollary 6, some improvement in  $S/S^+$  is only possible if  $Q$  is less than the lower bound in (1), followed by Braess' paradox until  $Q$  reaches the upper bound in (1), followed by the pseudo-paradox for larger values of  $Q$ .

## 6 Numerical Example

Let us consider the generalised network  $G^+$  shown in the left part of Figure 6, where the free flow travel times and delay parameters are indicated for all eleven links. As explained in Section 2, this network can be reduced to the network  $N^+$  shown in the right part of Figure 6. Now suppose we want to decide whether Braess' paradox occurs in  $G^+$  when removing any link on the path going from  $b$  to  $c$ ; let us denote the resulting network by  $G$ . This is equivalent to finding the conditions for Braess' paradox to occur in the network configuration  $N/N^+$ .



**Figure 6.** A generalised network  $G^+$  reduced to a four-node network  $N^+$ .

Thus, the network configuration  $N/N^+$  has the following parameters:

$$\alpha_1 = 2, \quad \alpha_2 = 36, \quad \alpha_3 = 6, \quad \alpha_4 = 40, \quad \alpha_5 = 2,$$

$$\beta_1 = 30, \quad \beta_2 = 32, \quad \beta_3 = 3, \quad \beta_4 = 8, \quad \beta_5 = 19.$$

Using the formulas in Section 2, we obtain

$$\alpha = \alpha_{45} - \alpha_{12} = 4, \quad \bar{\alpha} = \alpha_4 - \alpha_{13} = 32, \quad \hat{\alpha} = \alpha_2 - \alpha_{35} = 28,$$

and

$$\beta = \beta_1 + \beta_2 + \beta_4 + \beta_5 = 89, \quad \mu_1 = \frac{\hat{\alpha}\beta_{14} - \alpha\beta_3}{\beta_3\beta_{45} + \beta_5\beta_{14}} = 1.31, \quad \mu_2 = \frac{\bar{\alpha}\beta_{25} + \alpha\beta_3}{\beta_1\beta_{25} + \beta_3\beta_{12}} = 0.96.$$

Note that rounded numbers (to 2 dp) are used instead of exact values.

Let us apply Theorems 1–4, as shown in the following table:

Th.	Braess No.	Lower Bound for $Q$	Upper Bound for $Q$	Interval for $Q$
1	$\mathcal{B}_1 = 314$	$\max\{0.06; -0.15; 1.31; 0.96\} = 1.31$	8.59	$]1.31; 8.59[$
2	$\mathcal{B}_2 = 2954$	$\max\{0.06; -0.15; 0.93\} = 0.93$	$\min\{1.27; 0.97\} = 0.97$	$]0.93; 0.97]$
3	$\mathcal{B}_3 = 24193$	$\max\{0.06; -0.15; 0.97; -4; 0.76\} = 0.97$	1.31	$]0.97; 1.31]$
4	$\mathcal{B}_4 = 116440$	$\max\{0.06; -0.15; 1.27; -0.88; 0.80\} = 1.27$	0.96	$\emptyset$

Table 2: Practical application of Theorems 1–4.

Thus, Braess' paradox occurs in  $N/N^+$  in the following cases:

$$1.31 < Q < 8.59 \text{ by Theorem 1,}$$

$$0.93 < Q \leq 0.97 \text{ by Theorem 2,}$$

$$0.97 < Q \leq 1.31 \text{ by Theorem 3.}$$

Theorem 4 produces no interval. Therefore, Braess' paradox occurs if and only if

$$0.93 < Q < 8.59.$$

By Corollary 4 (c), the pseudo-paradox happens if

$$Q \geq 8.59,$$

i.e. under this condition the travel times at equilibria in  $N$  and  $N^+$  are the same and there is no improvement in the network.

Thus, some improvement of travel times in the network when adding the link  $(b, c)$  only occurs for small values of  $Q$  ( $Q < 0.93$ ). The extent of this improvement and of Braess' paradox can be seen from the equilibrium functions found (to 2 dp) by Lemmas 1 and 2:

$$T_{Eq} = \begin{cases} 38 + 62Q & \text{if } 0 < Q \leq 0.06; \\ 40.79 + 18.81Q & \text{if } Q > 0.06; \end{cases}$$

and

$$T_{Eq}^+ = \begin{cases} 10 + 52Q & \text{if } 0 < Q \leq 0.97; \\ 35.76 + 25.44Q & \text{if } 0.97 < Q \leq 1.31; \\ 45.10 + 18.31Q & \text{if } 1.31 < Q < 8.59; \\ 40.79 + 18.81Q & \text{if } Q \geq 8.59. \end{cases}$$

The above findings for the reduced network  $N/N^+$  also hold for the generalised network  $G/G^+$ . Let us summarise them in the following table:

Network	Improvement	Braess' Paradox	Pseudo-Paradox	Travel Times at Equilibria
$N/N^+$	$Q < 0.93$	$0.93 < Q < 8.59$	$Q \geq 8.59$	$T_{Eq}/T_{Eq}^+$
$G/G^+$	$Q < 0.93$	$0.93 < Q < 8.59$	$Q \geq 8.59$	$T_{Eq}/T_{Eq}^+$

Table 3: Summary of findings for the networks  $G/G^+$  and  $N/N^+$ .

The above equilibrium functions and Table 3 can be easily used to analyse the generalised network  $G^+$  for particular values of the total demand (flow)  $Q$ . For example, suppose that the following values of  $Q$  are of interest: 0.5, 5, 10. The results are given in Table 4 (to 1 dp).

Total Demand $Q$	Result	Travel Time in $G$ ( $T_{Eq}$ )	Travel Time in $G^+$ ( $T_{Eq}^+$ )
0.5	Improvement	50.2	36.0
5	Braess' Paradox	134.8	136.6
10	Pseudo-Paradox	228.9	228.9

Table 4: Analysis of particular values of  $Q$  in the network configuration  $G/G^+$ .

