The probabilistic approach to limited packings in graphs

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Abstract

We consider (closed neighbourhood) packings and their generalization in graphs. A vertex set X in a graph G is a *k-limited packing* if for every vertex $v \in V(G)$, $|N[v] \cap X| \leq k$, where $N[v]$ is the closed neighbourhood of v. The *k-limited packing number* $L_k(G)$ of a graph G is the largest size of a k-limited packing in G. Limited packing problems can be considered as secure facility location problems in networks.

In this paper, we develop a new application of the probabilistic method to limited packings in graphs, resulting in lower bounds for the k-limited packing number and a randomized algorithm to find k-limited packings satisfying the bounds. In particular, we prove that for any graph G of order n with maximum vertex degree Δ,

$$L_k(G) \geq \frac{kn}{(k+1)\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}}.$$

Also, some other upper and lower bounds for $L_k(G)$ are given.

Keywords: k-Limited packings, The probabilistic method, Lower and upper bounds, Randomized algorithm

1. Introduction

We consider simple undirected graphs. If not specified otherwise, standard graph-theoretic terminology and notations are used (e.g., see [1, 2]). We are interested in the classical packings and packing numbers of graphs as introduced in [9], and their generalization, called limited packings and limited packing numbers, respectively, as presented in [6]. In the literature, the classical packings are often referred to under different names: for example, as (distance) 2-packings [9, 13], closed neighborhood packings [11] or strong stable sets [8]. They can also be considered as generalizations of independent (stable) sets which, following the terminology of [9], would be (distance) 1-packings.

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Formally, a vertex set X in a graph G is a *k-limited packing* if for every vertex $v \in V(G)$,

$$|N[v] \cap X| \leq k,$$

where $N[v]$ is the closed neighbourhood of v. The *k-limited packing number* $L_k(G)$ of a graph G is the maximum size of a k-limited packing in G. In these terms, the classical (distance) 2-packings are 1-limited packings, and hence $\rho(G) = L_1(G)$, where $\rho(G)$ is the 2-packing number.

The problem of finding a 2-packing (1-limited packing) of maximum size is shown to be NP-complete by Hochbaum and Schmoys [8]. In [4], it is shown that the problem of finding a maximum size k-limited packing is NP-complete even for the classes of split and bipartite graphs.

Graphs usually serve as underlying models for networks. A number of interesting application scenarios of limited packings are described in [6], including network security, market saturation, and codes. These and others can be summarized as secure location or distribution of facilities in a network. In a more general sense, these problems can be viewed as (maximization) facility location problems to place/distribute in a given network as many resources as possible subject to some (security) constraints.

2-Packings (1-limited packings) are well-studied in the literature from the structural and algorithmic point of view (e.g., see [8, 9, 11, 12]) and in connection with other graph parameters (e.g., see [3, 7, 9, 11, 13]). In particular, several papers discuss connections between packings and dominating sets in graphs (e.g., see [3, 4, 6, 7, 11]). Although the formal definitions for packings and dominating sets may appear to be similar, the problems have a very different nature: one of the problems is a maximization problem not to break some (security) constraints, and the other is a minimization problem to satisfy some reliability requirements. For example, given a graph G, the definitions imply a simple inequality $\rho(G) \leq \gamma(G)$, where $\gamma(G)$ is the domination number of G (e.g., see [11]). However, the difference between $\rho(G)$ and $\gamma(G)$ can be arbitrarily large as illustrated in [3]: $\rho(K_n \times K_n) = 1$ for the Cartesian product of complete graphs, but $\gamma(K_n \times K_n) = n$.

In this paper, we develop an application of the probabilistic method to k-limited packings in general and to 2-packings (1-limited packings) in particular. In Section 2 we present the probabilistic construction and use it to derive two lower bounds for the k-limited packing number $L_k(G)$. Also, using a greedy algorithm approach, we provide an improved lower bound for the 2-packing (1-limited packing) number $\rho(G) = L_1(G)$. The probabilistic construction implies a randomized algorithm to find k-limited packings satisfying the lower bounds. The algorithm and its analysis are presented in Section 3. Section 4 shows that the main lower bound is asymptotically sharp, and discusses the improvement for 1-limited packings from the greedy algorithm approach. Finally, Section 5 provides upper bounds for $L_k(G)$, e.g. in terms of the k-tuple domination number $\gamma_{\times k}(G)$.

