Upper Bounds for α-Domination Parameters

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Abstract. We provide a new upper bound for the α-domination number in terms of a parameter α, $0 < \alpha \leq 1$, and graph vertex degrees. This result generalises the well-known Caro-Roditty bound for the domination number of a graph. The same probabilistic construction is used to generalise another well-known upper bound for the classical domination in graphs. Using a different probabilistic construction, we prove similar upper bounds for the α-rate domination number, which combines the concepts of α-domination and k-tuple domination.

Key words. Graph, Domination, α-Domination, α-Rate domination, Probabilistic method

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1. Introduction

Domination is one of the fundamental concepts in graph theory with various applications to ad hoc networks, biological networks, distributed computing, social networks and web graphs [1,5,7,12]. Dominating sets in graphs are natural models for facility location problems in operational research. An important role is played by multiple domination. For example, k-dominating sets can be used for balancing efficiency and fault tolerance [7].

We consider undirected simple finite graphs. If G is a graph of order n, then $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices of G, d_i denotes the degree of v_i, $i = 1, 2, \ldots, n$, and d_v stands for the degree of a vertex $v \in V(G)$. Let $N(v)$ denote the neighbourhood of a vertex v in G, and $N[v] = N(v) \cup \{v\}$ be the closed neighbourhood of v. A set $X \subseteq V(G)$ is called a *dominating set* if every vertex not in X is adjacent to at least one vertex in X. The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$. A set X is called a *k-dominating set* if every vertex not in X has at least k neighbours in X. The minimum cardinality of a k-dominating set of G is the *k-domination number* $\gamma_k(G)$. A set X is called a *k-tuple dominating set* of G if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a k-tuple dominating set of G is the *k-tuple domination number* $\gamma_{\times k}(G)$. The k-tuple domination number is only defined for graphs with $\delta \geq k - 1$. A number of upper bounds for these two multiple domination numbers can be found in [4,9–11,16].

Let α be a real number satisfying $0 < \alpha \leq 1$. A set $X \subseteq V(G)$ is called an *α-dominating set* of G if for every vertex $v \in V(G) - X$, $|N(v) \cap X| \geq \alpha d_v$, i.e. v is adjacent to at least $\lceil \alpha d_v \rceil$ vertices of X. The minimum cardinality of an α-dominating set of G is called the

α-domination number $\gamma_\alpha(G)$. The α-domination was introduced by Dunbar et al. [8]. It is easy to see that $\gamma(G) \le \gamma_\alpha(G)$, and $\gamma_{\alpha_1}(G) \le \gamma_{\alpha_2}(G)$ for $\alpha_1 < \alpha_2$. Also, $\gamma(G) = \gamma_\alpha(G)$ if α is sufficiently close to 0.

For an arbitrary graph G with n vertices and m edges, denote by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ the minimum and maximum vertex degrees of G, respectively. The following results are proved in [8]:

$$\frac{\alpha\delta n}{\Delta + \alpha\delta} \le \gamma_\alpha(G) \le \frac{\Delta n}{\Delta + (1-\alpha)\delta} \tag{1}$$

and

$$\frac{2\alpha m}{(1+\alpha)\Delta} \le \gamma_\alpha(G) \le \frac{(2-\alpha)\Delta n - (2-2\alpha)m}{(2-\alpha)\Delta}. \tag{2}$$

Interesting results on α-domination perfect graphs can be found in [6]. The problem of deciding whether $\gamma_\alpha(G) \le k$ for a positive integer k is known to be NP-complete [8]. Therefore, it is important to have good upper bounds for the α-domination number and efficient approximation, randomized and heuristic algorithms for finding 'small' α-dominating sets.