## 7 Conclusions and Further Research

Braess' paradox has been investigated in symmetric four-link networks by a number of authors. This paper provides a generalisation of Pas and Principio's findings [13] to non-symmetric networks and shows, for a given congested network, that Braess' paradox occurs if and only if the total demand of travel ( $Q$ ) lies within a certain range of values. The motivation for such an extension was given by Pas and Principio in their conclusions in [13]. Also, in the context of volume-delay functions and their parameters, it can be argued that symmetry properties of networks are not very common in real-life situations.

A further research direction is to generalise results presented in [13] and this work, and find the necessary and sufficient conditions for the existence of Braess' paradox with arbitrary travel times of links in networks with complex topology. Further extensions of this work might also include the investigation of additional network features that were not illustrated in the classical problem introduced by Braess and his colleagues [3]. Of specific relevance to transport researchers is the study of networks in which volume-delay functions are not linear in their parameters. Another future direction suggested by Pas and Principio [13] would be the investigation of Braess' paradox in situations where the overall demand ( $Q$ ) for travel is not constant (i.e. inelastic). The effect of the network symmetry on the Price of Anarchy (the measure of the efficiency of the overall system which can be formulated as a ratio between 'user equilibrium' and 'system optimum' travel times) is another direction for further research.

## Appendix A: Proof of the Lemmas

### Lemma 1

**Proof:** Let us denote  $f_{ab}$  by  $h$ . Then  $f_{bd} = h$  and  $f_{ac} = f_{cd} = Q - h$ . We have

$$T_1 = \alpha_{12} + h\beta_{12} \quad \text{and} \quad T_2 = \alpha_{45} + (Q - h)\beta_{45},$$

where  $T_i$  is the travel time on the path  $P_i$ .

**Case (a):** Suppose that  $P_1$  has a vanishing flow and  $P_2$  has a non-vanishing flow. Then  $Q > h = 0$  and

$$T_1 = \alpha_{12} \quad \text{and} \quad T_2 = \alpha_{45} + Q\beta_{45}.$$

At equilibrium,  $T_1 \geq T_2$ , i.e.  $\alpha_{12} \geq \alpha_{45} + Q\beta_{45}$  or  $Q \leq -\alpha/\beta_{45}$ . The travel time at equilibrium is

$$T_{Eq} = T_2 = \alpha_{45} + Q\beta_{45}.$$

**Case (b):** Assume that  $P_1$  has a non-vanishing flow and  $P_2$  has a vanishing flow. We have  $Q = h > 0$  and

$$T_1 = \alpha_{12} + Q\beta_{12} \quad \text{and} \quad T_2 = \alpha_{45}.$$

At equilibrium,  $T_1 \leq T_2$ , i.e.  $\alpha_{12} + Q\beta_{12} \leq \alpha_{45}$  or  $Q \leq \alpha/\beta_{12}$ . The travel time at equilibrium is as follows:

$$T_{Eq} = T_1 = \alpha_{12} + Q\beta_{12}.$$

**Case (c):** Suppose that no path has a vanishing flow. We have  $Q > h > 0$ . At equilibrium,  $T_1 = T_2$ , i.e.

$$\alpha_{12} + h\beta_{12} = \alpha_{45} + (Q - h)\beta_{45}$$

or

$$h = (\alpha + Q\beta_{45})/\beta.$$

Therefore, the condition  $Q > h > 0$  is equivalent to

$$Q > (\alpha + Q\beta_{45})/\beta > 0$$

or

$$Q > \max\{\alpha/\beta_{12}; -\alpha/\beta_{45}\}.$$

Finally,

$$T_{Eq} = T_1 = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta.$$

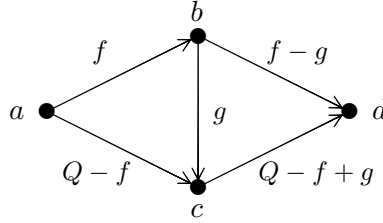
■

## Lemma 2

**Proof:** Let us denote  $f_{ab}$  by  $f$ , and  $f_{bc}$  by  $g$ . Then  $f_{ac} = Q - f$  and, using the conservation-of-flow constraints,  $f_{bd} = f - g$  and  $f_{cd} = Q - f + g$  (see Figure 7). We have

$$T_1 = \alpha_{12} + f\beta_{12} - g\beta_2, \quad T_2 = \alpha_{45} + Q\beta_{45} - f\beta_{45} + g\beta_5 \quad \text{and} \quad T_3 = \alpha_{135} + Q\beta_5 + f(\beta_1 - \beta_5) + g\beta_{35},$$

where  $T_i$  is the travel time on the path  $P_i$ .



**Figure 7.** The link flows in  $N^+$ .

**Case (a):** The only path with non-vanishing flow is  $P_3$ , i.e.  $P_1$  and  $P_2$  have a vanishing flow. Therefore,  $Q = f = g > 0$  and

$$T_1 = \alpha_{12} + Q\beta_1, \quad T_2 = \alpha_{45} + Q\beta_5 \quad \text{and} \quad T_3 = \alpha_{135} + Q\beta_{135}.$$

At equilibrium,  $T_1 \geq T_3$  and  $T_2 \geq T_3$ , i.e.

$$\begin{cases} \alpha_{12} + Q\beta_1 \geq \alpha_{135} + Q\beta_{135}; \\ \alpha_{45} + Q\beta_5 \geq \alpha_{135} + Q\beta_{135}. \end{cases}$$

Thus,

$$0 < Q \leq \min\{\hat{\alpha}/\beta_{35}; \bar{\alpha}/\beta_{13}\}$$

and

$$T_{Eq}^+ = T_3 = \alpha_{135} + Q\beta_{135}.$$

**Case (b):** The only path with non-vanishing flow is  $P_2$ . Hence  $Q > f = g = 0$  and

$$T_1 = \alpha_{12}, \quad T_2 = \alpha_{45} + Q\beta_{45} \quad \text{and} \quad T_3 = \alpha_{135} + Q\beta_5.$$