Notice that the probabilistic construction and approach are different from the well-known probabilistic constructions used for independent sets (e.g., see [1], p.27–28). In terms of packings, an independent set in a graph G is a distance 1-packing: for any two vertices in an independent set, the distance between them in G is greater than 1. To the best of our knowledge, the proposed application of the probabilistic method is a new approach to work with packings and related maximization problems.

2. The probabilistic construction and lower bounds

Let $\Delta = \Delta(G)$ denote the maximum vertex degree in a graph G. Notice that $L_k(G) = n$ when $k \geq \Delta + 1$. We define

$$c_t = c_t(G) = \binom{\Delta}{t} \quad \text{and} \quad \tilde{c}_t = \tilde{c}_t(G) = \binom{\Delta+1}{t}.$$

In what follows, we put $\binom{a}{b} = 0$ if $b > a$.

The following theorem gives a new lower bound for the k-limited packing number. It may be pointed out that the probabilistic construction used in the proof of Theorem 1 implies a randomized algorithm for finding a k-limited packing set, whose size satisfies the bound of Theorem 1 with a positive probability (see Algorithm 1 in Section 3).

Theorem 1. *For any graph G of order n with $\Delta \geq k \geq 1$,*

$$L_k(G) \geq \frac{kn}{\tilde{c}_{k+1}^{1/k}\,(1+k)^{1+1/k}}. \tag{1}$$

PROOF. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with the probability

$$p = \left(\frac{1}{\tilde{c}_{k+1}\,(1+k)}\right)^{1/k}. \tag{2}$$

For $m = k, ..., \Delta$, we denote

$$A_m = \{v \in A : |N(v) \cap A| = m\}.$$

For each set A_m, we form a set A'_m in the following way. For every vertex $v \in A_m$, we take $m-(k-1)$ neighbours from $N(v) \cap A$ and add them to A'_m. Such neighbours always exist because $m \geq k$. It is obvious that

$$|A'_m| \leq (m-k+1)|A_m|.$$

For $m = k+1, ..., \Delta$, let us denote

$$B_m = \{v \in V(G) - A : |N(v) \cap A| = m\}.$$

For each set B_m, we form a set B'_m by taking $m-k$ neighbours from $N(v) \cap A$ for every vertex $v \in B_m$. We have

$$|B'_m| \leq (m-k)|B_m|.$$

Let us construct the set X as follows:

$$X = A - \left(\bigcup_{m=k}^{\Delta} A'_m \right) - \left(\bigcup_{m=k+1}^{\Delta} B'_m \right).$$

It is easy to see that X is a k-limited packing in G. The expectation of $|X|$ is

$$\begin{aligned} \mathbf{E}[|X|] &\geq \mathbf{E}\left[|A| - \sum_{m=k}^{\Delta} |A'_m| - \sum_{m=k+1}^{\Delta} |B'_m| \right] \\ &\geq \mathbf{E}\left[|A| - \sum_{m=k}^{\Delta} (m-k+1)|A_m| - \sum_{m=k+1}^{\Delta} (m-k)|B_m| \right] \\ &= pn - \sum_{m=k}^{\Delta} (m-k+1)\mathbf{E}[|A_m|] - \sum_{m=k+1}^{\Delta} (m-k)\mathbf{E}[|B_m|]. \end{aligned}$$

Let us denote the vertices of G by $v_1, v_2, ..., v_n$ and the corresponding vertex degrees by $d_1, d_2, ..., d_n$. We will need the following lemma:

Lemma 2. *If $p = \left(\frac{1}{\tilde{c}_{k+1}\,(1+k)} \right)^{1/k}$, then, for any vertex $v_i \in V(G)$,*

$$\binom{d_i}{m} (1-p)^{d_i - m} \leq \binom{\Delta}{m} (1-p)^{\Delta - m}, \qquad m \geq k. \tag{3}$$

