The Nonlinear Stability Analysis of Double-Diffusive Convection with Viscous Dissipation Effect

N. Deepika[1], P.A.L. Narayana[2] and A.A.Hill[3]

[1]Department of Mathematics, National Institute of Technology Warangal, Warangal, 506004, Telangana, India.
[2]Department of Mathematics, Indian Institute of Technology Hyderabad, Sangareddy, 502285, Telangana, India.
[3]College of Health, Science and Society, University of the West of England, Bristol, BS16 1QY, United Kingdom.

Contributing authors: ndeepika@.nitw.ac.in; ananth@math.iith.ac.in; Antony.Hill@uwe.ac.uk;

Abstract

In this article, the onset of double-diffusive convection with the effect of viscous dissipation in a horizontal fluid-saturated porous layer is examined. Two impermeable isothermal and isosolutal walls bound the porous layer, and Darcy's law models the flow. The onset of convective instability is studied by two approaches: the linear stability analysis and the nonlinear stability analysis. The nonlinear stability analysis is performed by utilising the energy method. The literature on the nonlinear stability analysis of onset of convective instability with the viscous dissipation effect is limited. The present article aims to fill this gap. It is observed that, when the fluid is at rest, the effect of viscous dissipation does not influence the critical thermal Rayleigh number corresponding to both the linear and nonlinear stability analyses. Moreover, sub-critical instabilities do not occur when $\boldsymbol{Ra_S > 0}$ and the region of sub-critical instabilities increase along the negative $\boldsymbol{Ra_S}$ direction.

Keywords: Double-diffusive convection, fluid-saturated porous layer, viscous dissipation, energy method

1 Introduction

The onset of double-diffusive convection in a horizontal fluid-saturated porous layer heated from below has been extensively investigated in the last few decades due to its wide range of applications. Detailed discussions of the literature on this subject can be found in the textbooks by Nield and Bejan [1], Straughan [16] and Straughan [17]. The double-diffusive convection can occur when the density of the fluid is affected by at least two components with different diffusivities. The combined heat and mass transport (thermo-solutal convection) is one specific subset of the more general double-diffusive convection problem. In thermo-solutal phenomena, the order of magnitude of thermal diffusivity is approximately two orders of magnitude higher than the solute diffusivity. Under certain conditions, this may result in gravitational instabilities due to the lag between the faster diffusing heat and the slower diffusing solute (see Love et al. [6]). The interest in the thermo-solutal or the double-diffusive convective instability occurs relative to the contaminant transport in groundwater; and in the exploitation of geothermal reservoirs. The study of solar ponds is a very application of the convective instability in a binary-mixture fluid layer subject to thermal and concentration gradients. The convective instability of a thermal boundary layer in a fluid-saturated porous medium was studied by Wooding [18], and it was extended to a problem with large Péclet number was studied by Wooding [19].

The onset of thermohaline convection in a horizontal fluid-saturated porous layer was first studied by Nield [3]. Further studies on the combined heat and mass transfer by the natural convection in a porous medium were summarized by Trevisan and Bejan [12, 13]. The study of the nonlinear stability analysis via the energy method to the problem of double-diffusive convection in a horizontal porous layer was studied by Guo and Kaloni [4], and the same problem was extended with the Brinkmann effect by Guo and Kaloni [5]. In application to the thermohaline transport in ground water wells, Love et al. [6] examined double-diffusive convection in ground water well which was modelled as vertical circular cylinder. The combined effects of double-diffusion and viscous dissipation, along with the horizontal mass through flow, were examined by Nield and Barletta [9]. Later, Nield and Kuznetsov [14] investigated the effect of isoflux boundary conditions on the oscillatory instability.

From the previous studies, it has been identified that the effect of viscous dissipation is significant in large bodies, for example, on larger planets, larger masses of gas in space, geological processes, and in a more strong gravitational field, and in devices that operate at high rotative speeds. One of the earliest investigations on the effect of viscous dissipation in natural convection can be found in Gebhart [2] and they concluded that viscous dissipation in a natural convection is appreciable when induced kinetic energy becomes noticeable compared to the amount of heat transferred. A few recent works on the effect of viscous dissipation on the onset of convection in a fluid-saturated porous layer were given in (Barletta et. al [22], Deepika nad Narayana [23] and Nield et. al [24]) Later, Barletta et al. ([7], [21]) studied the effect of viscous dissipation on the thermal convective instability where there was no imposed temperature gradient across the layer, but the heat was generated internally by the action of viscous dissipation. Further, the investigation on thermoconvective instabilities along with the viscous dissipation effect was carried out by Barletta and Storesletten [8]. Subsequently, the effect of viscous dissipation in a clear fluid, and in a fluid-saturated porous medium is studied by Barletta [15] and summarized that viscous dissipation is the sole cause of convective instability.

All of the works mentioned above are only on the linear stability analysis of the onset of double-diffusive convection in a fluid-saturated porous layer. The nonlinear stability analysis utilising the energy method was first investigated by Joseph [25]. Later, the sufficient condition for the coincidence of the linear and nonlinear stability parameters for a thermal convection problem was proved by Rionero and Mulone [26]. The several works on energy method on the onset of double-diffusive convection were given in (Guo and Kaloni [4], Guo and Kaloni [5] and Deepika [20]). The nonlinear stability theory via the energy method for various mechanics problems was mentioned in the book by Straughan [16]. The implementation of the energy method in the presence of viscous dissipation needs to be improved in the the literature. The very first works on the nonlinear stability analysis of Darcy-Bénard problem with the effect of viscous dissipation was derived by Barletta and Mulone [11], where as in case of Brinkmann porous medium with the viscous dissipation effect is studied by Straughan [10].