At equilibrium,  $T_1 \geq T_2$  and  $T_3 \geq T_2$ , i.e.

$$\begin{cases} \alpha_{12} \geq \alpha_{45} + Q\beta_{45}; \\ \alpha_{135} + Q\beta_5 \geq \alpha_{45} + Q\beta_{45}. \end{cases}$$

Thus,

$$0 < Q \leq \min\{-\alpha/\beta_{45}; -\bar{\alpha}/\beta_4\}$$

and

$$T_{Eq}^+ = T_2 = \alpha_{45} + Q\beta_{45}.$$

**Case (c):** The only path with non-vanishing flow is  $P_1$ . Hence  $Q = f > g = 0$  and

$$T_1 = \alpha_{12} + Q\beta_{12}, \quad T_2 = \alpha_{45} \quad \text{and} \quad T_3 = \alpha_{135} + Q\beta_1.$$

At equilibrium,  $T_2 \geq T_1$  and  $T_3 \geq T_1$ , i.e.

$$\begin{cases} \alpha_{45} \geq \alpha_{12} + Q\beta_{12}; \\ \alpha_{135} + Q\beta_1 \geq \alpha_{12} + Q\beta_{12}. \end{cases}$$

Thus,

$$0 < Q \leq \min\{\alpha/\beta_{12}; -\hat{\alpha}/\beta_2\}$$

and

$$T_{Eq}^+ = T_1 = \alpha_{12} + Q\beta_{12}.$$

**Case (d):** The only path with vanishing flow is  $P_1$ . We obtain  $Q > f = g > 0$  and

$$T_1 = \alpha_{12} + f\beta_1, \quad T_2 = \alpha_{45} + Q\beta_{45} - f\beta_4 \quad \text{and} \quad T_3 = \alpha_{135} + Q\beta_5 + f\beta_{13}.$$

At equilibrium,  $T_2 = T_3$  and  $T_1 \geq T_2$ , i.e.

$$\begin{cases} \alpha_{45} + Q\beta_{45} - f\beta_4 = \alpha_{135} + Q\beta_5 + f\beta_{13}; \\ \alpha_{12} + f\beta_1 \geq \alpha_{45} + Q\beta_{45} - f\beta_4; \end{cases} \quad \text{or} \quad \begin{cases} f = (\bar{\alpha} + Q\beta_4)/\beta_{134}; \\ f \geq (\alpha + Q\beta_{45})/\beta_{14}. \end{cases}$$

Therefore,

$$(\bar{\alpha} + Q\beta_4)/\beta_{134} \geq (\alpha + Q\beta_{45})/\beta_{14}$$

or

$$Q(\beta_{45}\beta_{134} - \beta_4\beta_{14}) \leq \bar{\alpha}\beta_{14} - \alpha\beta_{134}.$$

Note that  $\beta_{45}\beta_{134} - \beta_4\beta_{14} = \beta_3\beta_{45} + \beta_5\beta_{14}$  and, using  $\bar{\alpha} = \alpha + \hat{\alpha}$ ,

$$\bar{\alpha}\beta_{14} - \alpha\beta_{134} = \hat{\alpha}\beta_{14} - \alpha\beta_3. \tag{2}$$

Hence

$$Q \leq \frac{\hat{\alpha}\beta_{14} - \alpha\beta_3}{\beta_3\beta_{45} + \beta_5\beta_{14}},$$

i.e.  $Q \leq \mu_1$ . Now, taking into account that  $Q > f > 0$ , the following is obtained:

$$Q > (\bar{\alpha} + Q\beta_4)/\beta_{134} > 0,$$

which is equivalent to

$$Q > \max\{\bar{\alpha}/\beta_{13}; -\bar{\alpha}/\beta_4\}.$$

Finally,

$$T_{Eq}^+ = T_2 = \alpha_{45} + Q\beta_{45} - (\bar{\alpha} + Q\beta_4)\beta_4/\beta_{134}.$$

**Case (e):** The only path with vanishing flow is  $P_2$ . We obtain  $Q = f > g > 0$  and

$$T_1 = \alpha_{12} + Q\beta_{12} - g\beta_2, \quad T_2 = \alpha_{45} + g\beta_5 \quad \text{and} \quad T_3 = \alpha_{135} + Q\beta_1 + g\beta_{35}.$$



At equilibrium,  $T_1 = T_3$  and  $T_2 \geq T_3$ , i.e.

$$\begin{cases} \alpha_{12} + Q\beta_{12} - g\beta_2 = \alpha_{135} + Q\beta_1 + g\beta_{35}; \\ \alpha_{45} + g\beta_5 \geq \alpha_{135} + Q\beta_1 + g\beta_{35}; \end{cases} \quad \text{or} \quad \begin{cases} g = (\hat{\alpha} + Q\beta_2)/\beta_{235}; \\ g \leq (\bar{\alpha} - Q\beta_1)/\beta_3. \end{cases}$$

Therefore,

$$(\hat{\alpha} + Q\beta_2)/\beta_{235} \leq (\bar{\alpha} - Q\beta_1)/\beta_3$$

or

$$Q(\beta_2\beta_3 + \beta_1\beta_{235}) \leq \bar{\alpha}\beta_{235} - \hat{\alpha}\beta_3.$$

Now, rearranging the left-hand side, and using  $\hat{\alpha} = \bar{\alpha} - \alpha$ , we obtain

$$Q(\beta_1\beta_{25} + \beta_3\beta_{12}) \leq \bar{\alpha}\beta_{25} + \alpha\beta_3.$$

