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Abstract

Introducing the approach by Masanao Aoki (1981) to time series econometrics, we show that the dynamics of symmetric linear possibly cointegrated two-country VAR models can be separated into two autonomous subsystems: the country averages and country differences, where the latter includes the exchange rate. The symmetric two-country cointegrated VAR model is synchronized, *i.e.*, the two countries are driven by the same common trends, if and only if the country-differences subsystem is stable. It is shown that separability carries over even under mild asymmetries such as difference in the size of the countries' economies. The possibilities of a recursive structural VECM representation under symmetry is evaluated. The derived conditions for symmetry and separability are easily testable and applied to nine-dimensional quarterly cointegrated VAR models for five different country pairs in the post-Bretton-Woods era. We find evidence for the symmetry of the cointegration space, which is of practical importance as it allows for the identification of the cointegration vectors in much smaller systems, and for the exchange rate equation in general.

Keywords: Multi-country modelling; Cointegration; Common trends; Structural VAR; Synchronization; Exchange rate; International Economics.

JEL classification: C32; C51; F41.

1 A generalization of Aoki (1981)

In this paper we investigate the applicability of the approach by Aoki (1981) frequently used in economic theory (see, *inter alia*, Turnovsky, 1986) for the construction and analysis of dynamic macroeconometric two-country models. Masanao Aoki showed that for a two-country model consisting of a system of stable linear differential equations, the assumption of country symmetry allows to separate the dynamics of the system into two autonomous subsystems of country averages and country differences.

This paper will explore the applicability of Aoki's approach to dynamic macroeconometric modelling, particularly with regard to its contribution to tackling the curse of dimensionality imminent in

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multi-country models due to the large number of variables. The vector autoregressive (VAR) model, the most commonly used modelling approach in empirical macroeconomics, is associated with an inherent problem that is the curse of dimensionality. With every additional variable, the number of parameters in the model to be estimated increases quadratically, leading to an inflation of estimation uncertainty. In practice, this problem is usually avoided by restricting the number of macroeconomic variables in the model. This, however, is not a viable strategy in the two-country context requiring the inclusion of time series for both countries as well as the exchange rate. The cointegrated VAR (CVAR) model offers the advantage of separating the long-run and the short-run dynamics of the macroeconomic system, but further highlights the limitation with regard to the number of variables due to the common problem of identifying the cointegration vectors in high-dimensional models.

From a methodological econometric view point, our approach of separating the analysis of high-dimensional systems into a set of tractable subsystems is most closely related to the integrated CVAR modelling approach of Juselius (2006), the G(lobal) VAR approach of Pesaran et al. (2004) and the infinite-dimensional VAR of Chudik and Pesaran (2011). In the integrated model of Juselius, the long-run structures are analysed separately for different sectors of the economy with the results then feeding into a large macroeconomic system. In the modelling of inflation, the subsystems included are the money market, an external sector and the labour market. This is extended in Tuxen (2007) by the addition of the public sector. In the GVAR approach, cointegrated VAR models are estimated for a large number of countries conditional on some country-specific global variables. The country models are then linked together, usually via a trade matrix. The validity of the GVAR approach crucially depends on the weak exogeneity of those global variables, an assumption that would be incompatible with the two-country system considered in this paper. Aoki's modelling approach could offer an alternative way to overcome the problem of dimensionality in multi-country models.

In the following we seek to develop a theory generalizing Aoki (1981) to the class of symmetric possibly cointegrated two-country linear vector autoregressive (VAR) processes: We generalize Aoki's approach to a stochastic setup in discrete time and demonstrate how it could be utilized for macroeconometric multi-country model building. While the transition from continuous to discrete time is straightforward as far as the concept of symmetry is concerned, the stochastic nature of the model adds a new dimension to the problem. A further major step forward is the consideration of non-stationarities in the system as well as the possible presence of cointegration.

The structure of the paper is as follows. In the next section we discuss the symmetry concept for two-country Vector Autoregressive (VAR) models, which is extended in §3 to two-country cointegrated VAR (CVAR) models and their Vector Equilibrium Correction Mechanism (VECM) representation. For the class of symmetric two-country CVAR models, we introduce in §4 the concept of synchronization, which requires not only symmetry of the parametric model structure of the two economies but also additional common features in form of cotrending. §5 investigates the existence of a recursive structural representation of the symmetric possibly cointegrated VAR model. In §6 the existing framework is extended for weak asymmetries in form of countries of different and possibly time-varying weights in the world economy. §7 then illustrates how the approach developed in the earlier sections can be utilized as a testing strategy for discovering symmetry features, which is applied to small two-country macroeconometric models for five different country pairs. Finally §8 concludes.

2 Symmetry in two-country VAR models

2.1 A balanced two-country VAR model

Suppose we have two countries and our focus is on a balanced set of K key macroeconomic variables for both economies, respectively, and their exchange rate. Let $\boldsymbol{y}_t$ be the $K \times 1$ vector of domestic variables, $\boldsymbol{y}_t^*$ the $K \times 1$ vector of the same variables for the foreign country, and e_t denotes the exchange rate. The full system vector $\boldsymbol{x}_t = (\boldsymbol{y}_t, \boldsymbol{y}_t^*, e_t)'$ is hence $2K+1$ dimensional.

Assuming linearity, the evolution of the two-country macroeconometric model is given by the following system of linear stochastic difference equations constituting a p-th order vector autoregressive process:

$$\begin{bmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_t^* \\ e_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{\nu}_1 \\ \boldsymbol{\nu}_2 \\ \nu_3 \end{bmatrix} + \sum_{i=1}^{p} \begin{bmatrix} \boldsymbol{A}_{11,i} & \boldsymbol{A}_{12,i} & \boldsymbol{a}_{13,i} \\ \boldsymbol{A}_{21,i} & \boldsymbol{A}_{22,i} & \boldsymbol{a}_{23,i} \\ \boldsymbol{a}_{31,i}' & \boldsymbol{a}_{32,i}' & a_{33,i} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_{t-i} \\ \boldsymbol{y}_{t-i}^* \\ e_{t-i} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_{1t} \\ \boldsymbol{\varepsilon}_{2t} \\ \varepsilon_{3t} \end{bmatrix}, \tag{1}$$

where the one-step prediction error $\boldsymbol{\varepsilon}_t$ is a Gaussian vector white noise process:

$$\begin{bmatrix} \boldsymbol{\varepsilon}_{1t} \\ \boldsymbol{\varepsilon}_{2t} \\ \varepsilon_{3t} \end{bmatrix} \sim \mathsf{NID}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\sigma}_{13} \\ \boldsymbol{\Sigma}_{12}' & \boldsymbol{\Sigma}_{22} & \boldsymbol{\sigma}_{23} \\ \boldsymbol{\sigma}_{13}' & \boldsymbol{\sigma}_{23}' & \sigma_{33} \end{bmatrix} \right).$$

For the following analysis, the NID assumption could be relaxed to a more general one such as that of a martingale difference sequence.

2.2 Country-averages-differences representation

With help of the isomorphic transformation of $\boldsymbol{y}_t$ and $\boldsymbol{y}_t^*$ into country averages, $\boldsymbol{y}_t^a = \frac{1}{2}\boldsymbol{y}_t + \frac{1}{2}\boldsymbol{y}_t^*$ and country differences, $\boldsymbol{y}_t^d = \boldsymbol{y}_t - \boldsymbol{y}_t^*$, the linear VAR($p$) in $\boldsymbol{x}_t = (\boldsymbol{y}_t, \boldsymbol{y}_t^*, e_t)'$ can always be rewritten in its country-averages-differences representation, which is the system of stochastic linear difference equations for country-averages, $\boldsymbol{y}_t^a$, country-differences, $\boldsymbol{y}_t^d$, and the exchange rate, e_t:

Proposition 1 *Due to its linearity, the system $\boldsymbol{x}_t = (\boldsymbol{y}_t, \boldsymbol{y}_t^*, e_t)'$ of equation (1) can be mapped to an isomorphic system in $\tilde{\boldsymbol{x}}_t = (\boldsymbol{y}_t^a, \boldsymbol{y}_t^d, e_t)'$. The VAR(p) process governing $\tilde{\boldsymbol{x}}_t$ is then given by:*

$$\tilde{\boldsymbol{x}}_t = \boldsymbol{K}\boldsymbol{x}_t = \tilde{\boldsymbol{\nu}} + \sum_{i=1}^{p} \tilde{\boldsymbol{A}}_i \tilde{\boldsymbol{x}}_{t-i} + \tilde{\boldsymbol{\varepsilon}}_t, \text{ with } \tilde{\boldsymbol{\varepsilon}}_t \sim \mathsf{NID}\left(\boldsymbol{0}, \tilde{\boldsymbol{\Sigma}}\right), \tag{2}$$

where $\tilde{\boldsymbol{\nu}} = \boldsymbol{K}\boldsymbol{\nu}$, $\tilde{\boldsymbol{A}}_i = \boldsymbol{K}\boldsymbol{A}_i\boldsymbol{K}^{-1}$, $\tilde{\boldsymbol{\varepsilon}}_t = \boldsymbol{K}\boldsymbol{\varepsilon}_t$ and $\tilde{\boldsymbol{\Sigma}} = \boldsymbol{K}\boldsymbol{\Sigma}\boldsymbol{K}'$. □

PROOF The result follows from $\tilde{\boldsymbol{x}}_t$ being a linear isomorphic transform of $\boldsymbol{x}_t$:

$$\tilde{\boldsymbol{x}}_t = (\boldsymbol{y}_t^a, \boldsymbol{y}_t^d, e_t)' = \boldsymbol{K}(\boldsymbol{y}_t, \boldsymbol{y}_t^*, e_t)' = \boldsymbol{K}\boldsymbol{x}_t, \tag{3}$$

where the $(2K+1) \times (2K+1)$ communication matrix $\boldsymbol{K}$ is of full rank:

$$\boldsymbol{K} = \begin{bmatrix} \frac{1}{2}\boldsymbol{I} & \frac{1}{2}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0}' & \boldsymbol{0}' & 1 \end{bmatrix}, \text{ with its inverse being } \boldsymbol{K}^{-1} = \begin{bmatrix} \boldsymbol{I} & \frac{1}{2}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & -\frac{1}{2}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0}' & \boldsymbol{0}' & 1 \end{bmatrix}. \tag{4}$$

The representation in (2) then results from substituting $\boldsymbol{x}_t = \boldsymbol{K}^{-1}\tilde{\boldsymbol{x}}_t$ in (1) and premultiplying by $\boldsymbol{K}$. ■

While the representation above is in general of limited interest on its own, it becomes crucially important when the two-country model is symmetric, which will be considered in the next section.