The literature on the nonlinear stability analysis of thermal convection with the viscous dissipation effect by utilising energy method is minimal. The motivation for the present work is to fill this gap. The present article aims to extend the work of Barletta and Mulone [11] to the problem of the stability analysis on the onset of double-diffusive convection. In this article, section 2 deals with the formulation of the problem followed by the linear stability analysis in section 3 and the nonlinear stability analysis in section 4.

2 Mathematical Formulation

Consider a horizontal fluid-saturated porous layer which is confined between two planes and extended to infinity in the horizontal directions. Let $Ox^*y^*z^*$ be the Cartesian co-ordinate system with x^*, y^* be the horizontal axes and z^* being the vertical axis. The height of the porous layer is H. This fluid-saturated porous layer is bounded by two impermeable planes, $z^* = 0$, $z^* = H$. Here, * is the dimensional quantity. The temperature and concentration at the lower plane be $T^* = T_1^*, C^* = C_1^*$, and at the upper plane, $T^* = T_0^*, C^* = C_0^*$. Let the temperature and concentration difference, $T_1^* - T_0^*, C_1^* - C_0^*$ is maintained throughout the porous layer.

Here, the fluid is assumed to be incompressible and Newtonian. The flow is modelled by Darcy'slaw with the assumption of the Oberbeck-Boussinesq approximation, i.e., the density variations are negligible everywhere except in the body force term and it can be expressed as

$$\rho_f = \rho_0 \left(1 - \gamma_T (T^* - T_0^*) - \gamma_C (C^* - C_0^*)\right), \tag{2.1}$$

where ρ_f is the density of the fluid, ρ_0 is the density of the fluid at the reference temperature T_0^*, the reference concentration C_0^*, and γ_T and γ_C are thermal and solutal expansion coefficients. The flow governing equations which express the conservation of mass, momentum, energy balance, mass balance are

$$\nabla^* \cdot \mathbf{v}^* = 0, \tag{2.2}$$

$$\frac{\mu}{K}\mathbf{v}^* = -\nabla^* p^* - \rho_f^* \mathbf{g}, \tag{2.3}$$

$$A\frac{\partial^* T^*}{\partial t^*} + \mathbf{v}^* \cdot \nabla^* T^* = \alpha \nabla^{*2} T^* + \frac{\mu}{\rho_0 K c}\mathbf{v}^* \cdot \mathbf{v}^*, \tag{2.4}$$

$$\phi\frac{\partial^* C^*}{\partial t^*} + \mathbf{v}^* \cdot \nabla^* C^* = D\nabla^{*2} C^*. \tag{2.5}$$

with the boundary conditions

$$w^* = 0, \quad T^* = T_1^* \quad C^* = C_1^* \quad \text{at} \quad z^* = 0,$$

$$w^* = 0, \quad T^* = T_0^* \quad C^* = C_0^* \quad \text{at} \quad z^* = H, \tag{2.6}$$

where $\mathbf{v}^* = (u^*, v^*, w^*)$ is the seepage velocity, t^* is the time, $\mathbf{g} = g\hat{e_z}$ (where $\hat{e_z}$ is the unit vector in the z^* direction) is the gravitational acceleration, p is the pressure, T is the temperature, and C is the concentration. Further, μ, c, K, α, ϕ and D, denote viscosity, specific heat, permeability, thermal diffusivity, porosity and solutal diffusivity, respectively. Introducing the following non-dimensional quantities to non-dimensionalize the above mentioned governing equations are

$$(x, y, z) = \frac{1}{H}(x^*, y^*, z^*), \quad t = \frac{\alpha}{AH^2}t^*, \quad (u, v, w) = \mathbf{v} = \frac{H}{\alpha}v^*, \quad p = \frac{K}{\mu\alpha}p^*,$$

$$\omega = \frac{\phi}{A} \quad \alpha = \frac{k}{(\rho c_p)_f}, \quad A = \frac{(\rho c)_m}{(\rho c_p)_f}, \quad T = \frac{T^* - T_0^*}{T_1^* - T_0^*}, \quad C = \frac{C^* - C_0^*}{C_1^* - C_0^*},$$

$$Ra_T = \frac{\rho_0 g \beta_T K H (T_1^* - T_0^*)}{\mu\alpha}, \quad Ra_S = \frac{\rho_0 g \beta_C K H (C_1^* - C_0^*)}{\mu D},$$

$$Le = \frac{\alpha}{D}, \quad Ge = \frac{g\beta_T H}{c},$$

where, A is the ratio of specific heats, Ra_T is the thermal Rayleigh number, Ra_S is the solutal Rayleigh number, Le is the Lewis number, Ge is the Gebhart number. By substituting the above non-dimensional quantities in (2.2)-(2.6), the obtained non-dimensional governing equations are