Thus,  $Q \leq \mu_2$ . Taking into account that  $Q > g > 0$ , we obtain

$$Q > (\hat{\alpha} + Q\beta_2)/\beta_{235} > 0,$$

which is equivalent to

$$Q > \max\{\hat{\alpha}/\beta_{35}; -\hat{\alpha}/\beta_2\}.$$

Finally,

$$T_{Eq}^+ = T_1 = \alpha_{12} + Q\beta_{12} - (\hat{\alpha} + Q\beta_2)\beta_2/\beta_{235}.$$

**Case (f):** The only path with vanishing flow is  $P_3$ . We obtain  $Q > f > g = 0$  and

$$T_1 = \alpha_{12} + f\beta_{12}, \quad T_2 = \alpha_{45} + Q\beta_{45} - f\beta_{45} \quad \text{and} \quad T_3 = \alpha_{135} + Q\beta_5 + f(\beta_1 - \beta_5).$$

At equilibrium,  $T_1 = T_2$  and  $T_3 \geq T_2$ , i.e.

$$\begin{cases} \alpha_{12} + f\beta_{12} = \alpha_{45} + Q\beta_{45} - f\beta_{45}; \\ \alpha_{135} + Q\beta_5 + f(\beta_1 - \beta_5) \geq \alpha_{45} + Q\beta_{45} - f\beta_{45}; \end{cases} \quad \text{or} \quad \begin{cases} f = (\alpha + Q\beta_{45})/\beta; \\ f \geq (\bar{\alpha} + Q\beta_4)/\beta_{14}. \end{cases}$$

Therefore,

$$(\alpha + Q\beta_{45})/\beta \geq (\bar{\alpha} + Q\beta_4)/\beta_{14}$$

or

$$Q(\beta_{14}\beta_{45} - \beta\beta_4) \geq \bar{\alpha}\beta - \alpha\beta_{14}.$$

It is not difficult to see that  $Q(\beta_{14}\beta_{45} - \beta\beta_4) = Q\mathcal{B}_1$  and, using  $\alpha = \bar{\alpha} - \hat{\alpha}$ ,

$$\bar{\alpha}\beta - \alpha\beta_{14} = \bar{\alpha}(\beta_{14} + \beta_{25}) - (\bar{\alpha} - \hat{\alpha})\beta_{14} = \hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}. \quad (3)$$

Thus,

$$\mathcal{B}_1 \geq \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{Q}.$$

Taking into account that  $Q > f > 0$ , we obtain

$$Q > (\alpha + Q\beta_{45})/\beta > 0,$$

which is equivalent to

$$Q > \max\{\alpha/\beta_{12}; -\alpha/\beta_{45}\}.$$

Finally,

$$T_{Eq}^+ = T_1 = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta.$$

**Case (g):** No path has a vanishing flow, and so  $Q > f > g > 0$ . At equilibrium,  $T_1 = T_2$  and  $T_2 = T_3$ , i.e.

$$\begin{cases} f\beta = \alpha + g\beta_{25} + Q\beta_{45}; \\ f\beta_{14} + g\beta_3 = \bar{\alpha} + Q\beta_4; \end{cases} \quad \text{or} \quad \begin{cases} f = (\alpha + g\beta_{25} + Q\beta_{45})/\beta; \\ g = (\bar{\alpha}\beta - \alpha\beta_{14} - Q(\beta_{45}\beta_{14} - \beta_4\beta))/(\beta_3\beta + \beta_{14}\beta_{25}). \end{cases}$$

It is easy to see that  $\beta_{45}\beta_{14} - \beta_4\beta = \mathcal{B}_1$ , so

$$g = \frac{\bar{\alpha}\beta - \alpha\beta_{14} - Q\mathcal{B}_1}{\beta_3\beta + \beta_{14}\beta_{25}}.$$

The condition  $g > 0$  is equivalent to  $\bar{\alpha}\beta - \alpha\beta_{14} - Q\mathcal{B}_1 > 0$  or  $Q\mathcal{B}_1 < \bar{\alpha}\beta - \alpha\beta_{14}$ . Using (3), we obtain

$$\mathcal{B}_1 < \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{Q}.$$

The condition  $f > g$  can be written as  $(\alpha + g\beta_{25} + Q\beta_{45})/\beta > g$  or  $g < (\alpha + Q\beta_{45})/\beta_{14}$ . Hence

$$\frac{\bar{\alpha}\beta - \alpha\beta_{14} - Q\mathcal{B}_1}{\beta_3\beta + \beta_{14}\beta_{25}} < \frac{\alpha + Q\beta_{45}}{\beta_{14}},$$

which is equivalent to

$$Q(\mathcal{B}_1\beta_{14} + \beta_{45}\beta_{25}\beta_{14} + \beta_3\beta_{45}\beta) > \bar{\alpha}\beta_{14}\beta - \alpha(\beta_{14}^2 + \beta_3\beta + \beta_{25}\beta_{14}).$$

It is easy to see that  $\mathcal{B}_1\beta_{14} + \beta_{45}\beta_{25}\beta_{14} = \beta_5\beta_{14}\beta$  and  $\beta_{14}^2 + \beta_3\beta + \beta_{25}\beta_{14} = \beta_{134}\beta$ . Therefore,

$$Q(\beta_3\beta_{45}\beta + \beta_5\beta_{14}\beta) > \bar{\alpha}\beta_{14}\beta - \alpha\beta_{134}\beta \quad \text{or} \quad Q(\beta_3\beta_{45} + \beta_5\beta_{14}) > \bar{\alpha}\beta_{14} - \alpha\beta_{134}.$$

Using (2), we obtain  $Q > \mu_1$ .