2.3 Symmetric two-country model

Definition 1 The two-country VAR model in (1) is called **symmetric** if and only if the following conditions are satisfied with regard to the dynamic,

$$\begin{aligned} \boldsymbol{A}_{11,i} &= \boldsymbol{A}_{22,i} = \boldsymbol{A}_i, & \text{(5a)}\\ \boldsymbol{A}_{12,i} &= \boldsymbol{A}_{21,i} = \boldsymbol{A}_i^*, & \text{(5b)}\\ \boldsymbol{a}_{13,i} &= -\boldsymbol{a}_{23,i} = \boldsymbol{a}_i, & \text{(5c)}\\ \boldsymbol{a}_{31,i} &= -\boldsymbol{a}_{32,i} = \boldsymbol{h}_i; & \text{(5d)} \end{aligned}$$

and contemporaneous properties of the VAR,

$$\begin{aligned} \boldsymbol{\Sigma}_{11} &= \boldsymbol{\Sigma}_{22} = \boldsymbol{\Sigma}, & \text{(6a)}\\ \boldsymbol{\Sigma}_{12} &= \boldsymbol{\Sigma}_{12}' = \boldsymbol{\Sigma}^*, & \text{(6b)}\\ \boldsymbol{\sigma}_{13} &= -\boldsymbol{\sigma}_{23} = \boldsymbol{\sigma}. & \text{(6c)} \end{aligned}$$

□

Remark 1 Note that the definition of symmetry does not impose restrictions on the deterministic terms.□

Lemma 1 *The symmetric two-country VAR model is characterized by the following parametric structure:*

$$\begin{bmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_t^* \\ e_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{\nu}_1 \\ \boldsymbol{\nu}_2 \\ \nu_3 \end{bmatrix} + \sum_{i=1}^{p} \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{A}_i^* & \boldsymbol{a}_i \\ \boldsymbol{A}_i^* & \boldsymbol{A}_i & -\boldsymbol{a}_i \\ \boldsymbol{h}_i' & -\boldsymbol{h}_i' & h_i \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_{t-i} \\ \boldsymbol{y}_{t-i}^* \\ e_{t-i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}, \tag{7}$$

$$\textit{where} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix} \sim \mathsf{NID}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Sigma}^* & \boldsymbol{\sigma} \\ \boldsymbol{\Sigma}^* & \boldsymbol{\Sigma} & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma}' & -\boldsymbol{\sigma}' & \sigma^2 \end{bmatrix} \right).$$

□

PROOF The Lemma follows directly from imposing the restrictions in (5)-(6) on the system in (1)-(2).■

The next proposition confirms that the result of Aoki (1981) derived for a system of linear differential equations holds for a system of linear stochastic difference equations: Under symmetry of the two economies and the feedbacks of the exchange rate, the laws of motion of the $2K+1$ dimensional two-country model can be studied in two separate autonomous dynamic subsystems of lower dimension: the K dimensional subsystem of country averages, $\tilde{\boldsymbol{x}}_t^a \equiv \boldsymbol{y}_t^a$, and the $K+1$ dimensional subsystem of country differences and the exchange rate, $\tilde{\boldsymbol{x}}_t^d \equiv (\boldsymbol{y}_t^{d\prime}, e_t)'$.

Proposition 2 *The symmetric two-country VAR model in (7) is given by:*

$$\begin{bmatrix} \boldsymbol{y}_t^a \\ \boldsymbol{y}_t^d \\ e_t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\boldsymbol{\nu}_1+\boldsymbol{\nu}_2) \\ \boldsymbol{\nu}_1-\boldsymbol{\nu}_2 \\ \nu_3 \end{bmatrix} + \sum_{i=1}^{p} \begin{bmatrix} \boldsymbol{A}_i+\boldsymbol{A}_i^* & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_i-\boldsymbol{A}_i^* & 2\boldsymbol{a}_i \\ \boldsymbol{0}' & \boldsymbol{h}_i' & h_i \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_{t-i}^a \\ \boldsymbol{y}_{t-i}^d \\ e_{t-i} \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{\varepsilon}}_{1t} \\ \tilde{\boldsymbol{\varepsilon}}_{2t} \\ \varepsilon_{3t} \end{bmatrix}, \tag{8}$$

$$\text{where} \begin{bmatrix} \tilde{\boldsymbol{\varepsilon}}_{1t} \\ \tilde{\boldsymbol{\varepsilon}}_{2t} \\ \varepsilon_{3t} \end{bmatrix} \sim \mathsf{NID}\left(\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2}(\boldsymbol{\Sigma}+\boldsymbol{\Sigma}^*) & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 2(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}^*) & 2\boldsymbol{\sigma} \\ \boldsymbol{0}' & 2\boldsymbol{\sigma}' & \sigma^2 \end{bmatrix} \right),$$

where $\tilde{\boldsymbol{\varepsilon}}_{1t} = \frac{1}{2}\boldsymbol{\varepsilon}_{1t} + \frac{1}{2}\boldsymbol{\varepsilon}_{2t}$ *and* $\tilde{\boldsymbol{\varepsilon}}_{2t} = \boldsymbol{\varepsilon}_{1t} - \boldsymbol{\varepsilon}_{2t}$. □

PROOF The result follows straightforwardly from the representation theorem in Proposition 1 with

$$\tilde{\boldsymbol{\Sigma}} = \boldsymbol{K}\boldsymbol{\Sigma}\boldsymbol{K}' = \begin{bmatrix} \frac{1}{2}(\boldsymbol{\Sigma}+\boldsymbol{\Sigma}^*) & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 2(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}^*) & 2\boldsymbol{\sigma} \\ \boldsymbol{0}' & 2\boldsymbol{\sigma}' & \sigma^2 \end{bmatrix},$$

$$\tilde{\boldsymbol{A}}_i = \boldsymbol{K}\boldsymbol{A}_i\boldsymbol{K}^{-1} = \begin{bmatrix} \boldsymbol{A}_i+\boldsymbol{A}_i^* & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_i-\boldsymbol{A}_i^* & 2\boldsymbol{a}_i \\ \boldsymbol{0}' & \boldsymbol{h}_i' & h_i \end{bmatrix}. \qquad \blacksquare$$

The block-diagonal structure of $\tilde{\boldsymbol{\Sigma}}$ and $\tilde{\boldsymbol{A}}_i, i = 1,\ldots,p$, leads directly to the separability result stated in Corollary 1:

Corollary 1 *For the symmetric two-country VAR model in (7), the dynamics of the corresponding* $\tilde{\boldsymbol{x}}_t$ *can be separated in two autonomous dynamic subsystems of* $\tilde{\boldsymbol{x}}_t^a = \boldsymbol{y}_t^a$ *and* $\tilde{\boldsymbol{x}}_t^d = (\boldsymbol{y}_t^{d\prime}, e_t)'$:

$$\tilde{\boldsymbol{x}}_t^a = \boldsymbol{\nu}^a + \sum_{i=1}^{p} \boldsymbol{A}_i^a \tilde{\boldsymbol{x}}_{t-i}^a + \tilde{\boldsymbol{\varepsilon}}_t^a, \quad \tilde{\boldsymbol{\varepsilon}}_t^a \sim \mathsf{NID}(\boldsymbol{0}, \boldsymbol{\Sigma}^a), \tag{9}$$

$$\tilde{\boldsymbol{x}}_t^d = \boldsymbol{\nu}^d + \sum_{i=1}^{p} \boldsymbol{A}_i^d \tilde{\boldsymbol{x}}_{t-i}^d + \tilde{\boldsymbol{\varepsilon}}_t^d, \quad \tilde{\boldsymbol{\varepsilon}}_t^d \sim \mathsf{NID}\left(\boldsymbol{0}, \boldsymbol{\Sigma}^d\right), \tag{10}$$

with $\tilde{\boldsymbol{\varepsilon}}_t^a = \frac{1}{2}(\boldsymbol{\varepsilon}_{1t} + \boldsymbol{\varepsilon}_{2t})$ *and* $\tilde{\boldsymbol{\varepsilon}}_t^d = (\boldsymbol{\varepsilon}_{1t}' - \boldsymbol{\varepsilon}_{2t}', \varepsilon_{3t})'$ *being mutually independent.* □

PROOF The corollary follows directly from Proposition 2 with

$$\boldsymbol{\nu}^a = \frac{1}{2}(\boldsymbol{\nu}_1+\boldsymbol{\nu}_2),\ \boldsymbol{A}_i^a = \boldsymbol{A}_i + \boldsymbol{A}_i^*,\ \boldsymbol{\Sigma}^a = \frac{1}{2}(\boldsymbol{\Sigma}+\boldsymbol{\Sigma}^*),$$

$$\boldsymbol{\nu}^d = \begin{bmatrix} \boldsymbol{\nu}_1-\boldsymbol{\nu}_2 \\ \nu_3 \end{bmatrix},\ \boldsymbol{A}_i^d = \begin{bmatrix} \boldsymbol{A}_i-\boldsymbol{A}_i^* & 2\boldsymbol{a}_i \\ \boldsymbol{h}_i' & h_i \end{bmatrix},\ \boldsymbol{\Sigma}^d = \begin{bmatrix} 2(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}^*) & 2\boldsymbol{\sigma} \\ 2\boldsymbol{\sigma}' & \sigma^2 \end{bmatrix}. \qquad \blacksquare$$

Remark 2 If the two-country model is symmetric with regard to the deterministic terms, all variables in the country-difference system except the exchange rate have zero mean: $\tilde{\boldsymbol{\nu}}^d = 0$. □

3 Two-country cointegrated VAR model

With Proposition 2 and Corollary 1 we have extended the results of Masanao Aoki to a system of now *stochastic* linear differential equations. However, to confront the non-stationarities inherent in capitalistic economies, we allow in the following for common stochastic trends. For sake of simplicity, we will restrict our analysis on processes where $\boldsymbol{x}_t$ is integrated of order 1, $\boldsymbol{x}_t \sim \mathsf{I}(1)$, such that $\Delta\boldsymbol{x}_t$ is stationary while $\boldsymbol{x}_t$ is nonstationary. The focus is on the presence of cointegration relationships, which can be interpreted as the long-run equilibrium of the system.