For $0 < \alpha \le 1$, the *α-degree* of a graph G is defined as follows:

$$\widehat{d}_\alpha = \widehat{d}_\alpha(G) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{d_i}{\lceil \alpha d_i \rceil - 1}\right).$$

In this paper, we use a probabilistic approach to prove that

$$\gamma_\alpha(G) \le \left(1 - \frac{\widehat{\delta}}{(1+\widehat{\delta})^{1+1/\widehat{\delta}}\,\widehat{d}_\alpha^{\,1/\widehat{\delta}}}\right) n,$$

where $\widehat{\delta} = \lfloor \delta(1-\alpha) \rfloor + 1$. This result generalises the well-known upper bound of Caro and Roditty ([12], p. 48). Using the same probabilistic construction, we also show that

$$\gamma_\alpha(G) \le \frac{\ln(\widehat{\delta}+1) + \ln \widehat{d}_\alpha + 1}{\widehat{\delta}+1} n,$$

which generalises another well-known upper bound of Alon and Spencer [2], Arnautov [3], Lovász [14] and Payan [15]. Finally, we introduce the α-rate domination number, which combines together the concepts of α-domination and k-tuple domination, and show that the α-rate domination number satisfies two similar upper bounds. The random constructions used in this paper also imply randomized algorithms to find α-dominating and α-rate dominating sets satisfying corresponding bounds.

2. New Upper Bounds for the α-Domination Number

One of the strongest known upper bounds for the domination number is due to Caro and Roditty:

Theorem 1. (Caro and Roditty [12], p. 48) *For any graph G with $\delta \ge 1$,*

$$\gamma(G) \le \left(1 - \frac{\delta}{(1+\delta)^{1+1/\delta}}\right) n. \tag{3}$$

The upper bound (3) is generalised for the α-domination number in Theorem 2. Indeed, if d_i are fixed for all $i = 1, 2, \ldots, n$, and α is sufficiently close to 0, then $\widehat{\delta} = \delta$ (provided $\delta \geq 1$) and $\widehat{d}_\alpha = 1$.

Theorem 2. *For any graph G,*

$$\gamma_\alpha(G) \leq \left(1 - \frac{\widehat{\delta}}{(1+\widehat{\delta})^{1+1/\widehat{\delta}}\, \widehat{d}_\alpha^{\,1/\widehat{\delta}}}\right) n, \tag{4}$$

where $\widehat{\delta} = \lfloor \delta(1-\alpha) \rfloor + 1$.

Proof. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with probability

$$p = 1 - \left(\frac{1}{(1+\widehat{\delta})\widehat{d}_\alpha}\right)^{1/\widehat{\delta}}. \tag{5}$$

We denote

$$B = \{v_i \in V(G) - A \ : \ |N(v_i) \cap A| \leq \lceil \alpha d_i \rceil - 1\}.$$

It is obvious that the set $D = A \cup B$ is an α-dominating set. The expectation of $|D|$ is

$$\begin{aligned} E(|D|) &= E(|A|) + E(|B|) \\ &= \sum_{i=1}^{n} P(v_i \in A) + \sum_{i=1}^{n} P(v_i \in B) \\ &= pn + \sum_{i=1}^{n} \sum_{r=0}^{\lceil \alpha d_i \rceil - 1} \binom{d_i}{r} p^r (1-p)^{d_i - r + 1}. \end{aligned}$$

It is easy to see that, for $0 \leq r \leq \lceil \alpha d_i \rceil - 1$,

$$\binom{d_i}{r} \leq \binom{d_i}{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{r}.$$

Also,

$$d_i - \lceil \alpha d_i \rceil \geq \lfloor \delta(1-\alpha) \rfloor.$$

Therefore,

$$\begin{aligned} E(|D|) &\leq pn + \sum_{i=1}^{n} \binom{d_i}{\lceil \alpha d_i \rceil - 1} (1-p)^{d_i - \lceil \alpha d_i \rceil + 2} \sum_{r=0}^{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{r} p^r (1-p)^{\lceil \alpha d_i \rceil - 1 - r} \\ &= pn + \sum_{i=1}^{n} \binom{d_i}{\lceil \alpha d_i \rceil - 1} (1-p)^{d_i - \lceil \alpha d_i \rceil + 2} \\ &\leq pn + (1-p)^{\lfloor \delta(1-\alpha) \rfloor + 2} \widehat{d}_\alpha n \\ &= pn + (1-p)^{\widehat{\delta}+1} \widehat{d}_\alpha n \qquad (6) \\ &= \left(1 - \frac{\widehat{\delta}}{(1+\widehat{\delta})^{1+1/\widehat{\delta}}\, \widehat{d}_\alpha^{\,1/\widehat{\delta}}}\right) n. \end{aligned}$$