Now let us consider the condition  $Q > f$ , which can be written as  $Q > (\alpha + g\beta_{25} + Q\beta_{45})/\beta$  or  $(Q\beta_{12} - \alpha)/\beta_{25} > g$ . Hence

$$\frac{Q\beta_{12} - \alpha}{\beta_{25}} > \frac{\bar{\alpha}\beta - \alpha\beta_{14} - Q\mathcal{B}_1}{\beta_3\beta + \beta_{14}\beta_{25}}$$

or

$$Q(\beta_{12}\beta_{25}\beta_{14} + \beta_{25}\mathcal{B}_1 + \beta_3\beta_{12}\beta) > \bar{\alpha}\beta_{25}\beta + \alpha\beta_3\beta.$$

It is not difficult to check that  $\beta_{12}\beta_{25}\beta_{14} + \beta_{25}\mathcal{B}_1 = \beta_1\beta_{25}\beta$  and hence

$$Q(\beta_1\beta_{25}\beta + \beta_3\beta_{12}\beta) > \bar{\alpha}\beta_{25}\beta + \alpha\beta_3\beta,$$

i.e.  $Q > \mu_2$ .

Finally,

$$T_{Eq}^+ = T_1 = \alpha_{12} + f\beta_{12} - g\beta_2 = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta + g(\beta_{25}\beta_{12}/\beta - \beta_2).$$

It is easy to see that  $\beta_{25}\beta_{12}/\beta - \beta_2 = \mathcal{B}_1/\beta$ , and so

$$T_{Eq}^+ = \alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta + g\mathcal{B}_1/\beta.$$

■

## Appendix B: Proof of the Theorems

We will consider the cases (i,j), where (i) is one of the cases of Lemma 1, and (j) is one of the cases of Lemma 2. First of all, suppose that at equilibrium the path  $P_3$  has a vanishing flow in  $N^+$ , i.e. the flow at equilibrium may only use  $P_1$  and  $P_2$ , or just one of those paths. It is easy to see that the same flow is an equilibrium flow in  $N$ , and the travel times at equilibria are equal. Thus, Braess' paradox cannot happen in the following cases: (a,b), (a,c), (a,f), (b,b), (b,c), (b,f), (c,b), (c,c), (c,f). Note that  $T_{Eq}^+ = T_{Eq}$  in the cases (a,b), (b,c) and (c,f), while the conditions in other cases are inconsistent. For example, in the case (a,c), Lemma 1 (a) implies  $\alpha < 0$ , while Lemma 2 (c) implies  $\alpha > 0$ , so the bounds are inconsistent.

Thus, there are 12 cases to consider. We will see that Braess' paradox may occur in four cases only: (c,a), (c,d), (c,e) and (c,g). This proves Mega-Theorem, while the cases themselves provide proofs of Theorems 2, 3, 4 and 1, respectively.

**Case (a,a):** Assume that  $T_{Eq}^+ > T_{Eq}$ , which implies  $Q(\beta_{13} - \beta_4) > \bar{\alpha}$ . Since  $0 < Q \leq \bar{\alpha}/\beta_{13}$ , we obtain  $\bar{\alpha} > 0$ . Therefore,  $\beta_{13} - \beta_4 > 0$  and

$$Q > \frac{\bar{\alpha}}{\beta_{13} - \beta_4} > \frac{\bar{\alpha}}{\beta_{13}},$$

contrary to  $Q \leq \bar{\alpha}/\beta_{13}$ .

**Case (a,d):**  $T_{Eq}^+ > T_{Eq}$  implies  $\bar{\alpha} + Q\beta_4 < 0$ , i.e.  $Q < -\bar{\alpha}/\beta_4$ , but  $Q > -\bar{\alpha}/\beta_4$  in the case (d).

**Case (a,e):** We will show that the upper and lower bounds imposed on  $Q$  in this case are inconsistent.

Let us first assume that  $\hat{\alpha} \geq 0$ . We have  $Q \leq -\alpha/\beta_{45}$  from (a) and  $Q > \hat{\alpha}/\beta_{35}$  from (e). Hence  $\hat{\alpha}/\beta_{35} < -\alpha/\beta_{45}$  or

$$\hat{\alpha}\beta_{45} + \alpha\beta_{35} < 0. \quad (4)$$

From (e), we obtain

$$\frac{\hat{\alpha}}{\beta_{35}} < \frac{\bar{\alpha}\beta_{25} + \alpha\beta_3}{\beta_1\beta_{25} + \beta_3\beta_{12}}.$$

Using  $\bar{\alpha} = \alpha + \hat{\alpha}$ , the above inequality is equivalent to

$$\hat{\alpha}(\beta_1\beta_{25} + \beta_3\beta_{12} - \beta_{25}\beta_{35}) - \alpha\beta_{35}\beta_{235} < 0.$$

Since  $\beta_1\beta_{25} + \beta_3\beta_{12} - \beta_{25}\beta_{35} = (\beta_1 - \beta_5)\beta_{235}$ , we have

$$\hat{\alpha}(\beta_1 - \beta_5) - \alpha\beta_{35} < 0. \quad (5)$$

Adding (4) and (5), we obtain  $\hat{\alpha}\beta_{14} < 0$ , contrary to the assumption  $\hat{\alpha} \geq 0$ .

Now suppose that  $\hat{\alpha} < 0$ . We have  $Q \leq -\alpha/\beta_{45}$  from (a) and  $Q > -\hat{\alpha}/\beta_2$  from (e). Hence  $-\hat{\alpha}/\beta_2 < -\alpha/\beta_{45}$  or

$$\hat{\alpha}\beta_{45} - \alpha\beta_2 > 0. \quad (6)$$

From (e), we obtain

$$\frac{-\hat{\alpha}}{\beta_2} < \frac{\bar{\alpha}\beta_{25} + \alpha\beta_3}{\beta_1\beta_{25} + \beta_3\beta_{12}}.$$

Hence, using  $\bar{\alpha} = \alpha + \hat{\alpha}$ ,

$$\hat{\alpha}(\beta_1\beta_{25} + \beta_3\beta_{12} + \beta_2\beta_{25}) + \alpha\beta_2\beta_{235} > 0.$$

It is easy to see that  $\beta_1\beta_{25} + \beta_3\beta_{12} + \beta_2\beta_{25} = \beta_{12}\beta_{235}$ , so

$$\hat{\alpha}\beta_{12} + \alpha\beta_2 > 0. \quad (7)$$

Adding (6) and (7), we obtain  $\hat{\alpha}\beta > 0$ , a contradiction.