Proof. The inequality (3) holds if $d_i = \Delta$. It is also true if $d_i < m$ because in this case $\binom{d_i}{m} = 0$. Thus, we may assume that

$$m \leq d_i < \Delta.$$

Now, it is easy to see that inequality (3) is equivalent to the following:

$$(1-p)^{\Delta - d_i} \geq \binom{d_i}{m} \Big/ \binom{\Delta}{m} = \frac{(\Delta - m)!/(d_i - m)!}{\Delta!/d_i!} = \prod_{i=0}^{\Delta - d_i - 1} \frac{\Delta - m - i}{\Delta - i}. \tag{4}$$

Further, $\Delta \geq k$ implies $\frac{\Delta}{k} \leq \frac{\Delta - i}{k - i}$, where $0 \leq i \leq k-1$. Taking into account that $\Delta > 0$, we obtain

$$\left(\frac{\Delta}{k} \right)^k \leq \prod_{i=0}^{k-1} \frac{\Delta - i}{k - i} = c_k < \tilde{c}_{k+1}(1+k)$$

or

$$\frac{1}{\tilde{c}_{k+1}\,(1+k)} < \left(\frac{k}{\Delta} \right)^k.$$

Thus,

$$p^k < \left(\frac{k}{\Delta} \right)^k \quad \text{or} \quad p < \frac{k}{\Delta} \leq \frac{m}{\Delta}.$$

We have $p < \frac{m}{\Delta}$, which is equivalent to $1-p > \frac{\Delta - m}{\Delta}$. Therefore,

$$(1-p)^{\Delta - d_i} > \left(\frac{\Delta - m}{\Delta}\right)^{\Delta - d_i} \geq \prod_{i=0}^{\Delta - d_i - 1} \frac{\Delta - m - i}{\Delta - i},$$

as required in (4). □

Now we go on with the proof of Theorem 1. By Lemma 2,

$$\begin{aligned} \mathbf{E}[|A_m|] &= \sum_{i=1}^{n} \mathbf{P}[v_i \in A_m] \\ &= \sum_{i=1}^{n} p \binom{d_i}{m} p^m (1-p)^{d_i - m} \\ &\leq p^{m+1} \sum_{i=1}^{n} \binom{\Delta}{m} (1-p)^{\Delta - m} \\ &= p^{m+1}(1-p)^{\Delta - m} c_m n, \end{aligned}$$

where $p \binom{d_i}{m} p^m (1-p)^{d_i - m}$ is the probability of having vertex v_i, $i = 1, \ldots, n$, in the set A_m, $m = k, \ldots, \Delta$. Again, by Lemma 2,

$$\begin{aligned} \mathbf{E}[|B_m|] &= \sum_{i=1}^{n} \mathbf{P}[v_i \in B_m] \\ &= \sum_{i=1}^{n} (1-p) \binom{d_i}{m} p^m (1-p)^{d_i - m} \\ &\leq p^{m} \sum_{i=1}^{n} \binom{\Delta}{m} (1-p)^{\Delta - m + 1} \\ &= p^{m}(1-p)^{\Delta - m + 1} c_m n, \end{aligned}$$

where $(1-p) \binom{d_i}{m} p^m (1-p)^{d_i - m}$ is the probability of having vertex v_i, $i = 1, \ldots, n$, in the set B_m, $m = k+1, \ldots, \Delta$.

Taking into account that $c_{\Delta+1} = \binom{\Delta}{\Delta + 1} = 0$, we obtain

$$\begin{aligned} \mathbf{E}[|X|] &\geq pn - \sum_{m=k}^{\Delta} (m-k+1) p^{m+1} (1-p)^{\Delta - m} c_m n - \sum_{m=k+1}^{\Delta+1} (m-k) p^m (1-p)^{\Delta - m + 1} c_m n \\ &= pn - \sum_{m=0}^{\Delta - k} (m+1) p^{m+k+1} (1-p)^{\Delta - m - k} c_{m+k} n \\ &\qquad - \sum_{m=0}^{\Delta - k} (m+1) p^{m+k+1} (1-p)^{\Delta - m - k} c_{m+k+1} n \end{aligned}$$

$$
\begin{aligned}
&= pn - \sum_{m=0}^{\Delta-k} (m+1) p^{m+k+1} (1-p)^{\Delta-m-k} n \left(c_{m+k} + c_{m+k+1}\right) \\
&= pn - p^{k+1} n \sum_{m=0}^{\Delta-k} (m+1) \tilde{c}_{m+k+1} p^m (1-p)^{\Delta-k-m}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
(m+1)\tilde{c}_{m+k+1} &= \binom{\Delta-k}{m} \frac{(m+1)!(\Delta+1)!}{(m+k+1)!(\Delta-k)!} \\
&\leq \binom{\Delta-k}{m} \frac{(\Delta+1)!}{(k+1)!(\Delta-k)!} = \binom{\Delta-k}{m} \tilde{c}_{k+1}.
\end{aligned}
$$