Note that the value of p in (5) is chosen to minimize the expression in line (6). Since the expectation is an average value, there exists a particular α-dominating set of order at most $\left(1-\frac{\widehat{\delta}}{(1+\widehat{\delta})^{1+1/\widehat{\delta}}\,\widehat{d}_\alpha^{1/\widehat{\delta}}}\right)n$, as required. The proof of the theorem is complete. □

Notice that in some cases Theorem 2 provides a much better bound than the upper bound in (1). For example, if G is a 1000-regular graph, then Theorem 2 gives $\gamma_{0.1}(G) < 0.305n$, while (1) yields only $\gamma_{0.1}(G) < 0.527n$.

Corollary 1. *For any graph G,*

$$\gamma_\alpha(G) \le \frac{\ln(\widehat{\delta}+1) + \ln \widehat{d}_\alpha + 1}{\widehat{\delta}+1} n. \tag{7}$$

Proof. We put

$$p = \min\left\{1, \frac{\ln(\widehat{\delta}+1) + \ln \widehat{d}_\alpha}{\widehat{\delta}+1}\right\}.$$

Using the inequality $1-p \le e^{-p}$, we can estimate the expression in (6) as follows:

$$E(|D|) \le pn + e^{-p(\widehat{\delta}+1)}\widehat{d}_\alpha n.$$

If $p=1$, then the result easily follows. If $p = \frac{\ln(\widehat{\delta}+1)+\ln \widehat{d}_\alpha}{\widehat{\delta}+1}$, then

$$E(|D|) \le \frac{\ln(\widehat{\delta}+1) + \ln \widehat{d}_\alpha + 1}{\widehat{\delta}+1} n,$$

as required. □

Corollary 1 generalises the following well-known upper bound independently proved by several authors [2,3,14,15]:

$$\gamma(G) \le \frac{\ln(\delta+1)+1}{\delta+1} n. \tag{8}$$

3. α-Rate Domination

Define a set $X \subseteq V(G)$ to be an *α-rate dominating set* of G if for any vertex $v \in V(G)$,

$$|N[v] \cap X| \ge \alpha d_v.$$

We call the minimum cardinality of an α-rate dominating set of G the *α-rate domination number* $\gamma_{\times\alpha}(G)$. It is easy to see that $\gamma_\alpha(G) \le \gamma_{\times\alpha}(G)$. The concept of α-rate domination is similar to the well-known k-tuple domination (for example, see [11,13,16]). For $0<\alpha\le 1$, the *closed α-degree* of a graph G is defined as follows:

$$\widetilde{d}_\alpha = \widetilde{d}_\alpha(G) = \frac{1}{n}\sum_{i=1}^{n}\binom{d_i+1}{\lceil \alpha d_i\rceil - 1}.$$

In fact, the only difference between the α-degree and the closed α-degree is that to compute the latter we choose from d_i+1 vertices instead of d_i, i.e. from the *closed* neighbourhood $N[v_i]$ of v_i instead of $N(v_i)$.

The following theorem provides an analogue of the Caro-Roditty bound (Theorem 1) for the α-rate domination number:

Theorem 3. *For any graph G and $0 < \alpha \le 1$,*

$$\gamma_{\times\alpha}(G) \le \left(1 - \frac{\widehat{\delta}}{(1+\widehat{\delta})^{1+1/\widehat{\delta}}\, \widetilde{d}_{\alpha}^{1/\widehat{\delta}}}\right) n, \tag{9}$$

where $\widehat{\delta} = \lfloor \delta(1-\alpha) \rfloor + 1$.