3.1 Vector Equilibrium Correction Mechanism Representation

The VAR(p) in (1) can be rewritten as a VECM($p-1$), which decomposes the dynamics of the system into adjustments towards the long-run equilibrium and short-run momentum:

$$\Delta\boldsymbol{x}_t = \boldsymbol{\nu} + \boldsymbol{\Pi}\boldsymbol{x}_{t-1} + \sum_{i=1}^{p-1} \boldsymbol{\Gamma}_i \Delta\boldsymbol{x}_{t-i} + \varepsilon_t, \tag{11}$$

where $\boldsymbol{\Pi} = \sum_{i=1}^{p} \boldsymbol{A}_i - \boldsymbol{I}$, $\boldsymbol{\Gamma}_i = -\sum_{j=i+1}^{p} \boldsymbol{A}_j$ for $i = 1,\dots,p-1$, and ε_t is the innovation process in (1). The VECM has the following parametric structure:

$$\begin{bmatrix}\Delta\boldsymbol{y}_t\\ \Delta\boldsymbol{y}^*_t\\ \Delta e_t\end{bmatrix} = \begin{bmatrix}\boldsymbol{\nu}_1\\ \boldsymbol{\nu}_2\\ \nu_3\end{bmatrix} + \begin{bmatrix}\boldsymbol{\Pi}_{11,i} & \boldsymbol{\Pi}_{12,i} & \boldsymbol{\pi}_{13,i}\\ \boldsymbol{\Pi}_{21,i} & \boldsymbol{\Pi}_{22,i} & \boldsymbol{\pi}_{23,i}\\ \boldsymbol{\pi}'_{31,i} & \boldsymbol{\pi}'_{32,i} & \pi_{33,i}\end{bmatrix} \begin{bmatrix}\boldsymbol{y}_{t-1}\\ \boldsymbol{y}^*_{t-1}\\ e_{t-1}\end{bmatrix} + \sum_{i=1}^{p-1} \begin{bmatrix}\boldsymbol{\Gamma}_{11,i} & \boldsymbol{\Gamma}_{12,i} & \boldsymbol{\gamma}_{13,i}\\ \boldsymbol{\Gamma}_{21,i} & \boldsymbol{\Gamma}_{22,i} & \boldsymbol{\gamma}_{23,i}\\ \boldsymbol{\gamma}'_{31,i} & \boldsymbol{\gamma}'_{32,i} & \gamma_{33,i}\end{bmatrix} \begin{bmatrix}\Delta\boldsymbol{y}_{t-i}\\ \Delta\boldsymbol{y}^*_{t-i}\\ \Delta e_{t-i}\end{bmatrix} + \begin{bmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\\ \varepsilon_{3t}\end{bmatrix}. \tag{12}$$

The number of cointegrating relations is given by the rank r of the matrix $\boldsymbol{\Pi}$, where it is assumed that $0 < r < 2K+1$.

3.2 Symmetry conditions

The symmetry conditions on $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}_i$ are equivalent to those imposed on $\boldsymbol{A}_i$ in (5):

Definition 2 The two-country VECM($p-1$) model in (11) is called **symmetric** if and only if the following conditions are satisfied with regard to all its three features:

(i) Equilibrium correction:

$$\boldsymbol{\Pi}_{11} = \boldsymbol{\Pi}_{22} = \boldsymbol{\Pi}, \tag{13a}$$
$$\boldsymbol{\Pi}_{12} = \boldsymbol{\Pi}_{21} = \boldsymbol{\Pi}^*, \tag{13b}$$
$$\boldsymbol{\pi}_{13} = -\boldsymbol{\pi}_{23} = \boldsymbol{\pi}_{ye}, \tag{13c}$$
$$\boldsymbol{\pi}_{31} = -\boldsymbol{\pi}_{32} = \boldsymbol{\pi}_{ey}. \tag{13d}$$

(ii) Short-run dynamics ($i = 1,\dots,p-1$):

$$\boldsymbol{\Gamma}_{11,i} = \boldsymbol{\Gamma}_{22,i} = \boldsymbol{\Gamma}_i, \tag{14a}$$
$$\boldsymbol{\Gamma}_{12,i} = \boldsymbol{\Gamma}_{21,i} = \boldsymbol{\Gamma}^*_i, \tag{14b}$$
$$\boldsymbol{\gamma}_{13,i} = -\boldsymbol{\gamma}_{23,i} = \boldsymbol{\gamma}_{ye,i}, \tag{14c}$$
$$\boldsymbol{\gamma}_{31,i} = -\boldsymbol{\gamma}_{32,i} = \boldsymbol{\gamma}_{ey,i}. \tag{14d}$$

(iii) Contemporaneous structure: restrictions as defined in (6). □

Remark 3 Under the symmetry conditions in (13), the long-run matrix has the following design:

$$\begin{bmatrix} \mathbf{\Pi}_{11} & \mathbf{\Pi}_{12} & \boldsymbol{\pi}_{13} \\ \mathbf{\Pi}_{21} & \mathbf{\Pi}_{22} & \boldsymbol{\pi}_{23} \\ \boldsymbol{\pi}'_{31} & \boldsymbol{\pi}'_{32} & \pi_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi} & \mathbf{\Pi}^* & \boldsymbol{\pi}_{ye} \\ \mathbf{\Pi}^* & \mathbf{\Pi} & -\boldsymbol{\pi}_{ye} \\ \boldsymbol{\pi}'_{ey} & -\boldsymbol{\pi}'_{ey} & \pi_{ee} \end{bmatrix}. \tag{15}$$

□

3.3 Cointegration under symmetry

For the symmetric two-country model, the cointegration properties of the system are directly linked to that of its country-averages-differences representation.

Lemma 2 *As stated by Corollary 1, the dynamics of the $\tilde{\boldsymbol{x}}_t$ can be separated in two autonomous dynamic subsystems of $\tilde{\boldsymbol{x}}^a_t = \boldsymbol{y}^a_t$ and $\tilde{\boldsymbol{x}}^d_t = (\boldsymbol{y}^{d\prime}_t, e_t)'$. For the corresponding VECM($p-1$) representation we get:*

$$\begin{bmatrix} \Delta\tilde{\boldsymbol{x}}^a_t \\ \Delta\tilde{\boldsymbol{x}}^d_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{\nu}^a \\ \boldsymbol{\nu}^d \end{bmatrix} + \begin{bmatrix} \mathbf{\Pi}^a & \mathbf{0} \\ \mathbf{0} & \mathbf{\Pi}^d \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}^a_{t-1} \\ \tilde{\boldsymbol{x}}^d_{t-1} \end{bmatrix} + \sum_{i=1}^{p-1} \begin{bmatrix} \mathbf{\Gamma}^a & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}^d \end{bmatrix} \begin{bmatrix} \Delta\tilde{\boldsymbol{x}}^a_{t-i} \\ \Delta\tilde{\boldsymbol{x}}^d_{t-i} \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{\varepsilon}}^a_t \\ \tilde{\boldsymbol{\varepsilon}}^d_t \end{bmatrix}, \tag{16}$$

where $\tilde{\boldsymbol{\varepsilon}}^a_t = \frac{1}{2}(\boldsymbol{\varepsilon}_{1t} + \boldsymbol{\varepsilon}_{2t})$ and $\tilde{\boldsymbol{\varepsilon}}^d_t = (\boldsymbol{\varepsilon}'_{1t} - \boldsymbol{\varepsilon}'_{2t}, \varepsilon_{3t})'$. □

The system in (16) determines the cointegration rank and cointegration space of the two-country model. Let $r^a = \text{rank}(\mathbf{\Pi}^a) \leq K$ and $r^d = \text{rank}(\mathbf{\Pi}^d) \leq K+1$ with $0 < r^a + r^d < 2K+1$. Then $\mathbf{\Pi}^a = \boldsymbol{\alpha}^a \boldsymbol{\beta}^{a\prime}$ and $\mathbf{\Pi}^d = \boldsymbol{\alpha}^d \boldsymbol{\beta}^{d\prime}$, where $\boldsymbol{\alpha}^a$ and $\boldsymbol{\beta}^a$ are $K \times r^a$ matrices of rank r^a and $\boldsymbol{\alpha}^d$ and $\boldsymbol{\beta}^d$ are $(K+1) \times r^d$ matrices of rank r^d.

Proposition 3 *For the symmetric two-country CVAR model, the cointegration rank r is determined by the sum of the number of stable long-run relations in the country-average subsystem, $\boldsymbol{y}^a_t$, and in the country-differences subsystem $\tilde{\boldsymbol{x}}^d_t = (\boldsymbol{y}^{d\prime}_t, e_t)'$: $r = \text{rank}(\mathbf{\Pi}) = r^a + r^d$.* □

PROOF The rank of the block-diagonal long-run matrix associated with $\tilde{\boldsymbol{x}}_t$,

$$\tilde{\mathbf{\Pi}} = \begin{bmatrix} \mathbf{\Pi}^a & \mathbf{0} \\ \mathbf{0} & \mathbf{\Pi}^d \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}^a & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha}^d \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}^{a\prime} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta}^{d\prime} \end{bmatrix} = \tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\beta}}', \tag{17}$$

is given by the sum of the ranks of the matrices on the diagonal: $\text{rank}(\tilde{\mathbf{\Pi}}) = \text{rank}(\mathbf{\Pi}^a) + \text{rank}(\mathbf{\Pi}^d) = r^a + r^d$. Furthermore, since $\mathbf{\Pi} = \boldsymbol{K}^{-1}\tilde{\mathbf{\Pi}}\boldsymbol{K} = \boldsymbol{K}^{-1}\tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\beta}}'\boldsymbol{K}$ with $\boldsymbol{K}$ being of full rank, we have that $\text{rank}(\mathbf{\Pi}) = \text{rank}(\tilde{\mathbf{\Pi}}) = r$. ■

Cointegration relations are invariant to linear transformation, allowing to rewrite $\boldsymbol{\beta}^{d\prime}$ as in the following proposition.

Proposition 4 *Suppose that the exchange rate cointegrates with the country differences, so that $\boldsymbol{\beta}^{d\prime}$ can be normalized as follows with $\boldsymbol{\alpha}^d$ being partitioned accordingly:*

$$\boldsymbol{\beta}^{d\prime} = \begin{bmatrix} \boldsymbol{\beta}^{d\,\prime}_{yy} & \mathbf{0} \\ \boldsymbol{\beta}^{d\,\prime}_{ey} & 1 \end{bmatrix} \text{ and } \boldsymbol{\alpha}^d = \begin{bmatrix} \boldsymbol{\alpha}^d_{yy} & \boldsymbol{\alpha}^d_{ye} \\ \boldsymbol{\alpha}^d_{ey} & \boldsymbol{\alpha}^d_{ee} \end{bmatrix}. \tag{18}$$

Then the loading matrix, $\boldsymbol{\alpha}$, and cointegration matrix, $\boldsymbol{\beta}'$, of the symmetric two-country VAR model are given by:

