# The $k$-Tuple Domination Number Revisited

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**Abstract**

The following fundamental result for the domination number $\gamma(G)$ of a graph $G$ was proved by Alon and Spencer, Arnautov, Lovász and Payan:

$$\gamma(G) \le \frac{\ln(\delta+1)+1}{\delta+1}n,$$

where $n$ is the order and $\delta$ is the minimum degree of vertices of $G$. A similar upper bound for the double domination number was found by Harant and Henning [On double domination in graphs. *Discuss. Math. Graph Theory* **25** (2005) 29–34], and for the triple domination number by Rautenbach and Volkmann [New bounds on the $k$-domination number and the $k$-tuple domination number. *Applied Math. Letters* **20** (2007) 98–102], who also posed the interesting conjecture on the $k$-tuple domination number: for any graph $G$ with $\delta \ge k-1$,

$$\gamma_{\times k}(G) \le \frac{\ln(\delta-k+2)+\ln(\widehat{d}_{k-1}+\widehat{d}_{k-2})+1}{\delta-k+2}n,$$

where $\widehat{d}_m = \sum_{i=1}^{n} \binom{d_i}{m} /n$ is the $m$-degree of $G$. This conjecture, if true, would generalise all the mentioned upper bounds and improve an upper bound proved in [A. Gagarin and V. Zverovich, A generalised upper bound for the $k$-tuple domination number. *Discrete Math.* (to appear)].

In this paper, we prove Rautenbach–Volkmann's conjecture.

Keywords: *graphs, domination number, double domination, triple domination, $k$-tuple domination.*

## 1 Notation

All graphs will be finite and undirected without loops and multiple edges. If $G$ is a graph of order $n$, then $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices in $G$, $d_i$ denotes the degree of $v_i$ and $d = \sum_{i=1}^{n} d_i/n$ is the average degree of $G$. Let $N(x)$ denote the neighbourhood of a vertex $x$. Also let $N(X) = \cup_{x\in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of vertices of $G$, respectively. Put $\delta = \delta(G)$ and $\Delta = \Delta(G)$. A set $X$ is called a dominating set if every vertex not in $X$ is adjacent to a vertex in $X$. The minimum cardinality of a dominating set of $G$ is the domination number $\gamma(G)$. A set $X$ is called a $k$-tuple dominating set of $G$ if for every vertex $v \in V(G)$, $|N[v] \cap X| \ge k$. The minimum cardinality of a $k$-tuple dominating set of $G$ is the $k$-tuple domination number $\gamma_{\times k}(G)$. The $k$-tuple domination number is only defined for graphs with $\delta \ge k-1$. It is easy to see that $\gamma(G) = \gamma_{\times 1}(G)$ and $\gamma_{\times k}(G) \le \gamma_{\times k'}(G)$ for $k \le k'$. The 2-tuple domination number $\gamma_{\times 2}(G)$ is called the double domination number and the 3-tuple domination number $\gamma_{\times 3}(G)$ is called the triple domination number. A number of interesting results on the $k$-tuple domination number can be found in [3]–[8] and [11].

## 2 Introduction

The following fundamental result was proved by many authors:

**Theorem 1 ([1, 2, 9, 10])** *For any graph $G$,*

$$\gamma(G) \leq \frac{\ln(\delta+1)+1}{\delta+1}n.$$

A similar upper bound for the double domination number was found by Harant and Henning [4]:

**Theorem 2 ([4])** *For any graph $G$ with $\delta \geq 1$,*

$$\gamma_{\times 2}(G) \leq \frac{\ln\delta + \ln(d+1)+1}{\delta}n.$$

Rautenbach and Volkmann posed the following interesting conjecture for the $k$-tuple domination number:

**Conjecture 1 ([11])** *For any graph $G$ with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2) + \ln\left(\sum_{i=1}^{n}\binom{d_i+1}{k-1}\right) - \ln(n) + 1}{\delta-k+2}n.$$

For $m \leq \delta$, let us define the $m$-degree $\widehat{d}_m$ of a graph $G$ as follows:

$$\widehat{d}_m = \widehat{d}_m(G) = \sum_{i=1}^{n}\binom{d_i}{m}/n.$$

Note that $\widehat{d}_1$ is the average degree $d$ of a graph and $\widehat{d}_0 = 1$. Also, we put $\widehat{d}_{-1} = 0$.