We obtain, by the binomial theorem,

$$
\begin{aligned}
\mathbf{E}[|X|] &\geq pn - p^{k+1} n \sum_{m=0}^{\Delta-k} \binom{\Delta-k}{m} \tilde{c}_{k+1} p^m (1-p)^{\Delta-k-m} \\
&= pn - p^{k+1} n \tilde{c}_{k+1} \\
&= pn(1 - p^k \tilde{c}_{k+1}) \\
&= \frac{kn}{\tilde{c}_{k+1}^{1/k}\,(1+k)^{1+1/k}}.
\end{aligned}
$$

Since the expectation is an average value, there exists a particular k-limited packing of size at least $\frac{kn}{\tilde{c}_{k+1}^{1/k}\,(1+k)^{1+1/k}}$, as required. The proof of the theorem is complete. □

The lower bound of Theorem 1 can be written in a simpler but weaker form as follows:

Corollary 3. *For any graph G of order n,*

$$
L_k(G) > \frac{kn}{e(1+\Delta)^{1+1/k}}.
$$

Proof. It is not difficult to see that

$$
\tilde{c}_{k+1} \leq \frac{(\Delta+1)^{k+1}}{(k+1)!}
$$

and, using Stirling's formula,

$$
(k!)^{1/k} > \left(\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\right)^{1/k} = \sqrt[2k]{2\pi k}\,\frac{k}{e}.
$$

By Theorem 1,

$$
L_k(G) \geq \frac{kn\left((k+1)!\right)^{1/k}}{(\Delta+1)^{1+1/k}\,(1+k)^{1+1/k}} > \frac{kn}{e(1+\Delta)^{1+1/k}} \times \frac{\sqrt[2k]{2\pi k}\;k}{1+k} > \frac{kn}{e(1+\Delta)^{1+1/k}}.
$$

Note that $\frac{\sqrt[2k]{2\pi k}\,k}{1+k} = \frac{\sqrt[2k]{2\pi k}}{1+1/k} > 1$. The last inequality is obviously true for $k = 1$, while for $k \geq 2$ it can be rewritten in the equivalent form: $2\pi k > (1 + 1/k)^{2k} = e^2 - o(1)$. □

In the case $k = 1$, Theorem 1 gives the following lower bound for the 2-packing (1-limited packing) number:

Corollary 4. *For any graph G of order n with $\Delta \geq 1$,*

$$\rho(G) = L_1(G) \geq \frac{n}{2\Delta(\Delta+1)}. \tag{5}$$

Let $\delta = \delta(G)$ denote the minimum vertex degree in a graph G. The lower bound of Corollary 4 can be improved as follows:

Theorem 5. *For any graph G of order n,*

$$\rho(G) = L_1(G) \geq \frac{n + \Delta(\Delta - \delta)}{\Delta^2 + 1} \geq \frac{n}{\Delta^2 + 1}. \tag{6}$$

Proof. Choose any vertex $v \in V(G)$ of the minimum degree δ in G. Then add v to a set X and remove vertices of $N[N[v]]$ from the graph to obtain $G' = G - N[N[v]]$, where $N[N[v]] = \{w : w \in N[u] \text{ for some } u \in N[v]\}$ is the so-called *second closed neighbourhood* of v in G. Recursively apply the same procedure to the remaining graph G' until it is empty. It is not difficult to see that X is a 1-limited packing (distance 2-packing) of size at least $\left\lceil \frac{n+\Delta(\Delta-\delta)}{\Delta^2+1} \right\rceil$: we remove at most $1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2$ vertices at each iteration, but at most $1 + \delta + \delta(\Delta - 1) = 1 + \delta\Delta$ vertices at the first iteration, and $(1 + \Delta^2) - (1 + \delta\Delta) = \Delta(\Delta - \delta)$. □

The proof of Theorem 5 provides a greedy algorithm to find a distance 2-packing (1-limited packing) satisfying bound (6). We explain later in Section 4 why the lower bound of Theorem 5 is as good as lower bound (5) of Corollary 4 for almost all graphs.