$$\nabla \cdot \mathbf{v} = 0, \tag{2.7}$$

$$\mathbf{v} = -\nabla p + \left(Ra_T + \frac{1}{Le} Ra_S \right) \hat{e_z}, \tag{2.8}$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla^2 T + \frac{Ge}{Ra_T} \mathbf{v} \cdot \mathbf{v}, \tag{2.9}$$

$$\omega \frac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = \frac{1}{Le} \nabla^2 C. \tag{2.10}$$

along with the boundary conditions

$$w = 0, \quad T = 1 \quad C = 1 \quad \text{at} \quad z = 0,$$

$$w = 0, \quad T = 0 \quad C = 0 \quad \text{at} \quad z = 1. \tag{2.11}$$

The steady-state solution for the velocity is in quiescent state, and temperature and concentration are given by

$$\mathbf{v}_B = (u_B, v_B, w_B) = (0,0,0), \quad P = P_B(z), \quad T = 1 - z, \quad C = 1 - z. \tag{2.12}$$

The onset of instability of the system is studied by perturbing basic state solution (2.12) as

$$\mathbf{v} = \mathbf{v}_B + \mathbf{V}, \quad p = P_B + P, \quad T = T_B + \theta, \quad C = C_B + \Phi, \tag{2.13}$$

where $\mathbf{V} = (U, V, W), P, \theta$, and Φ are perturbations contributions to the velocity, pressure, temperature and concentration fields, respectively. Substituting (2.13) in (2.7)-(2.11) yields the following perturbations equations.

$$\nabla \cdot \mathbf{V} = 0, \tag{2.14}$$

$$\mathbf{V} = -\nabla P + \left(Ra_T \theta + \frac{1}{Le} Ra_S \Phi \right) \hat{e_z}, \tag{2.15}$$

$$\frac{\partial \theta}{\partial t} + W \frac{dT_B}{dz} + \mathbf{V} \cdot \nabla \theta = \nabla^2 \theta + \frac{Ge}{Ra_T} \mathbf{V} \cdot \mathbf{V}, \tag{2.16}$$

$$\omega \frac{\partial \Phi}{\partial t} + W \frac{dC_B}{dz} + \mathbf{V} \cdot \nabla \Phi = \frac{1}{Le} \nabla^2 \Phi. \tag{2.17}$$

with the boundary conditions

$$z = 0, 1 \quad W = \theta = \Phi = 0. \tag{2.18}$$

Here, it is assumed that the perturbations $(\mathbf{V}, P, \theta, \Phi)$ which are defined on $(x, y, z) \in \mathbb{R}^2 \times [0, 1]$, are periodic functions in x and y directions with period $\frac{2\pi}{a_x}$ and $\frac{2\pi}{a_y}$, respectively (with $a_x > 0, a_y > 0$). The overall wave number is given by $a = \sqrt{{a_x}^2 + {a_y}^2}$. And, the periodicity cell Ω is denoted as (as in Kumar et. al [27])

$$\Omega = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0, 1]$$

.

3 Linear instability analysis

To proceed with the linear stability analysis, the product of perturbations $\mathbf{V}\cdot\nabla\theta, \mathbf{V}\cdot\nabla\Phi, \mathbf{V}\cdot\mathbf{V}$ are neglected from perturbation equations (2.14)-(2.18). Since, this system of equations constitutes a linear boundary value problem for the perturbations, we may seek solutions in the form of normal modes (as in Deepika [20], Kumar et. al [27])

$$(\mathbf{V}, P, \theta, \Phi) = (\mathbf{V}(z), P(z), \theta(z), \Phi(z))e^{i(a_x x + a_y y)}e^{\sigma t}. \tag{3.1}$$

Here, $\sigma = \sigma_r + i\sigma_i$ is the growth rate parameter and $a^2 = {a_x}^2 + {a_y}^2$, where a_x, a_y are wave numbers in x, y directions, respectively. Also, $\sigma_r = 0$ gives the condition on marginal stability or neutral stability. Substituting (3.1) into linearized perturbation equations yields the eigenvalue problem for linear stability theory:

$$(D^2 - a^2)W + a^2(Ra_T\theta + \frac{1}{Le}Ra_S\Phi) = 0, \tag{3.2}$$

$$(D^2 - a^2 - \sigma)\theta + W = 0, \tag{3.3}$$

$$(D^2 - a^2 - \omega Le\sigma)\Phi + LeW = 0, \tag{3.4}$$

with the boundary conditions

$$z = 0, 1 \qquad W = \theta = \Phi = 0. \tag{3.5}$$

Here, $D = \frac{d}{dz}$. It is noted that contribution of the viscous dissipation term is absent in (3.2)-(3.5) since the steady-state velocity is in quiescent. Hence, the viscous dissipation has no effect in the linear stability analysis.