Since

$$\binom{d_i+1}{k-1} = \binom{d_i}{k-1} + \binom{d_i}{k-2},$$

we see that the above conjecture can be re-formulated as follows:

**Conjecture 1′** *For any graph $G$ with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2) + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\delta-k+2}n.$$

It may be pointed out that this conjecture, if true, would generalise Theorem 2 and also Theorem 1 taking into account that $\widehat{d}_{-1} = 0$. Rautenbach and Volkmann proved the above conjecture for the triple domination number:

**Theorem 3 ([11])** *For any graph $G$ with $\delta \geq 2$,*

$$\gamma_{\times 3}(G) \leq \frac{\ln(\delta-1) + \ln(\widehat{d}_2 + d) + 1}{\delta-1}n.$$

The next result generalises all the above theorems, but it is still far from Conjecture 1′.

**Theorem 4 ([3])** *For any graph $G$ with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2) + \ln\left(\sum_{m=1}^{k-1}(k-m)\widehat{d}_m + \epsilon\right) + 1}{\delta-k+2}n,$$

*where $\epsilon = 1$ if $k = 1$ or 2, and $\epsilon = -d$ if $k \geq 3$.*

## 3 Proof of the Conjecture

The following theorem proves Rautenbach–Volkmann's conjecture.

**Theorem 5** *For any graph $G$ with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\delta - k + 2} n.$$

**Proof:** Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability $p$, $0 \leq p \leq 1$. For $m = 0, 1, ..., k-1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N(v_i) \cap A| = m\}.$$

Also, for $m = 0, 1, ..., k-2$, we denote

$$A_m = \{v_i \in A : |N(v_i) \cap A| = m\}.$$

For each set $A_m$, we form a set $A'_m$ in the following way. For every vertex in the set $A_m$, we take $k-m-1$ neighbours not in $A$ and add them to $A'_m$. Such neighbours always exist because $\delta \geq k-1$. It is obvious that $|A'_m| \leq (k-m-1)|A_m|$. For each set $B_m$, we form a set $B'_m$ by taking $k-m-1$ neighbours not in $A$ for every vertex in $B_m$. We have $|B'_m| \leq (k-m-1)|B_m|$.

We construct the set $D$ as follows:

$$D = A \cup \left( \bigcup_{m=0}^{k-2} A'_m \right) \cup \left( \bigcup_{m=0}^{k-1} B_m \cup B'_m \right).$$

The set $D$ is a $k$-tuple dominating set. Indeed, if there is a vertex $v$ which is not $k$-tuple dominated by $D$, then $v$ is not $k$-tuple dominated by $A$. Therefore, $v$ would belong to $A_m$ or $B_m$ for some $m$, but all such vertices are $k$-tuple dominated by the set $D$ by construction.

The expected value of $|D|$ is

$$\begin{aligned} E(|D|) &\leq E\left( |A| + \sum_{m=0}^{k-2} |A'_m| + \sum_{m=0}^{k-1} |B_m| + \sum_{m=0}^{k-1} |B'_m| \right) \\ &\leq E\left( |A| + \sum_{m=0}^{k-2} (k-m-1)|A_m| + \sum_{m=0}^{k-1} (k-m)|B_m| \right) \\ &= E(|A|) + \sum_{m=0}^{k-2} (k-m-1)E(|A_m|) + \sum_{m=0}^{k-1} (k-m)E(|B_m|). \end{aligned}$$

We have

$$E(|A|) = \sum_{i=1}^{n} P(v_i \in A) = pn.$$

Also,

$$\begin{aligned} E(|A_m|) &= \sum_{i=1}^{n} P(v_i \in A_m) \\ &= \sum_{i=1}^{n} p \binom{d_i}{m} p^m (1-p)^{d_i - m} \\ &\leq p^{m+1}(1-p)^{\delta - m} \sum_{i=1}^{n} \binom{d_i}{m} \\ &= p^{m+1}(1-p)^{\delta - m} \widehat{d}_m n \end{aligned}$$

and

$$\begin{aligned}
E(|B_m|) &= \sum_{i=1}^{n} P(v_i \in B_m) \\
&= \sum_{i=1}^{n} (1-p) \binom{d_i}{m} p^m (1-p)^{d_i - m} \\
&\leq p^m (1-p)^{\delta - m + 1} \sum_{i=1}^{n} \binom{d_i}{m} \\
&= p^m (1-p)^{\delta - m + 1} \widehat{d}_m n.
\end{aligned}$$