3. Randomized algorithm

A pseudocode presented in Algorithm 1 explicitly describes a randomized algorithm to find a k-limited packing set, whose size satisfies bound (1) with a positive probability. Notice that Algorithm 1 constructs a (preliminary) k-limited packing X' by recursively removing unwanted vertices from a randomly generated set A. This is different from the probabilistic construction used in the proof of Theorem 1. The recursive removal of vertices from the set A may be more effective and efficient, especially if one tries to remove overall as few vertices as possible from A by maximizing intersections of the sets A'_m ($m = k, \ldots, \Delta$) and B'_m ($m = k+1, \ldots, \Delta$).

At the final stage, Algorithm 1 does a (greedy) extension of the preliminary k-limited packing X' derived from the randomly generated set A. Our experimental tests with randomly generated problem instances show the following: although the randomized part of Algorithm 1 may eventually return a preliminary k-limited packing set slightly smaller than lower bound (1), the extension of this set to a maximal k-limited packing always satisfies (1). This is of no surprise, because the expectation of the size of randomly formed set A in Algorithm 1 is $\mathbf{E}[|A|] = pn$, where $p = \left(\binom{\Delta}{k}(\Delta+1)\right)^{-1/k}$, while the expression for lower bound in (1) yields a smaller value:

$$\frac{kn}{\tilde{c}_{k+1}^{1/k}\,(1+k)^{1+1/k}} = \frac{k}{k+1}pn = \frac{k}{k+1}\mathbf{E}[|A|] < \mathbf{E}[|A|].$$

From the experimental tests, an initially formed set A may contain only few redundant vertices to be removed to obtain the preliminary k-limited packing X'. As a result, the preliminary k-limited packing X' in many cases satisfies lower bound (1), and the extension of X' to a maximal k-limited packing X seems to always satisfy (1). In our view, since the problem is NP-hard, Algorithm 1 constitutes a simple efficient approach to tackle the problem in practice and, hopefully, can be useful to solve some hard instances of the problem.

Algorithm 1: Randomized k-limited packing

Input: Graph G and integer k, $1 \le k \le \Delta$.
Output: k-Limited packing X in G.

```
begin
  Compute p = ( 1 / (c̃_{k+1} (1+k)) )^{1/k} ;
  Initialize A = ∅;                                  /* Form a set A ⊆ V(G) */
  foreach vertex v ∈ V(G) do
    with the probability p, decide whether v ∈ A or v ∉ A;
  end
                            /* Recursively remove redundant vertices from A */
  foreach vertex v ∈ V(G) do
    Compute r = |N(v) ∩ A|;
    if v ∈ A and r ≥ k then
      remove any r − k + 1 vertices of N(v) ∩ A from A;
    end
    if v ∉ A and r > k then
      remove any r − k vertices of N(v) ∩ A from A;
    end
  end
  Put X' = A;                              /* X' is a k-limited packing */
  Extend X' to a maximal k-limited packing X;
  return X;
end
```

Algorithm 1 can be implemented to run in $O(n^2)$ time. To compute the probability $p = \left(\binom{\Delta}{k}(\Delta+1)\right)^{-1/k}$, the binomial coefficient $\binom{\Delta}{k}$ can be computed by using the dynamic programming and Pascal's triangle in $O(k\Delta) = O(\Delta^2)$ time using $O(k) = O(\Delta)$ memory. The maximum vertex degree Δ of G can be computed in $O(m)$ time, where m is the number of edges in G. Then p can be computed in $O(m + \Delta^2) = O(n^2)$ steps. It takes $O(n)$ time to find the initial set A. Computing the intersection numbers $r = |N(v) \cap A|$ and removing unwanted vertices of $N(v) \cap A$'s from A can be done in $O(n + m)$ steps. Finally, checking whether X' is maximal or extending X' to a maximal k-limited packing X can be done in $O(n+m)$ time: try to add vertices of $V(G) - X'$ to X' recursively one by one, and check whether the addition of a new vertex $v \in V(G) - X'$ to X' violates the conditions of a k-limited packing for v or at least one of its neighbours in G with respect to $X' \cup \{v\}$. Thus, overall Algorithm 1 takes $O(n^2)$ time, and, since $m = O(n^2)$ in general, it is linear in the graph size $(m + n)$ when $m = \theta(n^2)$.