**Case (a,g):** We will show that the upper and lower bounds imposed on  $Q$  in this case are inconsistent. We have  $\mu_1 < Q \leq -\alpha/\beta_{45}$ , i.e.

$$\hat{\alpha}\beta_{45} + \alpha\beta_5 < 0. \quad (8)$$

Also, from (g),

$$Q\mathcal{B}_1 < \hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}. \quad (9)$$

Suppose that  $\mathcal{B}_1 = 0$ , i.e.

$$\beta_1\beta_5 = \beta_2\beta_4 \quad (10)$$

and

$$0 < \hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}.$$

From (10), we obtain  $\beta_1 = \beta_2\beta_4/\beta_5$ ,  $\beta_2 = \beta_1\beta_5/\beta_4$  and  $\beta_1/\beta_4 = \beta_2/\beta_5$ . Therefore,

$$0 < \hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25} = \hat{\alpha}\beta_4(1 + \frac{\beta_2}{\beta_5}) + \bar{\alpha}\beta_5(1 + \frac{\beta_1}{\beta_4}) = (\hat{\alpha}\beta_4 + \bar{\alpha}\beta_5)(1 + \frac{\beta_2}{\beta_5}).$$

Thus,  $\hat{\alpha}\beta_4 + \bar{\alpha}\beta_5 > 0$ . Now

$$0 < \hat{\alpha}\beta_4 + \bar{\alpha}\beta_5 = \hat{\alpha}\beta_4 + (\alpha + \hat{\alpha})\beta_5 = \hat{\alpha}\beta_{45} + \alpha\beta_5,$$

contrary to (8).

Using  $\bar{\alpha} = \alpha + \hat{\alpha}$ , we can re-write (9) in the following form:

$$Q\mathcal{B}_1 < \alpha\beta_{25} + \hat{\alpha}\beta.$$

Assume now that  $\mathcal{B}_1 < 0$ , i.e.

$$Q > \frac{\alpha\beta_{25} + \hat{\alpha}\beta}{\mathcal{B}_1}.$$

Since  $Q \leq -\alpha/\beta_{45}$ , we obtain

$$\frac{\alpha\beta_{25} + \hat{\alpha}\beta}{\mathcal{B}_1} < \frac{-\alpha}{\beta_{45}}$$

or

$$\alpha(\mathcal{B}_1 + \beta_{25}\beta_{45}) + \hat{\alpha}\beta_{45}\beta > 0.$$

It is not difficult to see that the last inequality is equivalent to

$$\alpha\beta_5\beta + \hat{\alpha}\beta_{45}\beta > 0$$

or

$$\alpha\beta_5 + \hat{\alpha}\beta_{45} > 0,$$

contrary to (8).

Finally, let us suppose that  $\mathcal{B}_1 > 0$ , i.e.

$$Q < \frac{\alpha\beta_{25} + \hat{\alpha}\beta}{\mathcal{B}_1}.$$

Since  $Q > \mu_1$ , we obtain

$$\frac{\hat{\alpha}\beta_{14} - \alpha\beta_3}{L} < \frac{\alpha\beta_{25} + \hat{\alpha}\beta}{\mathcal{B}_1},$$

where  $L = \beta_3\beta_{45} + \beta_5\beta_{14}$ . The last inequality is equivalent to

$$\hat{\alpha}(\beta L - \beta_{14}\mathcal{B}_1) + \alpha(\beta_{25}L + \beta_3\mathcal{B}_1) > 0.$$

It is easy to see that

$$\beta L - \beta_{14}\mathcal{B}_1 = \beta_{45}(\beta\beta_3 + \beta_{14}\beta_{25})$$

and

$$\beta_{25}L + \beta_3\mathcal{B}_1 = \beta_5(\beta\beta_3 + \beta_{14}\beta_{25}).$$

Thus,

$$\hat{\alpha}\beta_{45} + \alpha\beta_5 > 0,$$

contrary to (8).

**Case (b,a):** Assume that  $T_{Eq}^+ > T_{Eq}$ , which implies  $Q(\beta_{35} - \beta_2) > \hat{\alpha}$ . Since  $0 < Q \leq \hat{\alpha}/\beta_{35}$ , we obtain  $\hat{\alpha} > 0$ . Therefore,  $\beta_{35} - \beta_2 > 0$  and

$$Q > \frac{\hat{\alpha}}{\beta_{35} - \beta_2} > \frac{\hat{\alpha}}{\beta_{35}},$$

contrary to  $Q \leq \hat{\alpha}/\beta_{35}$ .

**Case (b,d):** We will show that the upper and lower bounds imposed on  $Q$  in this case are inconsistent.

Let us first assume that  $\bar{\alpha} \geq 0$ . We have  $Q \leq \alpha/\beta_{12}$  from (b) and  $Q > \bar{\alpha}/\beta_{13}$  from (d). Hence  $\bar{\alpha}/\beta_{13} < \alpha/\beta_{12}$  or

$$\bar{\alpha}\beta_{12} - \alpha\beta_{13} < 0. \quad (11)$$

From (d,) we obtain

$$\frac{\bar{\alpha}}{\beta_{13}} < \frac{\hat{\alpha}\beta_{14} - \alpha\beta_3}{L},$$

where  $L = \beta_3\beta_{45} + \beta_5\beta_{14}$ . Using  $\hat{\alpha} = \bar{\alpha} - \alpha$ , the above inequality is equivalent to

$$\bar{\alpha}(L - \beta_{13}\beta_{14}) + \alpha\beta_{13}\beta_{14} < 0$$

or

$$\bar{\alpha}(\beta_5 - \beta_1) + \alpha\beta_{13} < 0. \quad (12)$$

Adding (11) and (12), we obtain  $\bar{\alpha}\beta_{25} < 0$ , contrary to the assumption  $\bar{\alpha} \geq 0$ .