Taking into account that $\widehat{d}_{-1} = 0$, we obtain

$$\begin{aligned}
E(|D|) &\leq pn + \sum_{m=0}^{k-2} (k-m-1) p^{m+1} (1-p)^{\delta - m} \widehat{d}_m n + \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{\delta - m + 1} \widehat{d}_m n \\
&= pn + \sum_{m=1}^{k-1} (k-m) p^m (1-p)^{\delta - m + 1} \widehat{d}_{m-1} n + \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{\delta - m + 1} \widehat{d}_m n \\
&= pn + \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{\delta - m + 1} (\widehat{d}_{m-1} + \widehat{d}_m) n \\
&= pn + (1-p)^{\delta - k + 2} n \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{k-m-1} (\widehat{d}_{m-1} + \widehat{d}_m).
\end{aligned}$$

Let us denote

$$\mu = \delta - k + 2.$$

Using the inequality $1 - x \leq e^{-x}$, we obtain

$$(1-p)^{\delta - k + 2} = (1-p)^{\mu} \leq e^{-p\mu}.$$

Thus,

$$E(|D|) \leq pn + e^{-p\mu} n \Theta,$$

where

$$\Theta = \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{k-m-1} (\widehat{d}_m + \widehat{d}_{m-1}). \tag{1}$$

We will prove that

$$\Theta \leq \widehat{d}_{k-1} + \widehat{d}_{k-2}.$$

We have

$$\begin{aligned}
\Theta &= \sum_{m=0}^{k-1} (k-m)(\widehat{d}_m + \widehat{d}_{m-1}) \sum_{i=0}^{k-m-1} (-1)^i \binom{k-m-1}{i} p^{m+i} \\
&= k(\widehat{d}_0 + \widehat{d}_{-1}) \binom{k-1}{0} p^0 - k(\widehat{d}_0 + \widehat{d}_{-1}) \binom{k-1}{1} p^1 + \ldots + k(\widehat{d}_0 + \widehat{d}_{-1}) \binom{k-1}{k-1} (-1)^{k-1} p^{k-1} \\
&\quad + (k-1)(\widehat{d}_1 + \widehat{d}_0) \binom{k-2}{0} p^1 + \ldots + (k-1)(\widehat{d}_1 + \widehat{d}_0) \binom{k-2}{k-2} (-1)^{k-2} p^{k-1} \\
&\quad \ldots \\
&\quad \ldots \\
&\quad \ldots
\end{aligned}$$

$$+(1)(\widehat{d}_{k-1}+\widehat{d}_{k-2})\binom{0}{0}(-1)^0p^{k-1}$$

$$= \sum_{j=0}^{k-1}\Big(\sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}(i+j+1)(\widehat{d}_{k-i-j-1}+\widehat{d}_{k-i-j-2})\Big)p^{k-j-1}$$

$$= \sum_{j=0}^{k-1} s_j p^{k-j-1},$$

where

$$\begin{aligned}
s_j &= \sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}(i+j+1)(\widehat{d}_{k-i-j-1}+\widehat{d}_{k-i-j-2}) \\
&\quad \text{(taking into account that } \widehat{d}_{-1}=0\text{)} \\
&= \sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}(i+j+1)\widehat{d}_{k-i-j-1}+\sum_{i=0}^{k-j-2}(-1)^i\binom{i+j}{i}(i+j+1)\widehat{d}_{k-i-j-2} \\
&= \binom{j}{0}(j+1)\widehat{d}_{k-j-1}+\sum_{i=1}^{k-j-1}(-1)^i\binom{i+j}{i}(i+j+1)\widehat{d}_{k-i-j-1} \\
&\qquad +\sum_{i=1}^{k-j-1}(-1)^{i-1}\binom{i+j-1}{i-1}(i+j)\widehat{d}_{k-i-j-1} \\
&= (j+1)\widehat{d}_{k-j-1}+\sum_{i=1}^{k-j-1}(-1)^i(j+1)\binom{i+j}{i}\widehat{d}_{k-i-j-1} \\
&= (j+1)\sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}\widehat{d}_{k-i-j-1} \\
&= (j+1)\sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}\sum_{l=1}^{n}\binom{d_l}{k-i-j-1}/n \\
&= (j+1)\sum_{l=1}^{n}\sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}\binom{d_l}{k-i-j-1}/n \\
&= (j+1)\sum_{l=1}^{n}\binom{d_l-j-1}{k-j-1}/n \qquad \text{(by Lemma 3)} \\
&\geq 0.
\end{aligned}$$