4 Nonlinear stability analysis

The linear instability analysis gives limited information on the onset of convection since the potential growth of nonlinear perturbation terms in (2.14)-(2.18) are neglected. Therefore, the linear instability theory gives sufficient conditions for instability only. However, the nonlinear stability theory by employing energy method provides a sufficient condition for stability (see [16]). In order to study the nonlinear stability, the energy functional is defined as

$$E(t) = \frac{\xi_1}{2}\|\theta\|^2 + \frac{\xi_2}{2}\omega Le\|\beta\|^2, \tag{4.1}$$

where ξ_1, ξ_2 are coupling parameters which are chosen to be optimally and $\beta = \frac{\Phi}{Le}$. Multiplying equation (2.15) by $\mathbf{V}$, (2.16) by θ, (2.17) by β and integrating over periodicity cell Ω and by using Gauss-divergence theorem along with the boundary conditions (3.5), the obtained equations are as follows

$$\|\mathbf{V}\|^2 = Ra_T\langle\theta W\rangle + Ra_S\langle\beta W\rangle, \tag{4.2}$$

$$\frac{1}{2}\frac{d}{dt}\left\|\theta\right\|^2 = -\left\|\nabla\theta\right\|^2 + \langle W\theta\rangle + \frac{Ge}{Ra_T}\left\langle(\mathbf{V}\cdot\mathbf{V})\theta\right\rangle, \tag{4.3}$$

$$\omega Le\frac{1}{2}\frac{d}{dt}\left\|\beta\right\|^2 = -\left\|\nabla\beta\right\|^2 + \langle W\beta\rangle, \tag{4.4}$$

where, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are inner product and norm in $L^2(\Omega)$ space. From (4.2)-(4.4), we write

$$\frac{dE}{dt} = I(t) - D(t) + N(t), \tag{4.5}$$

where

$$I(t) = \xi_1\langle W\theta\rangle + \xi_2\langle W\beta\rangle + Ra_T\langle\theta W\rangle + Ra_S\langle\beta W\rangle,$$

$$D(t) = \xi_1\left\|\nabla\theta\right\|^2 + \xi_2\left\|\nabla\beta\right\|^2 + \left\|\mathbf{V}\right\|^2,$$

$$N(t) = \xi_1\frac{Ge}{Ra_T}\left\langle(\mathbf{V}\cdot\mathbf{V})\theta\right\rangle. \tag{4.6}$$

The maximization problem is defined as

$$m = \max_H\frac{I}{D}, \tag{4.7}$$

where H is a set of all admissible solutions over which we seek for maximum. H is the space of all perturbations such that, $H = \{(\mathbf{V},\theta,\Phi)\in L^2(\Omega) : \nabla\cdot\mathbf{V} = 0, W = \theta = \Phi = 0 \text{ at } z = 0,1\}$. From equations (4.5)-(4.7), one can write

$$\frac{dE}{dt} \leq -D(1-m) + N. \tag{4.8}$$

Applying Cauchy-Schwarz inequality and Hölder's inequality on the term $\left\langle(\mathbf{V}\cdot\mathbf{V})\theta\right\rangle$,

$$\begin{aligned}\left\langle(\mathbf{V}\cdot\mathbf{V})\theta\right\rangle &\leq \|\theta\|\,\|\mathbf{V}\|_4^2\\ &= \|\theta\|\left(\int_\Omega V^4 d\Omega\right)^{1/2}\\ &\leq \|\theta\|\left(|\Omega|^{1/3}\left(\int_\Omega V^6 d\Omega\right)^{2/3}\right)^{1/2}\\ &= \|\theta\|\,|\Omega|^{1/6}\left(\int_\Omega V^6 d\Omega\right)^{1/3}\\ &= \|\theta\|\,|\Omega|^{1/6}\left(\left(\int_\Omega V^6 d\Omega\right)^{1/6}\right)^2\\ &= c_1\|\theta\|\,\|\mathbf{V}\|_6^2 \qquad (\text{where } \ c_1 = |\Omega|^{1/6}).\end{aligned} \tag{4.9}$$

Here, $c_1 = |\Omega|^{1/6}$, ($|\Omega|$ is the measure of Ω). Using the Sobolev and Poincaré inequalities on $\|\mathbf{V}\|_6$,

$$\|\mathbf{V}\|_6 \leq c_2 \|\mathbf{V}\|_{H'(\Omega)} \leq c_3 \|\nabla \mathbf{V}\|, \tag{4.10}$$

where, c_2 and c_3 are constants. Our next step is to prove $\|\nabla \mathbf{V}\|^2 \leq c_4(\|\nabla\theta\|^2 + \|\nabla\beta\|^2)$. To obtain this, take Laplacian of (2.15)

$$V_{i,jj} = -P_{,ijj} + Ra_T \theta_{,jj} \hat{e_z} + Ra_S \beta_{,jj} \hat{e_z}, \tag{4.11}$$

where, $(U = V_1, V = V_2, W = V_3)$. Multiplying with -$V_i$ and integrating over Ω, yields

$$\begin{aligned} -\langle V_{i,jj} V_i \rangle &= \langle P_{,ijj} V_i \rangle - Ra_T \langle \theta_{,jj} W \rangle - Ra_S \langle \beta_{,jj} W \rangle \\ &= Ra_T \langle \theta_{,j} W_{,j} \rangle + Ra_S \langle \beta_{,j} W_{,j} \rangle . \end{aligned} \tag{4.12}$$