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}^a & \frac{1}{2}\boldsymbol{\alpha}^d_{yy} & \frac{1}{2}\boldsymbol{\alpha}^d_{ye} \\ \boldsymbol{\alpha}^a & -\frac{1}{2}\boldsymbol{\alpha}^d_{yy} & -\frac{1}{2}\boldsymbol{\alpha}^d_{ye} \\ \mathbf{0} & \boldsymbol{\alpha}^d_{ey} & \boldsymbol{\alpha}^d_{ee} \end{bmatrix}, \quad \boldsymbol{\beta}' = \begin{bmatrix} \frac{1}{2}\boldsymbol{\beta}^{a\prime} & \frac{1}{2}\boldsymbol{\beta}^{a\prime} & \mathbf{0} \\ \boldsymbol{\beta}^{d\,\prime}_{yy} & -\boldsymbol{\beta}^{d\,\prime}_{yy} & \mathbf{0} \\ \boldsymbol{\beta}^{d\,\prime}_{ey} & -\boldsymbol{\beta}^{d\,\prime}_{ey} & 1 \end{bmatrix}, \tag{19}$$

and the long-run matrix $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$ results as:

$$\boldsymbol{\Pi} = \begin{bmatrix} \frac{1}{2}\boldsymbol{\alpha}^a\boldsymbol{\beta}^{a\prime} + \frac{1}{2}\left(\boldsymbol{\alpha}^d_{yy}\boldsymbol{\beta}^{d\,\prime}_{yy} + \boldsymbol{\alpha}^d_{ye}\boldsymbol{\beta}^{d\,\prime}_{ey}\right) & \frac{1}{2}\boldsymbol{\alpha}^a\boldsymbol{\beta}^{a\prime} - \frac{1}{2}\left(\boldsymbol{\alpha}^d_{yy}\boldsymbol{\beta}^{d\,\prime}_{yy} + \boldsymbol{\alpha}^d_{ye}\boldsymbol{\beta}^{d\,\prime}_{ey}\right) & \frac{1}{2}\boldsymbol{\alpha}^d_{ye} \\ \frac{1}{2}\boldsymbol{\alpha}^a\boldsymbol{\beta}^{a\prime} - \frac{1}{2}\left(\boldsymbol{\alpha}^d_{yy}\boldsymbol{\beta}^{d\,\prime}_{yy} + \boldsymbol{\alpha}^d_{ye}\boldsymbol{\beta}^{d\,\prime}_{ey}\right) & \frac{1}{2}\boldsymbol{\alpha}^a\boldsymbol{\beta}^{a\prime} + \frac{1}{2}\left(\boldsymbol{\alpha}^d_{yy}\boldsymbol{\beta}^{d\,\prime}_{yy} + \boldsymbol{\alpha}^d_{ye}\boldsymbol{\beta}^{d\,\prime}_{ey}\right) & -\frac{1}{2}\boldsymbol{\alpha}^d_{ye} \\ \boldsymbol{\alpha}^d_{ey}\boldsymbol{\beta}^{d\,\prime}_{yy} + \boldsymbol{\alpha}^d_{ee}\boldsymbol{\beta}^{d\,\prime}_{ey} & -\left(\boldsymbol{\alpha}^d_{ey}\boldsymbol{\beta}^{d\,\prime}_{yy} + \boldsymbol{\alpha}^d_{ee}\boldsymbol{\beta}^{d\,\prime}_{ey}\right) & \boldsymbol{\alpha}^d_{ee} \end{bmatrix}. \tag{20}$$

□

PROOF The results follow straightforwardly from the design of matrices $\tilde{\boldsymbol{\alpha}}$ and $\tilde{\boldsymbol{\beta}}$ as defined in (17) and (18) with the usual mapping from $\tilde{\boldsymbol{x}}$ to $\boldsymbol{x}$:

$$\boldsymbol{\Pi} = \boldsymbol{K}^{-1}\tilde{\boldsymbol{\Pi}}\boldsymbol{K} = \boldsymbol{K}^{-1}\tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\beta}}'\boldsymbol{K} = \left(\boldsymbol{K}^{-1}\tilde{\boldsymbol{\alpha}}\right)\left(\tilde{\boldsymbol{\beta}}'\boldsymbol{K}\right) = \boldsymbol{\alpha}\boldsymbol{\beta}'. \tag{21}$$

■

It is worth noting that the $\boldsymbol{\Pi}$ matrix in (20) is consistent with the the symmetry conditions in (13) and the structure of the $\boldsymbol{\Pi}$ matrix in Remark 3. Also note that the derivation delivers as a byproduct the loading and cointegration matrix of the two-country model, which are unique conditional on the usual normalization constraints.

Remark 4 If the exchange rate does not cointegrate with the country differences, $\boldsymbol{\beta}^{d\prime}$ collapses to:

$$\boldsymbol{\beta}^{d\prime} = \begin{bmatrix} \boldsymbol{\beta}^{d\,\prime}_{yy} & \mathbf{0} \end{bmatrix},$$

and the long-run matrix $\boldsymbol{\Pi}$ simplifies to:

$$\boldsymbol{\Pi} = \begin{bmatrix} \frac{1}{2}\left(\boldsymbol{\alpha}^a\boldsymbol{\beta}^{a\prime} + \boldsymbol{\alpha}^d_{yy}\boldsymbol{\beta}^{d\,\prime}_{yy}\right) & \frac{1}{2}\left(\boldsymbol{\alpha}^a\boldsymbol{\beta}^{a\prime} - \boldsymbol{\alpha}^d_{yy}\boldsymbol{\beta}^{d\,\prime}_{yy}\right) & \mathbf{0} \\ \frac{1}{2}\left(\boldsymbol{\alpha}^a\boldsymbol{\beta}^{a\prime} - \boldsymbol{\alpha}^d_{yy}\boldsymbol{\beta}^{d\,\prime}_{yy}\right) & \frac{1}{2}\left(\boldsymbol{\alpha}^a\boldsymbol{\beta}^{a\prime} + \boldsymbol{\alpha}^d_{yy}\boldsymbol{\beta}^{d\,\prime}_{yy}\right) & \mathbf{0} \\ \boldsymbol{\alpha}^d_{ey}\boldsymbol{\beta}^{d\,\prime}_{yy} & -\boldsymbol{\alpha}^d_{ey}\boldsymbol{\beta}^{d\,\prime}_{yy} & 0 \end{bmatrix}. \tag{22}$$

□

4 Synchronisation under symmetry

The theoretical analysis undertaken in this paper has so far focussed on the VAR and VECM representations of the two-country model. The results derived in the previous section for the properties of the two autonomous subsystems $\tilde{\boldsymbol{x}}^a_t$ and $\tilde{\boldsymbol{x}}^d_t$, have strong but straightforward implications for the Granger representation of cointegrated two-country VAR under symmetry:

Proposition 5 *For the symmetric two-country CVAR model, the common stochastic trends of the subsystems $\tilde{x}^a_t$ and $\tilde{x}^d_t$, $\boldsymbol{\tau}^a_t = \boldsymbol{\alpha}^{a\prime}_{\perp}\sum_{s=1}^{t}\tilde{\varepsilon}^a_s$ and $\boldsymbol{\tau}^d_t = \boldsymbol{\alpha}^{d\prime}_{\perp}\sum_{s=1}^{t}\tilde{\varepsilon}^d_s$, are orthogonal to each other.* □

PROOF The orthogonality of the stochastic trends follows from the orthogonality of the error processes under symmetry:

$$\begin{aligned}
\mathsf{E}\left[\boldsymbol{\tau}_t^a \boldsymbol{\tau}_t^{d\prime}\right] &= \mathsf{E}\left[\boldsymbol{\alpha}_\perp^{a\prime}\left(\sum_{s=1}^{t}\tilde{\varepsilon}_s^a\right)\left(\sum_{s=1}^{t}\tilde{\varepsilon}_s^d\right)'\boldsymbol{\alpha}_\perp^d\right] \\
&= \mathsf{E}\left[\boldsymbol{\alpha}_\perp^{a\prime}\left(\sum_{s=1}^{t}\sum_{k=1}^{t}\tilde{\varepsilon}_s^a\tilde{\varepsilon}_k^{d\prime}\right)\boldsymbol{\alpha}_\perp^d\right] \\
&= \boldsymbol{\alpha}_\perp^{a\prime}\left(\sum_{s=1}^{t}\sum_{k=1}^{t}\mathsf{E}\left[\tilde{\varepsilon}_s^a\tilde{\varepsilon}_k^{d\prime}\right]\right)\boldsymbol{\alpha}_\perp^d = \mathbf{0},
\end{aligned} \tag{23}$$

where we use the result from Proposition 2 that $\mathsf{E}\left[\tilde{\varepsilon}_s^a\tilde{\varepsilon}_k^{d\prime}\right] = \mathbf{0}$. ■

Based on Proposition 5, we can now introduce the concept of synchronisation, which – based on the Granger moving-average representation of the system – offers additional insight into the symmetry of the two-country system with regard to its common stochastic trends. Synchronization in symmetric two-country cointegrated VAR models requires not only symmetry of the parametric model structure of the two economies but also additional common features in form of cotrending. The technical analysis is most closely related to the work of Konishi et al. (1993) and Granger and Haldrup (1997) for the one-country case.

Under symmetry, the home and foreign country are governed by the same law of motion. The parametric equivalence, however, does not guarantee that the two economies will share the same common trends, *i.e.* growing along each other.

Example 1 Suppose that $\boldsymbol{\beta}_o$ is a cointegration vector of the country-average subsystem, such that $\boldsymbol{\beta}_o'\boldsymbol{y}_t^a \sim \mathsf{I}(0)$, but that $\begin{bmatrix} \boldsymbol{\beta}_o & 0 \end{bmatrix}'\boldsymbol{x}_t^d$ lies not in the cointegration space of the country-differences subsystem. Then the two economies are diverging, *i.e.* $\boldsymbol{\beta}_o'\boldsymbol{y}_t^d \sim \mathsf{I}(1)$, due to their stochastic trends drifting in opposite directions. □

Note that the notion of synchronisation refers to the case where the two countries, $\boldsymbol{y}_t$ and $\boldsymbol{y}_t^*$, share the same common trends such that the country differences are converging: $\boldsymbol{y}_t - \boldsymbol{y}_t^* \sim \mathsf{I}(0)$.

Definition 3 The symmetric two-country cointegrated VAR model is called **synchronized** when the two countries share the same common trends. □

The definition above does not require the exchange rate dynamics to be stable, though it could be extended accordingly. For cotrending, the country-differences subsystem, $\boldsymbol{y}_t^d$, needs to be $\mathsf{I}(0)$, while the country averages, $\boldsymbol{y}_t^a$, may be $\mathsf{I}(1)$. Thus, the two economies $\boldsymbol{y}_t$ and $\boldsymbol{y}_t^*$ are $\mathsf{I}(d)$ with $0 \leq d \leq 1$.