Thus, the function $\Theta(p)=s_0p^{k-1}+s_1p^{k-2}+...+s_{k-1}$ is monotonously increasing in $0\leq p\leq 1$. Therefore, (1) implies

$$\Theta \leq \widehat{d}_{k-1}+\widehat{d}_{k-2}.$$

We obtain

$$E(|D|)\leq pn+e^{-p\mu}n\Theta \leq pn+e^{-p\mu}n(\widehat{d}_{k-1}+\widehat{d}_{k-2}).$$

Let us denote

$$f(p)=pn+e^{-p\mu}n(\widehat{d}_{k-1}+\widehat{d}_{k-2}).$$

For $p\in[0,1]$, the function $f(p)$ is minimised at the point $\min\{1,z\}$, where

$$z=\frac{\ln\mu+\ln(\widehat{d}_{k-1}+\widehat{d}_{k-2})}{\mu}.$$

There are two cases to consider.

If $z \leq 1$, then

$$E(|D|) \leq f(z) = \left(z + \frac{1}{\mu}\right) n = \frac{\ln \mu + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\mu} n.$$

Since the expected value is an average value, there exists a particular $k$-tuple dominating set of order at most $f(z)$, as required.

Suppose now that $z > 1$. Taking into account that $\mu > 0$, we obtain

$$\gamma_{\times k}(G) \leq n < \left(z + \frac{1}{\mu}\right) n = \frac{\ln \mu + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\mu} n,$$

as required. The proof of Theorem 5 is complete. ∎

For $s \geq 1$, let us denote

$$T_t^s = \binom{s}{t} - \binom{s}{t-1} + ... + (-1)^t \binom{s}{0}.$$

**Lemma 1**

$$T_t^s = \binom{s-1}{t}.$$

**Proof:** Induction on $t$:

$$T_t^s = \binom{s}{t} - T_{t-1}^s = \binom{s}{t} - \binom{s-1}{t-1} = \binom{s-1}{t}.$$

∎

**Lemma 2** *For $j \geq 1$,*

$$\binom{j-1}{0} + \binom{j}{1} + ... + \binom{j+i-1}{i} = \binom{j+i}{i}.$$

**Proof:** Induction on $i$:

$$\binom{j-1}{0} + \binom{j}{1} + ... + \binom{j+i-1}{i} = \binom{j+i-1}{i-1} + \binom{j+i-1}{i} = \binom{j+i}{i}.$$

∎

**Lemma 3**

$$\sum_{i=0}^{l} (-1)^i \binom{i+j}{i} \binom{r}{l-i} = \binom{r-j-1}{l}.$$

**Proof:** Induction on $j$. If $j = 0$, then

$$\sum_{i=0}^{l} (-1)^i \binom{i+j}{i} \binom{r}{l-i} = \sum_{i=0}^{l} (-1)^i \binom{r}{l-i} = T_l^r = \binom{r-1}{l},$$

as required.

Suppose that $j \geq 1$ and the equation of Lemma 3 is true for any $j' \leq j-1$. Applying Lemmas 1 and 2, we obtain:

$$\begin{aligned}
\sum_{i=0}^{l}(-1)^i\binom{i+j}{i}\binom{r}{l-i} &= \sum_{i=0}^{l}(-1)^i\left(\binom{j-1}{0}+\binom{j}{1}+\ldots+\binom{j+i-1}{i}\right)\binom{r}{l-i} \\
&= \binom{j-1}{0}\sum_{i=0}^{l}(-1)^i\binom{r}{l-i}+\binom{j}{1}\sum_{i=1}^{l}(-1)^i\binom{r}{l-i}+\ldots \\
&\qquad +\binom{j+l-1}{l}\sum_{i=l}^{l}(-1)^l\binom{r}{0} \\
&= \binom{j-1}{0}T_l^r-\binom{j}{1}T_{l-1}^r+\ldots+\binom{j+l-1}{l}(-1)^lT_0^r \\
&= \sum_{i=0}^{l}(-1)^i\binom{j+i-1}{i}T_{l-i}^r \\
&= \sum_{i=0}^{l}(-1)^i\binom{j+i-1}{i}\binom{r-1}{l-i} \\
&= \binom{r-j-1}{l}. \qquad\qquad \text{(by hypothesis)}
\end{aligned}$$

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