Proof. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with probability p, $0 \le p \le 1$. For $m \ge 0$, denote by B_m the set of vertices $v \in V(G)$ dominated by exactly m vertices of A and such that $|N[v] \cap A| < \alpha d_v$, i.e.

$$|N[v] \cap A| = m \le \lceil \alpha d_v \rceil - 1.$$

Note that each vertex $v \in V(G)$ is in at most one of the sets B_m and $0 \le m \le \lceil \alpha d_v \rceil - 1$. We form a set B in the following way: for each vertex $v \in B_m$, select $\lceil \alpha d_v \rceil - m$ vertices from $N(v)$ that are not in A and add them to B. Consider the set $D = A \cup B$. It is easy to see that D is an α-rate dominating set. The expectation of $|D|$ is:

$$\begin{aligned}
E(|D|) &\le E(|A|) + E(|B|) \\
&\le \sum_{i=1}^{n} P(v_i \in A) + \sum_{i=1}^{n} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} (\lceil \alpha d_i \rceil - m) P(v_i \in B_m) \\
&= pn + \sum_{i=1}^{n} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} (\lceil \alpha d_i \rceil - m) \binom{d_i + 1}{m} p^m (1-p)^{d_i + 1 - m} \\
&\le pn + \sum_{i=1}^{n} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{m} p^m (1-p)^{d_i + 1 - m} \\
&= pn + \sum_{i=1}^{n} \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} (1-p)^{d_i - \lceil \alpha d_i \rceil + 2} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{m} p^m (1-p)^{\lceil \alpha d_i \rceil - 1 - m} \\
&= pn + \sum_{i=1}^{n} \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} (1-p)^{d_i - \lceil \alpha d_i \rceil + 2} \\
&\le pn + (1-p)^{\lfloor \delta(1-\alpha) \rfloor + 2} \sum_{i=1}^{n} \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} \\
&= pn + (1-p)^{\widehat{\delta}+1} \widetilde{d}_{\alpha} n,
\end{aligned}$$

since

$$(\lceil \alpha d_i \rceil - m) \binom{d_i + 1}{m} \le \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{m}.$$

Thus,

$$E(|D|) \le pn + (1-p)^{\widehat{\delta}+1} \widetilde{d}_{\alpha} n. \tag{10}$$

Minimizing the expression (10) with respect to p, we obtain

$$E(|D|) \le \left(1 - \frac{\widehat{\delta}}{(1+\widehat{\delta})^{1+1/\widehat{\delta}}\, \widetilde{d}_{\alpha}^{1/\widehat{\delta}}}\right) n,$$

as required. The proof of Theorem 3 is complete. □

Corollary 2. *For any graph G,*

$$\gamma_{\times\alpha}(G) \le \frac{\ln(\widehat{\delta}+1) + \ln \widetilde{d}_\alpha + 1}{\widehat{\delta}+1} n. \tag{11}$$

Proof. Using an approach similar to that in the proof of Corollary 1, the result follows if we put

$$p = \min \left\{ 1, \frac{\ln(\widehat{\delta}+1) + \ln \widetilde{d}_\alpha}{\widehat{\delta}+1} \right\}$$

and use the inequality $1 - p \le e^{-p}$ to estimate the expression (10) as follows:

$$E(|D|) \le pn + e^{-p(\widehat{\delta}+1)} \widetilde{d}_\alpha n.$$

□

Note that, similar to Corollary 1, the bound of Corollary 2 also generalises the classical upper bound (8). However, the probabilistic construction used to obtain the bounds (9) and (11) is different from that to obtain the bounds (4) and (7).

4. Final Remarks and Open Problems

Notice that the concept of the α-rate domination number $\gamma_{\times\alpha}(G)$ is 'opposite' to the α-independent α-domination number $i_\alpha(G)$ as defined in [6]. It would be interesting to use a probabilistic construction to obtain an upper bound for $i_\alpha(G)$.

Also, the random constructions used to obtain the upper bounds (4), (7), (9) and (11) imply randomized algorithms to find corresponding dominating sets in a given graph G. It would be interesting to derandomize these algorithms or to obtain independent deterministic algorithms to find corresponding dominating sets satisfying the upper bounds (4), (7), (9) and (11). Algorithms approximating the α- and α-rate domination numbers up to a certain degree of precision would be interesting too. For the k-tuple domination number, an interesting approximation algorithm was found by Klasing and Laforest [13].

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