By considering the fact that $\mathbf{V} = (U, V, W)$ is a divergence free vector, using the boundary conditions as well as the symmetry conditions at the boundaries, we have that the normal component of the velocity vanishes on the whole boundary of Ω. Then, the first term on the right hand side of (4.12) vanishes. Integrating $\langle V_{i,jj} V_i \rangle$, by considering the boundary conditions and symmetry conditions, implies

$$\begin{aligned} -\langle V_{i,jj} V_i \rangle = \|U_{,1}\|^2 + \|U_{,2}\|^2 + \|V_{,1}\|^2 + \|V_{,2}\|^2 + \|\nabla W\|^2 \\ - \langle U U_{,33} \rangle - \langle V V_{,33} \rangle . \end{aligned} \tag{4.13}$$

From (2.15), one can write

$$U = -P_{,1}, \quad V = -P_{,2}, \quad W = -P_{,3} + Ra_T \theta + Ra_T \beta. \tag{4.14}$$

From (4.14), one can deduce

$$\begin{aligned} -\langle U U_{,33} \rangle - \langle V V_{,33} \rangle &= -\langle P_{,1} P_{,133} + P_{,2} P_{,233} \rangle \\ &= \langle P_{,11} P_{,33} + P_{,22} P_{,33} \rangle = \langle (P_{,11} + P_{,22}) P_{,33} \rangle \\ &= \langle (-U_{,1} - V_{,2}) P_{,33} \rangle = \langle (W_{,3}) P_{,33} \rangle \\ &= \langle W_{,3}(-W_{,3} + Ra_T \theta_{,3} + Ra_S \beta_{,3}) \rangle \\ &= -\|W_{,3}\|^2 + Ra_T \langle W_{,3} \theta_{,3} \rangle + Ra_S \langle W_{,3} \beta_{,3} \rangle , \end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\left\|U_{,3}\right\|^2+\left\|V_{,3}\right\|^2 &= \left\langle U_{,3}U_{,3}\right\rangle+\left\langle V_{,3}V_{,3}\right\rangle\\ &= \langle P_{,13}\,P_{,13}\,\rangle+\langle P_{,23}\,P_{,23}\,\rangle=-\langle P_{,3}\,P_{113}\rangle-\langle P_{,3}\,P_{223}\rangle\\ &= \left\langle P_{,3}\,(U_{,13}+V_{,23})\right\rangle\\ &= \left\langle P_{,3}\,(-W_{,33})\right\rangle\\ &= \left\langle (W-Ra_T\theta-Ra_S\beta)(W_{,33})\right\rangle\\ &= -\left\|W_{,3}\right\|^2+Ra_T\left\langle W_{,3}\theta_{,3}\right\rangle+Ra_S\left\langle W_{,3}\beta_{,3}\right\rangle\\ &.\end{aligned} \tag{4.16}$$

From the equations (4.13)-(4.16), one can write

$$\begin{aligned}\|\nabla\mathbf{V}\|^2 &= Ra_T\left\langle W_{,i}\theta_{,i}\right\rangle+Ra_S\left\langle W_{,i}\beta_{,i}\right\rangle\\ &= Ra_T\langle\nabla W\cdot\nabla\theta\rangle+Ra_S\langle\nabla W\cdot\nabla\beta\rangle\\ &\le Ra_T\left(\frac{\nu_1}{2}\|\nabla\theta\|^2+\frac{1}{2\nu_1}\|\nabla W\|^2\right)+Ra_S\left(\frac{\nu_2}{2}\|\nabla\beta\|^2+\frac{1}{2\nu_2}\|\nabla W\|^2\right)\\ &= Ra_T^2\|\nabla\theta\|^2+Ra_S^2\|\nabla\beta\|^2+\frac{1}{2}\|\nabla W\|^2 \qquad (\text{set }\ \nu_1=2Ra_T,\nu_2=2Ra_S)\\ &\le Ra_T^2\|\nabla\theta\|^2+Ra_S^2\|\nabla\beta\|^2+\frac{1}{2}\|\nabla V\|^2,\end{aligned} \tag{4.17}$$

which implies,

$$\begin{aligned}\|\nabla\mathbf{V}\|^2 &\le 2(Ra_T^2\|\nabla\theta\|^2+Ra_S^2\|\nabla\beta\|^2)\\ &\le c_4(\|\nabla\theta\|^2+\|\nabla\beta\|^2) \qquad (c_4=\max(2Ra_T^2,2Ra_S^2)).\end{aligned} \tag{4.18}$$

Therefore from the equations (4.9),(4.10), (4.17)-(4.18), one can be written as

$$\begin{aligned}\left\langle(\mathbf{V}\cdot\mathbf{V})\theta\right\rangle &\le c_1\|\theta\|\,\|\mathbf{V}\|_6^2\\ &\le c_1c_3^2\|\theta\|\,\|\nabla\mathbf{V}\|^2\\ &\le c_1c_3^2C_4\|\theta\|\,(\|\nabla\theta\|^2+\|\nabla\beta\|^2)\\ &= c_5\|\theta\|\,(\|\nabla\theta\|^2+\|\nabla\beta\|^2),\end{aligned} \tag{4.19}$$

where $c_5 = c_1 c_3^2 c_4$. Therefore,

$$\begin{aligned} N = \xi_1 \frac{Ge}{Ra_T} \left\langle (\mathbf{V} \cdot \mathbf{V})\theta \right\rangle \leq \xi_1 \frac{Ge}{Ra_T} c_5 \left\| \theta \right\| (\left\| \nabla \theta \right\|^2 + \left\| \nabla \beta \right\|^2) \\ = c_6 \left\| \theta \right\| (\left\| \nabla \theta \right\|^2 + \left\| \nabla \beta \right\|^2) \quad (\text{let } \; c_6 = \frac{Ge}{Ra_T} \sqrt{2} c_5) \\ \leq c_6 E^{1/2} D. \end{aligned} \tag{4.20}$$