Now suppose that  $\bar{\alpha} < 0$ . We have  $-\bar{\alpha}/\beta_4 < \alpha/\beta_{12}$  or

$$\alpha\beta_4 + \bar{\alpha}\beta_{12} > 0. \quad (13)$$

From (d) we obtain

$$\frac{-\bar{\alpha}}{\beta_4} < \frac{\hat{\alpha}\beta_{14} - \alpha\beta_3}{L},$$

where  $L = \beta_3\beta_{45} + \beta_5\beta_{14}$ . Using  $\hat{\alpha} = \bar{\alpha} - \alpha$ , the above inequality is equivalent to

$$\bar{\alpha}(L + \beta_4\beta_{14}) - \alpha\beta_4\beta_{14} > 0$$

or

$$\bar{\alpha}\beta_{45} - \alpha\beta_4 > 0. \quad (14)$$

Adding (13) and (14), we obtain  $\bar{\alpha}\beta > 0$ , contrary to the assumption  $\bar{\alpha} < 0$ .

**Case (b,e):**  $T_{Eq}^+ > T_{Eq}$  implies  $\hat{\alpha} + Q\beta_2 < 0$ , i.e.  $Q < -\hat{\alpha}/\beta_2$ , but  $Q > -\hat{\alpha}/\beta_2$  in the case (e).

**Case (b,g):** We will show that the upper and lower bounds imposed on  $Q$  in this case are inconsistent. We have  $\mu_2 < Q \leq \alpha/\beta_{12}$ , i.e.

$$\bar{\alpha}\beta_{12} - \alpha\beta_1 < 0,$$

or, using  $\alpha = \bar{\alpha} - \hat{\alpha}$ ,

$$\hat{\alpha}\beta_1 + \bar{\alpha}\beta_2 < 0. \quad (15)$$

From (g), we have

$$Q\mathcal{B}_1 < \hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}. \quad (16)$$

Suppose that  $\mathcal{B}_1 = 0$ , i.e.

$$\beta_1\beta_5 = \beta_2\beta_4 \quad (17)$$

and

$$0 < \hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}.$$

From (17), we obtain  $\beta_4 = \beta_1\beta_5/\beta_2$ ,  $\beta_5 = \beta_2\beta_4/\beta_1$  and  $\beta_4/\beta_1 = \beta_5/\beta_2$ . Therefore,

$$0 < \hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25} = \hat{\alpha}\beta_1(1 + \frac{\beta_5}{\beta_2}) + \bar{\alpha}\beta_2(1 + \frac{\beta_4}{\beta_1}) = (\hat{\alpha}\beta_1 + \bar{\alpha}\beta_2)(1 + \frac{\beta_5}{\beta_2}).$$

Thus,  $\hat{\alpha}\beta_1 + \bar{\alpha}\beta_2 > 0$ , contrary to (15).

Assume now that  $\mathcal{B}_1 < 0$ , i.e.

$$Q > \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{\mathcal{B}_1}.$$

Since  $Q \leq \alpha/\beta_{12} = (\bar{\alpha} - \hat{\alpha})/\beta_{12}$ , we obtain

$$(\bar{\alpha} - \hat{\alpha})\mathcal{B}_1 < \hat{\alpha}\beta_{12}\beta_{14} + \bar{\alpha}\beta_{12}\beta_{25}$$

or

$$\hat{\alpha}(\beta_{12}\beta_{14} + \mathcal{B}_1) + \bar{\alpha}(\beta_{12}\beta_{25} - \mathcal{B}_1) > 0.$$

The last inequality is equivalent to

$$\hat{\alpha}\beta_1\beta + \bar{\alpha}\beta_2\beta > 0,$$

contrary to (15).

Finally, let us suppose that  $\mathcal{B}_1 > 0$ , i.e.

$$Q < \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{\mathcal{B}_1}.$$

Since,  $Q > \mu_2$ , we obtain

$$\frac{\bar{\alpha}\beta_{25} + \alpha\beta_3}{\beta_1\beta_{25} + \beta_3\beta_{12}} < \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{\mathcal{B}_1}.$$

Using  $\alpha = \bar{\alpha} - \hat{\alpha}$ , the last inequality can be re-written as follows:

$$\hat{\alpha}X + \bar{\alpha}Y > 0, \quad (18)$$

where

$$X = \beta_{14}(\beta_1\beta_{25} + \beta_3\beta_{12}) + \beta_3\mathcal{B}_1$$

and

$$Y = \beta_{25}(\beta_1\beta_{25} + \beta_3\beta_{12} - \mathcal{B}_1) - \beta_3\mathcal{B}_1.$$

It is not difficult to see that

$$X = \beta_1(\beta_{14}\beta_{25} + \beta_3\beta)$$

and

$$Y = \beta_2(\beta_{14}\beta_{25} + \beta_3\beta).$$

Thus, (18) is equivalent to

$$\hat{\alpha}\beta_1 + \bar{\alpha}\beta_2 > 0,$$

contrary to (15).