Also, this randomized algorithm for finding k-limited packings in a graph G can be implemented in parallel or as a local distributed algorithm. As explained in [5], this kind of algorithms are especially important, e.g. in the context of ad hoc and wireless sensor networks. We hope that this approach can be also extended to design self-stabilizing or on-line algorithms for k-limited packings. For example, a self-stabilizing algorithm searching for maximal 2-packings in a distributed network system is presented in [12]. Notice that self-stabilizing algorithms are distributed and fault-tolerant, and use the fact that each node has only a local view/knowledge of the distributed network system. This provides another motivation for efficient distributed search and algorithms to find k-limited packings in graphs and networks.

4. Sharpness of the lower bounds

We now show that the lower bound of Theorem 1 is asymptotically best possible for some values of k. The bound of Theorem 1 can be rewritten in the following form for $\Delta \geq k$:

$$L_k(G) \geq \frac{kn}{(k+1)\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}}.$$

Combining this bound with the upper bound of Lemma 8 from [6], we obtain that for any connected graph G of order n with minimum degree $\delta(G) \geq k$,

$$\frac{1}{\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}} \times \frac{k}{k+1}n \;\leq\; L_k(G) \;\leq\; \frac{k}{k+1}n. \tag{7}$$

Notice that the upper bound in the inequality (7) is sharp (see [6]), so these bounds provide an interval of values for $L_k(G)$ in terms of k and Δ when $k \leq \delta$. For regular graphs, $\delta = \Delta$, and, when $k = \Delta$, we have

$$\frac{1}{\sqrt[k]{\binom{\Delta}{k}(\Delta+1)}} = \frac{1}{(k+1)^{1/k}} \longrightarrow 1 \quad \text{as} \quad k \to \infty.$$

Therefore, the bound of Theorem 1 is asymptotically sharp for regular connected graphs in the case $k = \Delta$. In other words, there are graphs whose k-limited packing number is arbitrarily close to the bound of Theorem 1. Thus, the following result holds:

Theorem 6. *When n is large, there exist graphs G such that*

$$L_k(G) \leq \frac{kn}{\tilde{c}_{k+1}^{1/k}\,(1+k)^{1+1/k}}(1+o(1)). \tag{8}$$

As shown above, the graphs satisfying Theorem 6 contain regular connected ones for $k = \Delta$. This class of graphs can be extended, because it is possible to prove that the bound of Theorem 1 is asymptotically sharp for connected graphs with $k = \Delta(1 - o(1))$, $\delta(G) \geq k$.

Notice that, for regular graphs, the condition $k = \Delta$ and Lemma 5 from [6] imply $L_k(G) = n - \gamma(G)$. Then the classical upper bound (9) for $\gamma(G)$ gives a weaker lower bound for $L_k(G)$ than Theorem 1.

As shown in Theorem 5, in contrast to the situation for relatively 'large' values of k, bound (1) of Theorem 1 (see Corollary 4) can be improved for distance 2-packings (1-limited packings), i.e. when $k = 1$. However, this improvement is irrelevant for almost all graphs. A 1-limited packing set X in G has a very strong property that any two vertices in X are at distance at least 3 in G. It is well known that almost every graph has diameter equal to 2 (e.g., see [10]). Therefore, $\rho(G) = L_1(G) = 1$ for almost all graphs. Thus, in the case $k = 1$, Theorem 1 yields a lower bound of 1 for almost all graphs and is as good as Theorem 5. Notice that the bound of Theorem 5 is sharp, for example, for any number of disjoint copies of the Petersen graph. In the other cases, when G has a diameter larger than 2, one is encouraged to use the greedy algorithm and lower bound (6) provided by Theorem 5, because it improves bound (5) of Corollary 4 by a factor of $2 + o(1)$.