From equations (4.8), (4.20) we write,

$$\frac{dE}{dt} \leq D(m - 1 + c_6 E^{1/2}). \tag{4.21}$$

Now, assuming that

$$E(0) \leq \left(\frac{1-m}{c_6} \right)^2, \tag{4.22}$$

Then, one can write

$$\begin{aligned} \frac{dE}{dt} &\leq D(m - 1 + c_6 E(0)^{1/2}), \qquad \forall t \geq 0 \\ &\leq c_7 D \qquad (\text{where } \; c_7 = (m - 1 + c_6 E(0)^{1/2} < 0) \\ &\leq 2 c_7 \pi^2 E. \qquad (\text{by the use of Poincaré inequality}) \end{aligned} \tag{4.23}$$

By integrating the inequality (4.23), we get

$$E(t) \leq E(0) e^{2 c_7 \pi^2 t} \qquad (\text{where } \; c_7 < 0) \tag{4.24}$$

which implies the decay of $E(t)$ with time t. This shows that $\|\theta\|^2$, $\|\beta\|^2$ decays with time. Applying the arithmetic-geometric mean inequality on (2.15) one can show

$$\|\mathbf{V}\|^2 \leq 2(\|\theta\|^2 + \|\beta\|^2). \tag{4.25}$$

As a consequence, the time decay of $\|\theta\|^2$, $\|\beta\|^2$ ensures the decay of $\|\mathbf{V}\|^2$. This implies nonlinear conditional stability for velocity, temperature and concentration fields.

4.1 Eigenvalue problem:

To study the onset of convection the critical argument $m = 1$ is considered. This requires the understanding of the maximization problem given in the equation (4.7). To study this we derive the Euler-Lagrange equations as described in Gelfand and Fomin ([29]). Therefore the expected Euler-lagrange equations are

$$\xi_1 \theta \hat{e_z} + \xi_2 \beta \hat{e_z} + (Ra_T \theta + Ra_S \beta) \hat{e_z} - 2\mathbf{v} = \nabla \lambda, \tag{4.26}$$

$$\xi_1 W + Ra_T W + 2\xi_1 \nabla^2 \theta = 0, \tag{4.27}$$

$$\xi_2 W + Ra_S W + 2\xi_2 \nabla^2 \beta = 0. \tag{4.28}$$

Here, λ is the Lagrange multiplier. To eliminate λ, applying curlcurl on (4.26) and taking third component of resulting equation, we obtain

$$\begin{aligned} &2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)W - \xi_1\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\theta - \xi_2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\beta \\ &- Ra_T\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - Ra_S\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) = 0. \end{aligned} \tag{4.29}$$

After applying normal modes (3.1) to the equations (4.29), (4.27) and (4.28), the obtained eigenvalue problem for the nonlinear stability theory is:

$$(D^2 - a^2)W + \frac{a^2}{2}\xi_1\theta + \frac{a^2}{2}\xi_2\beta + \frac{a^2}{2}Ra_T\theta + \frac{a^2}{2}Ra_S\beta = 0, \tag{4.30}$$

$$(D^2 - a^2)\theta + \frac{1}{2\xi_1}\left(\xi_1 + Ra_T\right)W = 0, \tag{4.31}$$

$$(D^2 - a^2)\beta + \frac{1}{2\xi_2}\left(\xi_2 + Ra_S\right)W = 0, \tag{4.32}$$

with the boundary conditions $z = 0, 1: \quad W = \theta = \Phi = 0$. Here, ξ_1, ξ_2 are positive coupling parameters which are calculated as $\xi_1 = Ra_T$, $\xi_2 = |Ra_S|$ (as given in Deepika [20]).

5 Results and Discussion

The eigenvalue problem for the linear stability theory is solved as given in Deepika [20]. The eigenvalue problems (3.2)-(3.5) and (4.30)-(4.32) reduces to the eigenvalue problems studied in Deepika [20] for Soret number, $Sr = 0$. And, the critical thermal Rayleigh number Ra_L is given by

$$Ra_L = \min_{a_x} \min_{a_y} Ra_T(a_x, a_y, Le, Ra_S) \tag{5.1}$$

The eigenvalue problem for nonlinear stability theory is solved by using shooting and Runge-Kutta methods as given in Barletta and Nield [9]. The critical thermal Rayleigh number for nonlinear stability theory Ra_E is given as

$$Ra_E = \max_{\xi_1} \max_{\xi_2} \min_{a_x} \min_{a_y} Ra_T(a_x, a_y, \xi_1, \xi_2, Le, Ra_S) \tag{5.2}$$

The linear instability analysis gives linear instability boundary and the nonlinear stability analysis gives nonlinear stability boundary. Here, a_L and a_E are the critical thermal Rayleigh numbers for the linear and nonlinear stability theory, respectively. The region between these two curves is the region of sub-critical instabilities.