Proposition 6 *The symmetric two-country cointegrated VAR model is synchronized if and only if the country-differences subsystem is stable,* $r^d = K+1$*, or it exhibits a single unit root isolated to the exchange rate equation,* $r^d = K$ *and* $\boldsymbol{\alpha}_\perp^d = \begin{bmatrix} \mathbf{0}' & \alpha_{ee\perp}^d \end{bmatrix}'$*, where* $\boldsymbol{\alpha}_\perp^d$ *is the orthogonal complement of* $\boldsymbol{\alpha}^d$. □

PROOF The symmetry of the stochastic trends in stochastic two-country models depends on the stability of the subsystem in country differences, $\tilde{\boldsymbol{x}}_t^d$. Let $\boldsymbol{\beta}^d$ have full rank, $r^d = K+1$. Then $\boldsymbol{\alpha}^d$ has full rank and there does not exist a stochastic trend in the country-differences subsystem, $\tilde{\boldsymbol{x}}_t^d$. For country-averages-differences system, $\tilde{\boldsymbol{x}}_t$, we then have

$$\tilde{\boldsymbol{\beta}}' = \begin{bmatrix} \boldsymbol{\beta}^{a\prime} & \mathbf{0}' \end{bmatrix}' \quad \text{and} \quad \tilde{\boldsymbol{\alpha}}'_{\perp} = \begin{bmatrix} \boldsymbol{\alpha}_{\perp}^{a\prime} & \mathbf{0}' \end{bmatrix}' \tag{24}$$

The stochastic trends in $\tilde{\boldsymbol{x}}_t$ are given by:

$$\begin{aligned} \tilde{\boldsymbol{z}}_t &= \tilde{\boldsymbol{\alpha}}'_{\perp} \sum_{s=1}^{t} \tilde{\boldsymbol{\varepsilon}}_s \\ &= \begin{bmatrix} \boldsymbol{\alpha}_{\perp}^{a\prime} & \mathbf{0}' \end{bmatrix} \begin{bmatrix} \sum_{s=1}^{t} \tilde{\boldsymbol{\varepsilon}}_s^a \\ \sum_{s=1}^{t} \tilde{\boldsymbol{\varepsilon}}_s^d \end{bmatrix} \\ &= \boldsymbol{\alpha}_{\perp}^{a\prime} \sum_{s=1}^{t} \tilde{\boldsymbol{\varepsilon}}_s^a. \end{aligned} \tag{25}$$

For the two-country system we then have $K - r^a$ common trends affecting the two countries equally:

$$\begin{aligned} \boldsymbol{z}_t = \boldsymbol{\alpha}'_{\perp} \sum_{s=1}^{t} \boldsymbol{\varepsilon}_s &= \tilde{\boldsymbol{\alpha}}'_{\perp} \boldsymbol{K} \sum_{s=1}^{t} \boldsymbol{\varepsilon}_s \\ &= \begin{bmatrix} \frac{1}{2}\boldsymbol{\alpha}_{\perp}^{a\prime} & \frac{1}{2}\boldsymbol{\alpha}_{\perp}^{a\prime} & 0 \end{bmatrix} \begin{bmatrix} \sum_{s=1}^{t} \varepsilon_{1s} \\ \sum_{s=1}^{t} \varepsilon_{2s} \\ \sum_{s=1}^{t} \varepsilon_{3s} \end{bmatrix} \\ &= \frac{1}{2}\boldsymbol{\alpha}_{\perp}^{a\prime} \sum_{s=1}^{t} (\varepsilon_{1s} + \varepsilon_{2s}). \end{aligned} \tag{26}$$

Suppose now that the rank$(\boldsymbol{\beta}^d) < K+1$. Then there exists a linear combination of $\boldsymbol{x}_t^d$ that is non-stationary, *i.e.* $\boldsymbol{\beta}_{\perp}^d \boldsymbol{y}_t^d \sim \mathsf{I}(1)$. Furthermore, if $\boldsymbol{\alpha}_{\perp}^d \neq \begin{bmatrix} \mathbf{0}' & \alpha_{ee\perp}^d \end{bmatrix}'$ then the stochastic trend lies in the space of $\boldsymbol{y}_t^d$, driving $\boldsymbol{y}_t$ and $\boldsymbol{y}_t^*$ in opposite directions. ■

5 Structural VECM representation

Since Sims (1980) it is common to represent the reduced-form VAR model with its correlated prediction errors as a *structural* VAR with orthogonal errors, which allow a structural interpretation. For the two-country VECM in (11) we get:

$$\boldsymbol{B}\Delta\boldsymbol{x}_t = \boldsymbol{\phi} + \boldsymbol{\Psi}\boldsymbol{x}_{t-1} + \sum_{i=1}^{p-1} \boldsymbol{\Phi}_i \Delta\boldsymbol{x}_{t-i} + \boldsymbol{\eta}_t, \tag{27}$$

with the innovation process $\boldsymbol{\eta} \sim \mathsf{NID}(\mathbf{0}, \boldsymbol{\Omega})$, where the variance-covariance matrix $\boldsymbol{\Omega}$ is diagonal and $\boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{\Omega}\boldsymbol{B}'$ holds. Furthermore, $\boldsymbol{\phi} = \boldsymbol{B}\boldsymbol{\nu}$, $\boldsymbol{\Psi} = \boldsymbol{B}\boldsymbol{\Pi}$ and $\boldsymbol{\Phi}_i = \boldsymbol{B}\boldsymbol{\Gamma}_i$ for $i = 1, \ldots, p-1$.

When $\boldsymbol{B}$ is a triangular matrix, the SVECM in equation (27) can be considered a particular simultaneous equation model in the spirit of the *Cowles approach*. Particularly, it is a recursive system of the sort proposed by Wold (1949) and Strotz and Wold (1960) and closely related to the concept of causal ordering introduced by Simon (1953).

For a symmetric two-country model, the matrix $\boldsymbol{B}$ is of a particular design:

Proposition 7 *For a symmetric two-country VECM model, there exists a recursive structural VECM representation if and only if the following conditions are met:*

(i) both countries have identical contemporaneous structures;

(ii) there are no instantaneous spill overs from one country to another;

(iii) the country differences instantaneously affect the exchange rate or vice versa *but not simultaneously.* □

PROOF The first condition follows directly from the definition of symmetry. The second and third condition can be shown by contradiction: If $\boldsymbol{y}_t$ were affecting $\boldsymbol{y}_t^*$, symmetry would require that $\boldsymbol{y}_t^*$ also affects $\boldsymbol{y}_t$. But this violates the recursive contemporaneous structure of the SVECM. This also excludes the possibility of simultaneity of country differences and the exchange rate. ■

Remark 5 The two-country recursive SVECM representation only exits if the off-diagonal matrix $\boldsymbol{\Sigma}^*$ in (7) is equal to zero. □

Lemma 3 *Suppose the recursive structural VECM representation of symmetric two-country VECM model exists, then the structure of the contemporaneous triangular contemporaneous matrix $\boldsymbol{B}$ is characterized by one of following two designs:*

$$\boldsymbol{B}^{(1)} = \begin{bmatrix} \boldsymbol{B}_{yy} & . & . \\ \boldsymbol{B}_{y^*y} & \boldsymbol{B}_{y^*y^*} & . \\ \boldsymbol{b}_{ey}{}' & \boldsymbol{b}_{ey^*}{}' & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{yy}^{(1)} & . & . \\ \boldsymbol{0} & \boldsymbol{B}_{yy}^{(1)} & . \\ \boldsymbol{b}_{ey}^{(1)\prime} & -\boldsymbol{b}_{ey}^{(1)\prime} & 1 \end{bmatrix}, \tag{28}$$

$$\boldsymbol{B}^{(2)} = \begin{bmatrix} \boldsymbol{B}_{yy} & \boldsymbol{B}_{yy^*} & \boldsymbol{b}_{ye} \\ . & \boldsymbol{B}_{y^*y^*} & \boldsymbol{b}_{y^*e} \\ . & . & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{yy}^{(2)} & \boldsymbol{0} & \boldsymbol{b}_{ye}^{(2)} \\ . & \boldsymbol{B}_{yy}^{(2)} & -\boldsymbol{b}_{ye}^{(2)} \\ . & . & 1 \end{bmatrix}, \tag{29}$$

□

where the ordering of the variables in $\boldsymbol{y}_t$ and $\boldsymbol{y}_t^*$ is identical for both countries, and can be conveniently chosen to reflect the causal chain $y_{1t} \to \ldots \to y_{Kt}$ for $\boldsymbol{B}_{yy}^{(1)}$ and in reversed order for $\boldsymbol{B}_{yy}^{(2)}$.

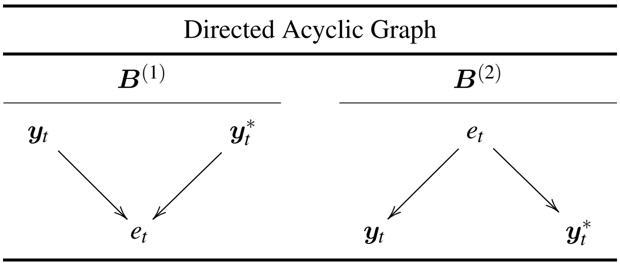
PROOF The lemma follows from Proposition 7. ■

The types of instantaneous causality consistent with Proposition 7 are illustated in Table 1, which depicts the directed acyclic graph for the three blocks of variables: $\boldsymbol{y}_t$, $\boldsymbol{y}_t^*$, and e_t.

6 Separability under mild asymmetries

So far we considered the relationship between symmetry and separability within a framework where the structure of one country is the mirror image of the other. In the following we introduce some mild asymmetries to the two-country model, which preserve the separability of the model dynamics into country averages and differences.

Figure 1 *Types of contemporaneous causal structure in symmetric two-country recursive SVECMs.*



6.1 Asymmetries in the size of the countries' economies

Suppose now that the countries are symmetric only to scale. So that the two countries are of a different size and consequently enter with different weights into the construction of the world economy:

$$\boldsymbol{y}_t^a = \lambda \boldsymbol{y}_t + (1-\lambda)\boldsymbol{y}_t^*, \text{ where } 0 < \lambda < 1. \tag{30}$$

The definition of country differences, $\boldsymbol{y}_t^d = \boldsymbol{y}_t - \boldsymbol{y}_t^*$, and the exchange rate, e_t, remain unchanged. Obviously this definition nests our previous assumption for $\lambda = \frac{1}{2}$.