**Case (c,a):** This case corresponds to Theorem 2. It is easy to see that  $T_{Eq}^+ > T_{Eq}$  is equivalent to

$$Q(\beta_{135}\beta - \beta_{12}\beta_{45}) > \hat{\alpha}\beta + \alpha\beta_{12}$$

or, using  $\alpha = \bar{\alpha} - \hat{\alpha}$ ,

$$Q\mathcal{B}_2 > \hat{\alpha}\beta_{45} + \bar{\alpha}\beta_{12}.$$

Moreover,

$$\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}\right\} < Q \leq \min\left\{\frac{\hat{\alpha}}{\beta_{35}}; \frac{\bar{\alpha}}{\beta_{13}}\right\},$$

which implies  $\hat{\alpha} > 0$  and  $\bar{\alpha} > 0$ . Therefore,  $\hat{\alpha}\beta_{45} + \bar{\alpha}\beta_{12} > 0$  and  $\mathcal{B}_2 > 0$ , i.e.

$$Q > \frac{\hat{\alpha}\beta_{45} + \bar{\alpha}\beta_{12}}{\mathcal{B}_2}.$$

**Case (c,d):** This case corresponds to Theorem 3. It is easy to see that  $T_{Eq} < T_{Eq}^+$  is equivalent to

$$\alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta < \alpha_{45} + Q\beta_{45} - (\bar{\alpha} + Q\beta_4)\beta_4/\beta_{134}$$

or

$$Q\mathcal{B}_3 > \bar{\alpha}\beta_4\beta - \alpha\beta_{134}\beta_{45}.$$

In addition, we have

$$\max\left\{\frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\bar{\alpha}}{\beta_{13}}; \frac{-\bar{\alpha}}{\beta_4}\right\} < Q \leq \mu_1.$$

Let us show that these inequalities are inconsistent if  $\mathcal{B}_3 \leq 0$ . Using (2), the inequality  $-\alpha/\beta_{45} < \mu_1$  can be written as follows:

$$\frac{-\alpha}{\beta_{45}} < \frac{\bar{\alpha}\beta_{14} - \alpha\beta_{134}}{\beta_3\beta_{45} + \beta_5\beta_{14}},$$

which is equivalent to

$$\bar{\alpha}\beta_{14}\beta_{45} - \alpha(\beta_{134}\beta_{45} - \beta_3\beta_{45} - \beta_5\beta_{14}) > 0$$

or

$$\bar{\alpha} > \frac{\alpha\beta_4}{\beta_{45}}.$$

Therefore,

$$Q\mathcal{B}_3 > \bar{\alpha}\beta_4\beta - \alpha\beta_{134}\beta_{45} > \frac{\alpha\beta_4}{\beta_{45}}\beta_4\beta - \alpha\beta_{134}\beta_{45} = \frac{-\alpha\mathcal{B}_3}{\beta_{45}}.$$

Thus,

$$Q\mathcal{B}_3 > \frac{-\alpha\mathcal{B}_3}{\beta_{45}},$$

which is not satisfied if  $\mathcal{B}_3 = 0$ . If  $\mathcal{B}_3 < 0$ , then we obtain

$$Q < \frac{-\alpha}{\beta_{45}},$$

which is inconsistent with the inequality  $Q > -\alpha/\beta_{45}$ . We conclude that  $\mathcal{B}_3 > 0$  and hence

$$Q > \frac{\bar{\alpha}\beta_4\beta - \alpha\beta_{134}\beta_{45}}{\mathcal{B}_3}.$$

**Case (c,e):** This case corresponds to Theorem 4. It is not difficult to see that  $T_{Eq} < T_{Eq}^+$  is equivalent to

$$\alpha_{12} + (\alpha + Q\beta_{45})\beta_{12}/\beta < \alpha_{12} + Q\beta_{12} - (\hat{\alpha} + Q\beta_2)\beta_2/\beta_{235}$$

or

$$Q\mathcal{B}_4 > \hat{\alpha}\beta_2\beta + \alpha\beta_{235}\beta_{12}.$$

Moreover, we obtain

$$\max \left\{ \frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \frac{\hat{\alpha}}{\beta_{35}}; \frac{-\hat{\alpha}}{\beta_2} \right\} < Q \leq \mu_2.$$

Let us show that these inequalities are inconsistent if  $\mathcal{B}_4 \leq 0$ . Using  $\bar{\alpha} = \alpha + \hat{\alpha}$ , the inequality  $\alpha/\beta_{12} < \mu_2$  can be written as follows:

$$\frac{\alpha}{\beta_{12}} < \frac{\alpha\beta_{235} + \hat{\alpha}\beta_{25}}{\beta_1\beta_{25} + \beta_3\beta_{12}},$$

which is equivalent to

$$\alpha(\beta_{12}\beta_{235} - \beta_1\beta_{25} - \beta_3\beta_{12}) + \hat{\alpha}\beta_{12}\beta_{25} > 0$$

or

$$\hat{\alpha} > \frac{-\alpha\beta_2}{\beta_{12}}.$$

Therefore,

$$Q\mathcal{B}_4 > \hat{\alpha}\beta_2\beta + \alpha\beta_{235}\beta_{12} > \frac{-\alpha\beta_2}{\beta_{12}}\beta_2\beta + \alpha\beta_{235}\beta_{12} = \frac{\alpha\mathcal{B}_4}{\beta_{12}}.$$

Thus,

$$Q\mathcal{B}_4 > \frac{\alpha\mathcal{B}_4}{\beta_{12}},$$

which is not satisfied if  $\mathcal{B}_4 = 0$ . If  $\mathcal{B}_4 < 0$ , then

$$Q < \frac{\alpha}{\beta_{12}},$$

which is inconsistent with the inequality  $Q > \alpha/\beta_{12}$ . Thus,  $\mathcal{B}_4 > 0$  and hence

$$Q > \frac{\hat{\alpha}\beta_2\beta + \alpha\beta_{235}\beta_{12}}{\mathcal{B}_4}.$$

**Case (c,g):** This case corresponds to Theorem 1. It is easy to see that  $T_{Eq}^+ > T_{Eq}$  is equivalent to

$$g\mathcal{B}_1/\beta > 0.$$

From the proof of Lemma 2 we know that  $g > 0$ . Hence  $\mathcal{B}_1 > 0$ . Therefore, the last inequality in the case (g) can be written as

$$Q < \frac{\hat{\alpha}\beta_{14} + \bar{\alpha}\beta_{25}}{\mathcal{B}_1}.$$

In addition, we have

$$\max \left\{ \frac{\alpha}{\beta_{12}}; \frac{-\alpha}{\beta_{45}}; \mu_1; \mu_2 \right\} < Q.$$

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