The analytical expression for the critical thermal Rayleigh number for the linear stability theory for both stationary and oscillatory modes are given in equations (15) and (20) in Deepika [20]. To get the analytical expression for the critical thermal Rayleigh number for the case of the nonlinear stability theory, one can apply the procedure given in Deepika [20]. Assuming the solution $W(z), \theta(z), \beta(z)$ of eigenvalue problem (4.30)-(4.32) as

$$(W(z), \theta(z), \beta(z)) = (W_0, \theta_0, \beta_0)\sin(n\pi z), \quad n = 1, 2, 3\cdot \tag{5.3}$$

which satisfies the boundary condition $z = 0, 1: \quad W = \theta = \Phi = 0$. Substituting (5.3) into eigenvalue problem (4.30)-(4.32), one can obtain the following matrix equation.

$$\begin{pmatrix} -(n^2\pi^2 + a^2) & a^2 Ra_T & \frac{a^2}{2}(Ra_S + |Ra_S|) \\ 1 & -(n^2\pi^2 + a^2) & 0 \\ \frac{(Ra_S + |Ra_S|)}{2|Ra_S|)} & 0 & -(n^2\pi^2 + a^2) \end{pmatrix} \begin{pmatrix} W_0 \\ \theta_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.4}$$

The condition of nontrivial solution of the above system (5.4), gives an expression on the eigenvalue Ra_T in the following form:

$$Ra_T = \frac{(n^2\pi^2 + a^2)^2}{a^2} - \frac{1}{4}\frac{(Ra_S + |Ra_S|)^2}{|Ra_S|} \tag{5.5}$$

Here, Ra_S is the solutal Rayleigh number which can be positive or negative. In addition, $n = 1$ is the most unstable mode (Straughan [16]). The minimum value of the Rayleigh number occurs at the critical wave number $a_E = \pi$ Then, the critical thermal Rayleigh number for nonlinear stability theory is

$$Ra_E = \begin{cases} 4\pi^2 - Ra_S & \text{when } \mathrm{Ra}_S > 0, \\ 4\pi^2 & \text{when } \mathrm{Ra}_S < 0. \end{cases} \tag{5.6}$$

When $Ra_S > 0$, the expression for the critical thermal Rayleigh number Ra_E is consistent with the critical Rayleigh number Ra_T^{St} (in case of stationary modes) for linear stability theory (Deepika [20]).

Table 1 shows the comparison between the critical thermal Rayleigh numbers of the linear and nonlinear stability theories. When $Ra_S > 0$, it has destabilizing effect. The critical thermal Rayleigh numbers Ra_L (when $Le = 0.1$), Ra_E (when $Le = 10$) gives a very good agreement with Guo and Kaloni [5]. This result is consistent with equation (5.6) and with (equation (15) of) Deepika [20]. The critical thermal Rayleigh number Ra_L (when $Le = 10$) meet with the result of Deepika et al.[28] (Figure 5). When Ra_S changes from positive to negative, the onset of convection transit from stationary modes to oscillatory modes. Detailed explanation on this transition of modes is given in Deepika et al.[28].

Ra_S	$Le = 10$		$Le = 0.1$		$Le = 10$	
	a_L	Ra_L	a_L	Ra_L	a_E	Ra_E
−30	3.14251	46.42626	3.14206	69.47842	3.1412	39.47841
−20	3.14262	45.42626	3.14161	59.47842	3.1412	39.47841
−10	3.1416	44.42626	3.14234	49.47842	3.1412	39.47841
0	3.14114	39.47842	3.14114	39.47842	3.1415	39.4782
10	3.14262	29.47842	3.14262	29.47842	3.1414	29.4782
20	3.14206	19.47842	3.14206	19.47842	3.1414	19.4782
30	3.14206	9.47842	3.14206	9.47842	3.1412	9.47841

Table 1: Comparison between the critical thermal Rayleigh numbers for linear and nonlinear theories when $Le = 10$.

When Ra_S is positive, the critical thermal Rayleigh numbers for both the linear and nonlinear theories coincides. Hence, the region of sub-critical instabilities do not exist. When Ra_S is negative, the region of sub-critical instabilities increases along negative Ra_S direction.

6 Conclusions

The onset of double-diffusive convection in a horizontal fluid-saturated porous layer has been investigated. The effect of viscous dissipation is considered. Momentum transfer is modelled by the use of Darcy's law and with the assumption of the Oberbeck–Boussinesq approximation. The nonlinear stability of present problem is investigated via energy method.

Euler–Lagrange equations are employed to solve the maximization problem. On condition (4.22), the nonlinear stability is achieved. Therefore, the viscous dissipation parameter do not play a major role on nonlinear stability threshold. The conclusion is that, when $Ra_S > 0$, the nonlinear stability threshold is same as the linear instability threshold, which implies the region of sub-critical instabilities do not exist. But, when $Ra_S < 0$, the region of sub-critical instabilities increases along with negative Ra_S direction.

The above conclusion is when the steady state velocity is zero. But, if the basic state velocity includes the horizontal mass throughflow, the effect of viscous dissipation plays a significant role on the onset of convection.

References

[1] Nield, D. A., and Bejan, A.: Convection in porous media. Vol. 3. New York: Springer, 2006.