Lemma 4 *Suppose the corresponding country-averages-differences representation in (2) is separable. Then the parameter matrices can be partitioned as follows:*

$$\tilde{\boldsymbol{A}}_i = \begin{bmatrix} \boldsymbol{A}^a_{yy,i} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^d_{yy,i} & \boldsymbol{a}^d_{ye,i} \\ \boldsymbol{0}' & \boldsymbol{a}^{d\,\prime}_{ey,i} & a^d_{ee,i} \end{bmatrix} \text{ and } \tilde{\boldsymbol{\Sigma}} = \begin{bmatrix} \boldsymbol{\Sigma}^a_{yy} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}^d_{yy} & \boldsymbol{\sigma}^d_{ye} \\ \boldsymbol{0}' & \boldsymbol{\sigma}^{d\,\prime}_{ye} & \sigma^d_{ee} \end{bmatrix}. \tag{31}$$

□

Proposition 8 *Under the assumption in Equation 30, the dynamics of the resulting two-country model can be separated into country averages and differences if and only if the countries are symmetric subject to scale such that we have for each country $j = 1, 2$:*

$$\begin{aligned} \boldsymbol{y}_t^j &= \boldsymbol{\nu}_j + w_j \sum_{i=1}^{p} \left(\boldsymbol{A}^a_{yy,i} + \frac{1-w_j}{w_j} \boldsymbol{A}^d_{yy,i} \right) \boldsymbol{y}^j_{t-i} \\ &\quad + (1-w_j) \sum_{i=1}^{p} \left(\boldsymbol{A}^a_{yy,i} - \boldsymbol{A}^d_{yy,i} \right) \boldsymbol{y}^{\neg j}_{t-i} + (1-w_j) \sum_{i=1}^{p} \boldsymbol{a}^d_{ye,i} e_{t-i} + \boldsymbol{\varepsilon}_{jt}. \end{aligned} \tag{32}$$

For the exchange rate equation, the weights naturally have no impact:

$$e_t = \nu_3 + \sum_{i=1}^{p} \boldsymbol{a}^{d\,\prime}_{ey,i} \left(\boldsymbol{y}_{t-i} - \boldsymbol{y}^*_{t-i} \right) + \sum_{i=1}^{p} a^d_{ee,i} e_{t-i} + \varepsilon_{3t}. \tag{33}$$

The autoregressive matrices have the following design, which no longer exhibits the block-symmetry found in (7):

$$\boldsymbol{A}_i = \begin{bmatrix} \boldsymbol{A}_i - (1-\lambda)\boldsymbol{A}_i^* & (1-\lambda)\boldsymbol{A}_i^* & (1-\lambda)\boldsymbol{a}_i \\ \lambda \boldsymbol{A}_i^* & \boldsymbol{A}_i - \lambda \boldsymbol{A}_i^* & -\lambda \boldsymbol{a}_i \\ \boldsymbol{h}_i' & -\boldsymbol{h}_i' & h_i \end{bmatrix} \text{ for } i = 1, \ldots, p, \tag{34}$$

where $\boldsymbol{A}_i^* = \boldsymbol{A}_{yy,i}^a - \boldsymbol{A}_{yy,i}^d$, $\boldsymbol{A}_i = \boldsymbol{A}_{yy,i}^a$, $\boldsymbol{a}_i = \boldsymbol{a}_{ye,i}^d$, $\boldsymbol{h}_i = \boldsymbol{a}_{ey,i}^d$, $h_i = a_{ee,i}^d$.

The variance-covariance matrix is given by:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma} + (1-\lambda)^2\boldsymbol{\Sigma}^* & \boldsymbol{\Sigma} - \lambda(1-\lambda)\boldsymbol{\Sigma}^* & (1-\lambda)\boldsymbol{\sigma} \\ \boldsymbol{\Sigma} - \lambda(1-\lambda)\boldsymbol{\Sigma}^* & \boldsymbol{\Sigma} + \lambda^2\boldsymbol{\Sigma}^* & -\lambda\boldsymbol{\sigma} \\ (1-\lambda)\boldsymbol{\sigma}' & -\lambda\boldsymbol{\sigma}' & \sigma^2 \end{bmatrix}, \tag{35}$$

where $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{yy}^a$, $\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}_{yy}^d$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{ye}^d$, *and* $\sigma^2 = \sigma_{ee}^d$. □

PROOF Under the weighting scheme in 30, the communication matrix and its inverse are given by:

$$\boldsymbol{K} = \begin{bmatrix} \lambda\boldsymbol{I} & (1-\lambda)\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0}' & \boldsymbol{0}' & 1 \end{bmatrix} \text{ and } \boldsymbol{K}^{-1} = \begin{bmatrix} \boldsymbol{I} & (1-\lambda)\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & -\lambda\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0}' & \boldsymbol{0}' & 1 \end{bmatrix}. \tag{36}$$

Since the mapping from $\boldsymbol{x}_t$ to $\tilde{\boldsymbol{x}}_t$ is again isomorphic, Proposition 1 still applies and the parametric structure of the untransformed system follows from $\boldsymbol{A}_i = \boldsymbol{K}^{-1}\tilde{\boldsymbol{A}}_i\boldsymbol{K}$ and $\boldsymbol{\Sigma} = \boldsymbol{K}^{-1}\tilde{\boldsymbol{\Sigma}}\boldsymbol{K}^{-1\prime}$:

$$\boldsymbol{A}_i = \begin{bmatrix} (1-\lambda)\boldsymbol{A}_{yy,i}^d + \lambda\boldsymbol{A}_{yy,i}^a & -(1-\lambda)\boldsymbol{A}_{yy,i}^d + (1-\lambda)\boldsymbol{A}_{yy,i}^a & (1-\lambda)\boldsymbol{a}_{ye,i}^d \\ -\lambda\boldsymbol{A}_{yy,i}^d + \lambda\boldsymbol{A}_{yy,i}^a & \lambda\boldsymbol{A}_{yy,i}^d + (1-\lambda)\boldsymbol{A}_{yy,i}^a & -\lambda\boldsymbol{a}_{ye,i}^d \\ \boldsymbol{a}_{ey,i}^{d\;\prime} & -\boldsymbol{a}_{ey,i}^{d\;\prime} & a_{ee,i}^d \end{bmatrix}, \tag{37}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} (1-\lambda)^2\boldsymbol{\Sigma}_{yy}^d + \boldsymbol{\Sigma}_{yy}^a & -\lambda(1-\lambda)\boldsymbol{\Sigma}_{yy}^d + \boldsymbol{\Sigma}_{yy}^a & (1-\lambda)\boldsymbol{\sigma}_{ye}^d \\ -\lambda(1-\lambda)\boldsymbol{\Sigma}_{yy}^d + \boldsymbol{\Sigma}_{yy}^a & \lambda^2\boldsymbol{\Sigma}_{yy}^d + \boldsymbol{\Sigma}_{yy}^a & -\lambda\boldsymbol{\sigma}_{ye}^d \\ (1-\lambda)\boldsymbol{\sigma}_{ye}^{d\;\prime} & -\lambda\boldsymbol{\sigma}_{ye}^{d\;\prime} & \sigma_{ee}^d \end{bmatrix}. \tag{38}$$

■

Remark 6 The smaller the country the greater the impact of the foreign economy and the foreign exchange rate, the lesser the dependency on its own economic record. □

6.2 Time-varying country weights

For the aggregation of country time series to global or regional measures, the use of the Divisia index has become increasingly popular. This continuous time aggregation of quantities with changing prices was originally proposed by Divisia (1925). Discrete-time approximations were developed by Törnqvist (1936) and Theil (1967). Bridging the gap of index number theory and aggregation theory by introducing the class of second-order 'superlative' index numbers, Diewert (1976) showed that the Divisia index is a superlative index number, i.e., it is exact for a flexible aggregator functional form. For the construction of monetary aggregates, Barnett (1980) and various follow-up papers made a strong case for the appropriateness of aggregation with the help of the Divisia index when compared to the traditional use of simple sum of levels. So it is worth to consider the case of time-varying country weights.

Proposition 9 *If the country weights in (30) are time-varying, the VAR representations of* $\boldsymbol{y}_t$ *in (1) and* $\tilde{\boldsymbol{y}}_t$ *in (2) cannot both be time-invariant.* □

PROOF Time variation in the weights of the countries in $\boldsymbol{y}_t^a$ results in the time-variation of the communication matrix $\boldsymbol{K}_t$. Now suppose the VAR representation of $\boldsymbol{x}_t$ is time-invariant, especially $\boldsymbol{A}_t = \boldsymbol{A}$ for all t, then $\tilde{\boldsymbol{A}}_t = \boldsymbol{K}_t\boldsymbol{A}\boldsymbol{K}_t^{-1}$ is varying over time. On the other hand, suppose that the country-averages-differences representation does not suffer from time-variation. Then $\boldsymbol{A}_t = \boldsymbol{K}_t^{-1}\tilde{\boldsymbol{A}}\boldsymbol{K}_t$ is time-varying. ■

The result of Proposition 9 is economically intuitive. Suppose, for example, the laws of the motion of the US macroeconomy depend on total exports only and the Chinese economy is growing at faster rate than the European one, then over time the dependency of the US economy on China will increase mirroring a decrease of the importance of Europe.

Proposition 10 *If the country-averages-differences system is structurally stable, the time-varying two-country VAR is given by:*

$$\boldsymbol{y}_t = \boldsymbol{\nu}(\lambda_t) + \sum_{i=1}^{p} \boldsymbol{A}_i(\lambda_t)\boldsymbol{y}_{t-i} + \varepsilon_t, \text{ with } \varepsilon_t \sim \mathsf{NID}(\boldsymbol{0}, \boldsymbol{\Sigma}(\lambda_t)), \tag{39}$$

where the parameter matrices have the following design::

$$\boldsymbol{A}_i(\lambda_t) = \begin{bmatrix} \boldsymbol{A}_i - (1-\lambda_t)\boldsymbol{A}_i^* & (1-\lambda_t)\boldsymbol{A}_i^* & (1-\lambda_t)\boldsymbol{a}_i \\ \lambda_t \boldsymbol{A}_i^* & \boldsymbol{A}_i - \lambda_t \boldsymbol{A}_i^* & -\lambda_t \boldsymbol{a}_i \\ \boldsymbol{h}_i' & -\boldsymbol{h}_i' & h_i \end{bmatrix} \text{ for } i = 1, \ldots, p, \tag{40}$$

where $\boldsymbol{A}_i^* = \boldsymbol{A}_{yy,i}^a - \boldsymbol{A}_{yy,i}^d$, $\boldsymbol{A}_i = \boldsymbol{A}_{yy,i}^a$, $\boldsymbol{a}_i = \boldsymbol{a}_{ye,i}^d$, $\boldsymbol{h}_i = \boldsymbol{a}_{ey,i}^d$, $h_i = a_{ee,i}^d$.

The variance-covariance matrix is given by:

$$\boldsymbol{\Sigma}(\lambda_t) = \begin{bmatrix} \boldsymbol{\Sigma} + (1-\lambda_t)^2 \boldsymbol{\Sigma}^* & \boldsymbol{\Sigma} - \lambda_t(1-\lambda_t)\boldsymbol{\Sigma}^* & (1-\lambda_t)\boldsymbol{\sigma} \\ \boldsymbol{\Sigma} - \lambda_t(1-\lambda_t)\boldsymbol{\Sigma}^* & \boldsymbol{\Sigma} + \lambda_t^2 \boldsymbol{\Sigma}^* & -\lambda_t \boldsymbol{\sigma} \\ (1-\lambda_t)\boldsymbol{\sigma}' & -\lambda_t \boldsymbol{\sigma}' & \boldsymbol{\sigma}^2 \end{bmatrix}, \tag{41}$$

where $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{yy}^a, \boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}_{yy}^d$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{ye}^d$, *and* $\boldsymbol{\sigma}^2 = \boldsymbol{\sigma}_{ee}^d$. □

7 Empirical illustration

For a short empirical illustration of the test of the symmetry conditions derived in the previous sections of this paper, we investigate two-country VAR models for five country pairs: the US with the Euro Area, with Japan, with the UK and with Canada, additionally the Euro Area with the UK. The nine dimensional (9D) system for each country pair include the rates of inflation, output growth, short- and long-term interest for each country as well as the nominal exchange rate. The sample consists of 157 quarterly observations for the post-Bretton-Woods era from 1972Q4 to 2011Q4. The aggregated Euro area time series for the whole sample period including the years preceding the creation of the European Monetary Union were drawn from the OECD data base for GDP and its deflator and from Reuters for the short- and long-term interest rate. The pre-1986 short-term interest rates were constructed with a Divisia index using the 3 month deposit rates of Germany, France, Italy and the Netherlands.