5. Upper bounds

As mentioned earlier, $\rho(G) = L_1(G) \leq \gamma(G)$. In [6], the authors provide several upper bounds for $L_k(G)$, e.g. $L_k(G) \leq k\gamma(G)$ for any graph G. Using the well-known bound (see e.g. [1])

$$\gamma(G) \leq \frac{\ln(\delta+1)+1}{\delta+1}n, \tag{9}$$

we obtain

$$L_k(G) \leq \frac{\ln(\delta+1)+1}{\delta+1}kn. \tag{10}$$

Even though this bound does not work well when k is 'close' to δ, it is very reasonable for small values of k.

We now prove an upper bound for the k-limited packing number in terms of the k-tuple domination number. A set X is called a *k-tuple dominating set* of G if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a k-tuple dominating set of G is the *k-tuple domination number* $\gamma_{\times k}(G)$. The k-tuple domination number is only defined for graphs with $\delta \geq k - 1$.

Theorem 7. *For any graph G of order n with $\delta \geq k-1$,*

$$L_k(G) \leq \gamma_{\times k}(G). \tag{11}$$

Proof. We prove inequality (11) by contradiction. Let X be a maximum k-limited packing in G of size $L_k(G)$, and let Y be a minimum k-tuple dominating set in G of size $\gamma_{\times k}(G)$. We denote $B = X \cap Y$, i.e. $X = A \cup B$ and $Y = B \cup C$, where A and C are disjoint. Assume to the contrary that $L_k(G) > \gamma_{\times k}(G)$, thus $|A| > |C|$.

Since Y is k-tuple dominating set, each vertex of A is adjacent to at least k vertices of Y. Hence the number of edges between A and $B \cup C$ is as follows:

$$e(A, B \cup C) \geq k|A|.$$

Now, every vertex of C is adjacent to at most k vertices of X, because X is a k-limited packing set. Therefore, the number of edges between C and $A \cup B$ satisfies

$$e(C, A \cup B) \leq k|C|.$$

We obtain

$$e(C, A \cup B) \leq k|C| < k|A| \leq e(A, B \cup C),$$

i.e. $e(C, A \cup B) < e(A, B \cup C)$. By eliminating the edges between A and C, we conclude that

$$e(C, B) < e(A, B).$$

Now, let us consider an arbitrary vertex $b \in B$ and denote $s = |N(b) \cap A|$. Since $X = A \cup B$ is a k-limited packing set, we obtain $|N(b) \cap X| \leq k-1$, and hence $|N(b) \cap B| \leq k-s-1$. On the other hand, $Y = B \cup C$ is k-tuple dominating set, so $|N(b) \cap Y| \geq k-1$. Therefore, $|N(b) \cap C| \geq s$. Thus, $|N(b) \cap C| \geq |N(b) \cap A|$ for any vertex $b \in B$. We obtain,

$$e(C, B) \geq e(A, B),$$

a contradiction. We conclude that $L_k(G) \leq \gamma_{\times k}(G)$. □

Notice that it is possible to have $k = \Delta + 1$ in the statement of Theorem 7, which is not covered by Theorem 1. Then $\delta = \Delta$, which implies the graph is regular. However, $L_k(G) = \gamma_{\times k}(G) = n$ for $k = \delta + 1 = \Delta + 1$. In non-regular graphs, $\delta + 1 \leq \Delta$, and $k \leq \Delta$ to satisfy the conditions of Theorem 1 as well.

For $t \leq \delta$, we define

$$\delta' = \delta - k + 1 \quad \text{and} \quad \tilde{b}_t = \tilde{b}_t(G) = \binom{\delta+1}{t}.$$

Using the upper bound for the k-tuple domination number from [5], we obtain:

Corollary 8. *For any graph G with $\delta \geq k$,*

$$L_k(G) \leq \left(1 - \frac{\delta'}{\tilde{b}_{k-1}^{1/\delta'}(1+\delta')^{1+1/\delta'}}\right) n. \tag{12}$$

In some cases, Theorem 1 and Corollary 8 simultaneously provide good bounds for the k-limited packing number. For example, for a 40-regular graph G:

$$0.312n < L_{25}(G) < 0.843n.$$

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