[2] Gebhart, B.: Effects of viscous dissipation in natural convection. J. Fluid Mech. 14(2), 225-232 (1962)

[3] Nield, D. A.: Onset of thermohaline convection in a porous medium. Water Resources Research 4(3), 553-560 (1968)

[4] Guo, J, and Kaloni, P. N.: Double-diffusive convection in a porous medium, nonlinear stability, and the Brinkman effect. Studies in Applied Mathematics 94(3), 341-358, (1995)

[5] Guo, J, and Kaloni, P. N.: Nonlinear stability of convection induced by inclined thermal and solutal gradients. Zeitschrift für angewandte Mathematik und Physik ZAMP 46(5), 645-654 (1995)

[6] Love, A. J., Simmons, C. T., and Nield, D. A.: Double-diffusive convection in groundwater wells. Water resources research 43(8), (2007).

[7] Barletta, A., Celli, M. and Rees, D.A. S.: The onset of convection in a porous layer induced by viscous dissipation: a linear stability analysis. International Journal of Heat and Mass Transfer 52(1-2), 337-344 (2009)

[8] Barletta, A., and Storesletten, L.: Viscous dissipation and thermoconvective instabilities in a horizontal porous channel heated from below. International journal of thermal sciences 49(4), 621-630 (2010)

[9] Barletta, A., and Nield, D. A.: Thermosolutal convective instability and viscous dissipation effect in a fluid-saturated porous medium. International Journal of Heat and Mass Transfer 54(7-8), 1641-1648 (2011)

[10] Straughan, B.: Nonlinear stability for thermal convection in a Brinkman porous material with viscous dissipation. Transport in Porous Media 134(2), 303-314 (2020)

[11] Barletta, A., and Mulone, G.: The energy method analysis of the Darcy–Bénard problem with viscous dissipation. Continuum Mechanics and Thermodynamics 33(1), 25-33 (2021)

[12] Trevisan, O. V., and Bejan, A.: Natural convection with combined heat and mass transfer buoyancy effects in a porous medium. International Journal of Heat and Mass Transfer 28 (8), 1597-1611 (1985)

[13] Trevisan, O. V., and Bejan, A.: Combined heat and mass transfer by natural convection in a porous medium. In Advances in heat transfer, 20, 315-352 (1990)

[14] Nield, D. A., and Kuznetsov, A. V.: Do isoflux boundary conditions inhibit oscillatory double-diffusive convection?. Transport in Porous Media 112(3), 609-618 (2016)

[15] Barletta, A.: On the thermal instability induced by viscous dissipation. International Journal of Thermal Sciences 88, 238-247 (2015)

[16] Straughan, B.: The energy method, stability, and nonlinear convection. 91. Springer Science Business Media, (2013)

[17] Straughan, B.: Stability and wave motion in porous media (Vol. 165). Springer Science Business Media, (2008).

[18] Wooding, R. A.: Rayleigh instability of a thermal boundary layer in flow through a porous medium. Journal of fluid mechanics, 9(2), 183-192 (1960)

[19] Wooding, R. A.: Convection in a saturated porous medium at large Rayleigh number or Peclet number. Journal of Fluid Mechanics 15(4), 527-544 (1963)

[20] Deepika, N.: Linear and nonlinear stability of double-diffusive convection with the Soret effect. Transport in Porous Media 121, 93-108 (2018)

[21] Barletta, A., Celli, M. and Rees, D. A. S.: Darcy–Forchheimer flow with viscous dissipation in a horizontal porous layer: onset of convective instabilities. Journal of heat transfer 131(7), (2009)

[22] Barletta, A., Rossi di Schio, E. and Storesletten, L.: Convective roll instabilities of vertical throughflow with viscous dissipation in a horizontal porous layer. Transport in porous media 81 461-477 (2010)

[23] Deepika, N., and Narayana, P. A. L.: Linear Stability of Double diffusive Convection of Hadley-Prats Flow with Viscous Dissipation in a Porous Media. Procedia Engineering 127, 193-200 (2015)

[24] Nield, D. A., Barletta, A. and Celli, M.: The effect of viscous dissipation on the onset of convection in an inclined porous layer. Journal of fluid mechanics 679, 544-558 (2011)

[25] Joseph, D. D.: Nonlinear stability of the Boussinesq equations by the method of energy. Archive for Rational Mechanics and Analysis 22, 163-184 (1966)

[26] Rionero, S., and Mulone, G.: On the non-linear stability of a thermo-diffusive fluid mixture in a mixed problem. Journal of mathematical analysis and applications 124(1), 165-188 (1987)

[27] Kumar, G., Narayana, P. A. L. and Sahu, K. C.: Linear and nonlinear thermosolutal instabilities in an inclined porous layer. Proceedings of the Royal Society A 476(2233), 20190705 (2020)

[28] Deepika, N., Murthy, P. V. S. N. and Narayana, P. A. L.: The effect of magnetic field on the stability of double-diffusive convection in a porous layer with horizontal mass throughflow. Transport in Porous Media 134, 435-452 (2020)

[29] Gelfand, I. M., and Fomin, S.V.: Calculus of Variations Prentice-Hall, Inc., Englewood Cliffs (1963).

7 Funding

The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

8 Acknowledgements

The authors thank the reviewers for their constructive comments for the improvement of the paper to a great extent.