Commencing from a 9D possibly cointegrated VAR(3) model, with statistically significant constant and restricted trend that were sufficient to capture the dynamics of the system, the Johansen trace test results show a rank of 4 for all systems except US-Canada where the rank is 3. These model specifications were also used for the country-average and the country-difference subsytems. The results are summarised in Table 1.

The tests for symmetry are performed separately for the long-run equilibrium, the adjustment to the long-run, the short-run dynamics and the contemporaneous effects in terms of restrictions on the VECM in the joint country-average-difference representation. The tests are formulated ignoring deterministic

Table 1 *Johansen likelihood ratio trace test results for 9D systems and for subsystems, models with constant and trend*

	r	r^a	r^d
$US-EA$	4	3	3
$US-JA$	4	3	1
$US-UK$	4	3	3
$US-CA$	3	3	2
$EA-UK$	4	3	2

terms and normalising the last cointegrating vector on the exchange rate, we consider the following system:[1]

$$\begin{bmatrix}\Delta \boldsymbol{y}_t^a \\ \Delta \boldsymbol{y}_t^d \\ \Delta e_t\end{bmatrix} = \begin{bmatrix}\boldsymbol{\alpha}_{yy}^{a} & \boldsymbol{\alpha}_{yy}^{ad} & \boldsymbol{\alpha}_{ye}^{ad} \\ \boldsymbol{\alpha}_{yy}^{da} & \boldsymbol{\alpha}_{yy}^{d} & \boldsymbol{\alpha}_{ye}^{d} \\ \boldsymbol{\alpha}_{ey}^{da} & \boldsymbol{\alpha}_{ey}^{d} & \alpha_{ee}^{d}\end{bmatrix} \begin{bmatrix}\boldsymbol{\beta}_{yy}^{a\,\prime} & \boldsymbol{\beta}_{yy}^{ad\prime} & \mathbf{0} \\ \boldsymbol{\beta}_{yy}^{da\prime} & \boldsymbol{\beta}_{yy}^{d\,\prime} & \mathbf{0} \\ \boldsymbol{\beta}_{ey}^{da\prime} & \boldsymbol{\beta}_{ey}^{d\,\prime} & \beta_{ee}^{d\,\prime}\end{bmatrix} \begin{bmatrix}\boldsymbol{y}_{t-1}^a \\ \boldsymbol{y}_{t-1}^d \\ e_{t-1}\end{bmatrix}$$

$$+\sum_{i=1}^{p-1} \begin{bmatrix}\boldsymbol{\Gamma}_{yy,i}^{a} & \boldsymbol{\Gamma}_{yy,i}^{ad} & \boldsymbol{\Gamma}_{ye,i}^{ad} \\ \boldsymbol{\Gamma}_{yy,i}^{da} & \boldsymbol{\Gamma}_{yy,i}^{d} & \boldsymbol{\Gamma}_{ye,i}^{d} \\ \boldsymbol{\Gamma}_{ey,i}^{da} & \boldsymbol{\Gamma}_{ey,i}^{d} & \Gamma_{ee,i}^{d}\end{bmatrix} \begin{bmatrix}\Delta \boldsymbol{y}_{t-i}^a \\ \Delta \boldsymbol{y}_{t-i}^d \\ \Delta e_{t-i}\end{bmatrix} + \begin{bmatrix}\tilde{\boldsymbol{\varepsilon}}_{1t} \\ \tilde{\boldsymbol{\varepsilon}}_{2t} \\ \varepsilon_{3t}\end{bmatrix}, \tag{42}$$

$$\text{where} \begin{bmatrix}\tilde{\boldsymbol{\varepsilon}}_{1t} \\ \tilde{\boldsymbol{\varepsilon}}_{2t} \\ \varepsilon_{3t}\end{bmatrix} \sim \text{NID}\left(\begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ 0\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{yy}^{a} & \boldsymbol{\Sigma}_{yy}^{ad} & \boldsymbol{\Sigma}_{ye}^{ad} \\ \boldsymbol{\Sigma}_{yy}^{da} & \boldsymbol{\Sigma}_{yy}^{d} & \boldsymbol{\Sigma}_{ye}^{d} \\ \boldsymbol{\Sigma}_{ey}^{da} & \boldsymbol{\Sigma}_{ey}^{d} & \Sigma_{ee}^{d}\end{bmatrix}\right).$$

First, we test whether the exchange rate is not part of the cointegration space, see Table 2, for $H_o^1 : \tilde{\boldsymbol{\beta}}'$ has a zero column in e_t. For all systems this hypothesis has to be rejected. So the exchange rate is part of a cointegrating space. Next, we test whether the exchange rate is not cointegrated with country average variables. Zero restrictions on the country average variables in the cointegration relationship containing the exchange rate are tested.[2] Throughout Hypothesis H_o^2 can not be rejected. Thus, in the long-run, the exchange rates are determined by the country differentials only.

Table 2 *9D two-country CVAR(3) models:*
Testing for symmetry of the cointegration space with regard to the exchange rate (Likelihood ratio tests).
H_o^1 : *e_t not in cointegration space,* $\beta_{ee}^d = 0$
H_o^2 : *e_t not cointegrated with* $\boldsymbol{y}_t^a$, $\boldsymbol{\beta}_{ey}^{da} = \mathbf{0}$.

Countries	H_o^1		H_o^2	
$US-EA$	$\chi^2(4) = 15.22$	[0.004]	$\chi^2(1) = 1.91$	[0.167]
$US-JA$	$\chi^2(4) = 24.23$	[0.000]	$\chi^2(1) = 1.24$	[0.266]
$US-UK$	$\chi^2(4) = 24.25$	[0.000]	$\chi^2(1) = 1.71$	[0.192]
$US-CA$	$\chi^2(3) = 8.79$	[0.032]	$\chi^2(2) = 1.68$	[0.431]
$EA-UK$	$\chi^2(4) = 30.03$	[0.000]	$\chi^2(1) = 3.45$	[0.063]

A stronger test for long-run symmetry is Hypothesis H_o^3, where we test whether there are any com-

[1] Hecq et al. (2002) suggest a similar multiple step procedure when testing for separation in VAR models.
[2] Due to $(r-1)$ additional possibilities of normalising $\tilde{\boldsymbol{\beta}}$, the number of degrees of freedom in this χ^2test is $k-(r-1)$.

posite cointegration vectors involving country averages and differences. Since the ranks of the subsystems do not add up all consistently to the rank of the full model, due to possibly latent asymmetries, we consider all combinations of r^a and r^d consistent with the cointegration rank of the full system, from one cointegration relationship with country-average variables and $r-1$ cointegration vectors in country differences, to $r-1$ cointegration relationships lying within the country-average space and only one in the country difference space. For all country pairs, full long-run symmetry has to be rejected, see Table 3, at the usual significance levels. Models with $r^a = 2$ are closest to accepting symmetry of the long-run.

Table 3 *9D two-country CVAR(3) models: Testing for symmetry of the cointegration space (Likelihood ratio tests).* $H_o^3 : (\beta_{yy}^{ad}, \beta_{yy}^{da}, \beta_{ey}^{da}) = \mathbf{0}$.

Countries	$r^a = 1, r^d = r - r^a$		$r^a = 2, r^d = r - r^a$		$r^a = 3, r^d = r - r^a$	
$US-EA$	$\chi^2(11) = 32.88$	[0.001]	$\chi^2(10) = 23.59$	[0.009]	$\chi^2(13) = 62.26$	[0.000]
$US-JA$	$\chi^2(11) = 62.63$	[0.000]	$\chi^2(10) = 43.79$	[0.000]	$\chi^2(13) = 62.27$	[0.000]
$US-UK$	$\chi^2(11) = 46.53$	[0.000]	$\chi^2(10) = 24.28$	[0.007]	$\chi^2(13) = 49.71$	[0.000]
$US-CA$	$\chi^2(9) = 34.68$	[0.000]	$\chi^2(10) = 57.26$	[0.000]		
$EA-UK$	$\chi^2(11) = 54.96$	[0.000]	$\chi^2(10) = 37.33$	[0.000]	$\chi^2(13) = 70.37$	[0.000]

For testing the symmetry of the adjustment process of the long-run equilibrium we use the symmetry long-run specification with the highest p-val in Table 3. We consider three different test hypothesis on $\tilde{\alpha}$: H_o^4 tests whether the exchange rate is not adjusting in the long-run to country average cointegration relationships, H_o^5 tests whether the country averages are not adjusting to the cointegration relationship containing the exchange rate, and H_o^6 evaluates whether overall symmetry in the adjustment to the long-run is present. Hypothesis H_o^4 can only be rejected for the US-Japan, for the other country pairs the test shows there is some symmetry in the adjustment at least for the exchange rate. However H_o^5 and H_o^6 have to be rejected for all models.

Table 4 *9D two-country VECM(2) model with symmetric cointegration space: Testing for symmetry in equilibrium correction (Wald tests).*
$H_o^4 : \Delta e_t \perp \beta^{a\prime} \boldsymbol{y}_{t-1}^a$: $\boldsymbol{\alpha}_{ey}^{da} = \mathbf{0}$
$H_o^5 : \Delta \boldsymbol{y}_t^a \perp (\beta_{ey}^{d\,\prime} \boldsymbol{y}_{t-1}^d + \beta_{ee}^{d\,\prime} e_{t-1})$: $\boldsymbol{\alpha}_{ye}^{ad} = \mathbf{0}$
$H_o^6 : \boldsymbol{\alpha}_{yy}^{ad} = \mathbf{0}, \boldsymbol{\alpha}_{ye}^{ad} = \mathbf{0}, \boldsymbol{\alpha}_{yy}^{da} = \mathbf{0}, \boldsymbol{\alpha}_{ey}^{da} = \mathbf{0}$.

Countries	H_o^4		H_o^5		H_o^6	
$US-EA$	$\chi^2(2) = 3.82$	[0.148]	$\chi^2(4) = 30.95$	[0.000]	$\chi^2(18) = 57.71$	[0.000]
$US-JA$	$\chi^2(2) = 10.97$	[0.004]	$\chi^2(4) = 29.54$	[0.000]	$\chi^2(18) = 75.04$	[0.000]
$US-UK$	$\chi^2(2) = 3.94$	[0.139]	$\chi^2(4) = 46.94$	[0.000]	$\chi^2(18) = 106.78$	[0.000]
$US-CA$	$\chi^2(1) = 1.99$	[0.159]	$\chi^2(4) = 31.81$	[0.000]	$\chi^2(13) = 66.49$	[0.000]
$EA-UK$	$\chi^2(2) = 0.17$	[0.920]	$\chi^2(4) = 34.32$	[0.000]	$\chi^2(18) = 66.34$	[0.000]

Analogue to the tests for the adjustment to the long-run on $\tilde{\alpha}$, we perform three tests for the short-run dynamics on $\tilde{\boldsymbol{\Gamma}}$. The country average short-run dynamics have no direct effect on the exchange rate, see Hypothesis H_o^7 in Table 5. Also the exchange rate short-run dynamics have no effect on the country average variables, see Hypothesis H_o^8. But overall symmetry for the short-run dynamics has to be rejected according to Hypothesis H_o^9.

Next we perform two Wald tests for symmetry on the variance-covariance matrix $\tilde{\boldsymbol{\Sigma}}$ (see Lütkepohl,

Table 5 *9D two-country VECM(2) model with symmetric cointegration space: Testing for symmetry in short-run dynamics (Wald tests).*
$H_o^7 : \Delta e_t \perp \{\Delta \boldsymbol{y}_{t-i}^a\}_{i=1}^{p-1}$: $\boldsymbol{\Gamma}_{ey,i}^{da} = \mathbf{0}$
$H_o^8 : \Delta \boldsymbol{y}_t^a \perp \{\Delta e_{t-i}\}_{i=1}^{p-1}$: $\boldsymbol{\Gamma}_{ye,i}^{ad} = \mathbf{0}$
$H_o^9 : (\boldsymbol{\Gamma}_{yy,i}^{ad}, \boldsymbol{\Gamma}_{ye,i}^{ad}, \boldsymbol{\Gamma}_{yy,i}^{da}, \boldsymbol{\Gamma}_{ey,i}^{da\,\prime}) = \mathbf{0}.$

Countries	H_o^7		H_o^8		H_o^9	
$US-EA$	$\chi^2(8) = 4.49$	[0.811]	$\chi^2(8) = 7.61$	[0.473]	$\chi^2(80) = 121.38$	[0.002]
$US-JA$	$\chi^2(8) = 12.35$	[0.136]	$\chi^2(8) = 3.71$	[0.882]	$\chi^2(80) = 187.96$	[0.000]
$US-UK$	$\chi^2(8) = 13.10$	[0.109]	$\chi^2(8) = 8.52$	[0.384]	$\chi^2(80) = 193.73$	[0.000]
$US-CA$	$\chi^2(8) = 6.51$	[0.590]	$\chi^2(8) = 13.79$	[0.088]	$\chi^2(80) = 176.93$	[0.000]
$EA-UK$	$\chi^2(8) = 2.57$	[0.958]	$\chi^2(8) = 4.08$	[0.850]	$\chi^2(80) = 199.39$	[0.000]

2005, p.104). Hypothesis H_o^{10} tests whether there is no contemporaneous correlation between the exchange rate and country average variables. For three models, the US - Euro Area, the US - UK and the Euro Area - UK, the absence of contemporaneous correlation of the exchange rate with country average variables can not be rejected, which can be interpreted as a short-run version of symmetry with respect to the exchange rate. Further our Wald tests of H_o^{11} stating the lack of contemporaneous correlation between country difference and country average variables is rejected for all models.

Table 6 *9D two-country VECM(2) model with symmetric cointegration space: Wald tests for symmetry in variance-covariance matrix.*
$H_o^{10} : \Delta e_t \perp \Delta \boldsymbol{y}_t^a$: $\boldsymbol{\Sigma}_{ey}^{da} = \mathbf{0}$
$H_o^{11} : (\boldsymbol{\Sigma}_{yy}^{ad}, \boldsymbol{\Sigma}_{ye}^{ad}, \boldsymbol{\Sigma}_{yy}^{da}, \boldsymbol{\Sigma}_{ey}^{da\prime}) = \mathbf{0}.$

Countries	H_o^{10}		H_o^{11}	
$US-EA$	$\chi^2(4) = 5.88$	[0.208]	$\chi^2(20) = 66.78$	[0.000]
$US-JA$	$\chi^2(4) = 14.14$	[0.007]	$\chi^2(20) = 112.77$	[0.000]
$US-UK$	$\chi^2(4) = 6.66$	[0.155]	$\chi^2(20) = 101.46$	[0.000]
$US-CA$	$\chi^2(4) = 17.42$	[0.002]	$\chi^2(20) = 114.83$	[0.000]
$EA-UK$	$\chi^2(4) = 5.88$	[0.208]	$\chi^2(20) = 155.07$	[0.000]

Finally we perform tests on the structural contemporaneous $\boldsymbol{B}$ matrix of a recursive structural VECM representation of the system. In Hypothesis H_o^{12}, see Table 7, we test for the domestic-foreign representation of the recursive structural model whether the exchange rate is symmetrically driven by $\boldsymbol{y}_t$ and $\boldsymbol{y}_t^*$. Contemporaneous symmetry of the exchange rate equation is only rejected for US - Japan and Euro Area - UK models. In Hypotheses H_o^{13} and H_o^{14}, see Table 7, we test whether $\boldsymbol{y}_t$ and $\boldsymbol{y}_t^*$ is symmetrically driven by the exchange rate. This has to be rejected for two models, US - Japan and US - Canada. Lastly we test for full contemporaneous symmetry in the recursive structural VECM, by testing the zero restrictions in (28) and (29) in four different model specifications: H_o^{15} to H_o^{18}. Full symmetry is constantly rejected.

To sum up our findings, there is strong support for symmetry as far as the exchange rate equation is concerned. This is especially pronounced in the US - Euro Area and US - UK system, where all tests of our stepwise procedure confirm, that exchange rates are driven by country differentials only. Asymmetry is found for the Dollar/Yen exchange rate, which adjusts to the country-average cointegrating vectors. Our tests for full symmetry of the two-country models are generally rejected for all country pairs of our study. The concept of full symmetry is very demanding. The focus in practice has to be on the

Table 7 *9D two-country recursive structural VECM(2) model with symmetric cointegration space: Wald tests for symmetry of the contemporaneous structure concerning e_t.*
$H_o^{12}: b_{ey} = -b_{ey^*}$ *in* $\boldsymbol{B}^{(1)}$ *of (28)*
$H_o^{13}: b_{ye} = -b_{y^*e}$ *in* $\boldsymbol{B}^{(2)}$ *of (29) with* $\boldsymbol{y} = (\boldsymbol{y}, \boldsymbol{y}^*, e_t)'$
$H_o^{14}: b_{ye} = -b_{y^*e}$ *in* $\boldsymbol{B}^{(2)}$ *of (29) with* $\boldsymbol{y} = (\boldsymbol{y}^*, \boldsymbol{y}, e_t)'$.

Countries	H_o^{12}		H_o^{13}		H_o^{14}	
$US-EA$	$\chi^2(4) = 5.74$	[0.220]	$\chi^2(4) = 7.21$	[0.125]	$\chi^2(4) = 5.74$	[0.220]
$US-JA$	$\chi^2(4) = 12.39$	[0.015]	$\chi^2(4) = 13.02$	[0.011]	$\chi^2(4) = 11.95$	[0.018]
$US-UK$	$\chi^2(4) = 3.86$	[0.426]	$\chi^2(4) = 4.41$	[0.353]	$\chi^2(4) = 5.36$	[0.252]
$US-CA$	$\chi^2(4) = 4.39$	[0.356]	$\chi^2(4) = 17.86$	[0.001]	$\chi^2(4) = 18.43$	[0.001]
$EA-UK$	$\chi^2(4) = 10.60$	[0.031]	$\chi^2(4) = 5.21$	[0.266]	$\chi^2(4) = 3.76$	[0.439]

Table 8 *9D two-country recursive structural VECM(2) model with symmetric cointegration space: Wald tests for symmetry of the contemporaneous structure.*
$H_o^{15}: \boldsymbol{B}^{(1)}$ *in (28) with* $\boldsymbol{y} = (\boldsymbol{y}, \boldsymbol{y}^*, e_t)'$
$H_o^{16}: \boldsymbol{B}^{(1)}$ *in (28) with* $\boldsymbol{y} = (\boldsymbol{y}^*, \boldsymbol{y}, e_t)'$
$H_o^{17}: \boldsymbol{B}^{(2)}$ *in (29) with* $\boldsymbol{y} = (\boldsymbol{y}, \boldsymbol{y}^*, e_t)'$
$H_o^{18}: \boldsymbol{B}^{(2)}$ *in (29) with* $\boldsymbol{y} = (\boldsymbol{y}^*, \boldsymbol{y}, e_t)'$.

Countries		H_o^{15}	H_o^{16}	H_o^{17}	H_o^{18}
$US-EA$	$\chi^2(16)$	101.49 [0.000]	94.88 [0.000]	110.86 [0.000]	108.89 [0.000]
$US-JA$	$\chi^2(16)$	62.62 [0.000]	63.39 [0.000]	56.65 [0.000]	56.49 [0.000]
$US-UK$	$\chi^2(16)$	62.81 [0.000]	62.24 [0.000]	61.56 [0.000]	61.50 [0.000]
$US-CA$	$\chi^2(16)$	483.49 [0.000]	442.80 [0.000]	573.71 [0.000]	515.76 [0.000]
$EA-UK$	$\chi^2(16)$	100.66 [0.000]	101.01 [0.000]	105.77 [0.000]	104.82 [0.000]

identification of symmetry features.

8 Conclusions

In this paper we investigated the applicability of Masanao Aoki's approach of separating the dynamics of a two-country model under symmetry into the autonomous subsystems of country averages and country differences to dynamic macroeconometric modelling. We were able to develop a general theory for the class of symmetric possibly cointegrated two-country linear vector autoregressive processes, and demonstrated how the theory encompasses two-country systems with asymmetric and even time-varying weights. Aoki's approach is frequently used in economic theory. In order to make it work for empirical modelling, it is essential to allow for only partial symmetry of the countries involved. For this purpose we implemented a testing strategy for discovering symmetry features in econometric two-country CVAR models.

Further advances are feasible. While the separation into two autonomous subsystems substantially reduces the complexity of the two-country system, the subsystems themselves may still suffer from the curse of dimensionality leading to imprecise parameter estimates and uninformative impulse responses. Automatic model selection procedures to reduce the complexity further by eliminating insignificant parameters without losing information such as general-to-specific approach proposed by Heinlein and Krolzig (2013) for the selection of a congruent parsimonious structural two-country vector equilibrium correction models.

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